MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 26 /1984

Order Geometry and Direct Differential Geometry

10. 6. bis 16. 6. 1984

The session took place under the direction of Professor Peter Scherk (Toronto). It reviewed the progress made in the fields of order geometry and direct differential geometry since 1978, the year of the last Oberwolfach meeting on this topic. The lectures and reports were planned by Professor Scherk to fall naturally into two categories: the theory of curves and the foundations of the theory of orders, both in the framework of direct differential geometry. In the case of curve theory, the differentiability conditions are always linear, but the curves considered are not necessarily planar: they can be imbedded in n-space, $n \geq 2$. In the foundations of the theory of order, only plane curves are considered, but differentiability is not necessarily linear: it can just as well be defined by means of circles, conics, graphs of polynomials or other kinds of "quasigraphs".

In curve theory, the emphasis during our session was on convex space curves. Certain known results of classical differential geometry appear to find a more natural setting in direct differential geometry, where the relevant notions are more naturally defined, while the proofs are simpler and lead to stronger generalizations. The study of plane curves of even order is also related to this problem. A lecture on vertex theorems and cusps of caustics completed the survey.

The major progress in the foundations of the theory of orders was the realization that matroid theory provides a natural way to define the dimension of the space of quasigraphs. This led to a reexamination of the particular cases that the theory of quasigraphs is meant to generalize, specially the conical case. These particular cases, "revisited", were thus reviewed, and other new results were presented.

Participants

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Lecture abstracts

T. BISZTRICZKY:

Curve theory I. Singularities of curves in the real projective plane

Let C be a directly differentiable curve in P^2. A point p of C is ordinary if C is locally convex at p (char. (1,1)); otherwise p is singular. We assume that the singular points of C are $n_1(C)$ inflections (char. (1,2)), $n_2(C)$ cusps of the first kind (thorns, char. (2,1)) and $n_3(C)$ cusps of the second kind (beaks, char. (2,2)). Let $n(C) = n_1(C) + n_2(C) + n_3(C)$ and $\bar{n}(C) = n_1(C) + 2n_2(C) + n_3(C)$. Under certain conditions (most notably that every line in P^2 meets C with a positive even multiplicity), we determine minimum values for $n(C)$ and $\bar{n}(C)$.

T. BISZTRICZKY:

Curve theory II. Inflectional convex space curves

In the context of classical differential geometry, B. Segre proved in 1968 that, if a simple curve is drawn on a sphere in Euclidean 3-space, then every interior point of the convex hull of the curve lies on at least four osculating planes of that curve. In 1977, J. L. Weiner generalized this property to curves with n vertices (points of the curve where the geodesic curvature has local extreme values) drawn on a sphere in 3-space and proved that, if the osculating planes at the vertices of the curve meet the curve exactly once, then every point of the interior of the convex hull of the curve lies on exactly n osculating planes of the curve. In 1979, P. Scherk conjectured the following generalization of these two theorems: that if a directly differentiable curve with exactly n singular points in affine 3-space lies on the boundary of its convex hull, then every point of the interior of this convex hull lies on exactly n osculating planes of the curve. We prove here that this conjecture is true in the case where the curve has finite order and no straight line meets the curve in more than two points.

J. SCHAER:

Curve theory III. Convex space curves in real projective 3-space

Let C be a directly differentiable elementary curve in affine 3-space A^3. C is convex if it is the set of extreme points of its convex hull. A point p on C is ordinary if C has a neighbourhood of p which meets every plane in at most three points (then p is regular and has char. (1,1,1)); otherwise p is singular. First we show that C has at least two singular points. Since C has a supporting plane at each point, singular points with char. (1,2,2) and (2,1,2) cannot occur. If the only singular points are of char. (1.1.2), inflection points, then C is called inflectional and has at least four such points.

T. BISZTRICZKY:

Curve theory IV. An n-vertex theorem for convex space curves

If a directly differentiable, elementary, convex, inflectional space curve in real affine 3-space A^3 intersects a plane in a certain way in n points, then it has at least n inflection points.

P. SCHERK:

Curve theory V. A convexity property of arcs of order n in n-space

Let C denote a differentiable arc of order n in real affine n-space, let $C_{n-1}(t)$ be the osculating (n-1)-flat of C at the point C(t) and let H(C) be the convex hull of C. Then

$$C_{n-1}(t_1) \cap C_{n-1}(t_2) \cap \operatorname{int} H(C) = \emptyset$$

if the points $C(t_1)$ and $C(t_2)$ are distinct.

P. SCHERK:

Curve theory VI. Convex curves in n-space; definitions, conjectures, discussion

In real affine n-space, let C denote an elementary curve which is the set of the extremal points of H(C), its convex hull. We know that

$$C_1(p) \cap \operatorname{int} H(C) = \emptyset.$$

Question: which is the number k(p) such that

$$C_{k(p)}(p) \cap \operatorname{int} H(C) = \emptyset,$$

but

$$C_{k(p)+1}(p) \cap \operatorname{int} H(C) \neq \emptyset \ ?$$

Conjecture: if p is regular, then

$$C_{n-2}(p) \cap \operatorname{int} H(C) = \emptyset, \text{ if } n \text{ is odd}$$

and

$$C_{n-3}(p) \cap \operatorname{int} H(C) = \emptyset, \text{ if } n \text{ is even.}$$

The definition of convex curves in real affine n-space is also discussed, together with several other questions and conjectures.

E. HEIL:

Curve Theory VII. Vertex theorems and cusps of caustics

J. W. Bruce, P. J. Giblin and G. G. Gibson (Caustics through the looking glass. The Mathematical Intelligencer 6, 1 (1984), pp. 47-58. Further references may be found there) consider a plane oval M as a mirror and investigate the generic forms of the (real or virtual) caustic C. The reflected bundle of rays possesses an orthogonal trajectory (wave front) W_0 which corresponds to the light source L. If L lies within M, then W_0 is starshaped with respect to L. The wave front W_0 has four vertices, and therefore C has four cusps (if C does not degenerate). If L lies outside M, then W_0 has a double point and C must have only two cusps. If a circle centered at L intersects M in 2n points, then there is a larger circle which intersects W_0 in 2n points too. The 2n-vertex theorem then gives 2n cusps of the caustic C.

P. SCHERK:

Foundations of the geometry of orders I. Introduction

Four examples of the relation between characteristic and order lead to the present investigation, by N. D. Lane, J. M. Turgeon and P. Scherk, of a general theory. These four studies involved linear, conformal, conic-sectional and polynomial differentiability. A fifth case, parabolic differentiability, also led to interesting results, but is so pathological that it cannot be included in a common generalization. All five cases will be presented by N. D. Lane. The results obtained in the search for a general theory will then be presented by J. M. Turgeon.

N. D. LANE:

Foundations of the geometry of orders II. Direct linear differentiability

In the linear case, the characteristic curves are lines in the real affine plane. An arc A is differentiable at an interior point p of finite linear order if A has an ordinary tangent at p. The non-tangent lines through p all intersect A at p or all of them support. The point p is assigned a characteristic (a_0,a_1), where a_0 and a_1 are equal to 1 or 2. The digit a_0 is equal to 1 or 2 according as the non-tangent lines through p all intersect A at p or all support. The digit a_1 is then determined by the condition that $a_0 + a_1$ is odd or even according as the tangent of A at p intersects or supports A at p. There are four kinds of differentiable points: (1,1), (1,2), (2,1) and (2,2). The order of the point p is not less than $a_0 + a_1$ and, if p is elementary (thus there is a neighbourhood of p on A which is decomposed by p into two convex one-sided neighbourhoods), then the order of p is equal to $a_0 + a_1$.

N. D. LANE:

Foundations of the geometry of orders III. Direct conformal differentiability

In the conformal case, the characteristic curves are circles in the conformal plane, together with the basic point p. An arc A is conformally differentiable at an interior point p of finite cyclic order if there is a tangent circle of A at p through each point Q different from p and an osculating circle at p. The non-tangent circles through p all intersect A at p or all support. The non-osculating tangent circles at p all intersect A at p or all support and, in the cases where the osculating circle is not the point circle p, they all support. There are four kinds of points for which the osculating circle at p is not a point circle and four kinds for which it is the point p. These eight points have the characteristics (1,1,1), (1,1,2), (2,2,1), (2,2,2), $(1,1,2)_0$, $(1,2,1)_0$, $(2,1,1)_0$ and $(2,2,2)_0$. If (a_0,a_1,a_2) or $(a_0,a_1,a_2)_0$ is the characteristic of p, then the order of p is not less than $a_0 + a_1 + a_2$ and, if p is an elementary point, its order is equal to $a_0 + a_1 + a_2$.

N. D. LANE:

Foundations of the geometry of orders IV. Direct conical differentiability

In the conical case, the characteristic curves are non-degenerate conic sections in the projective plane, the pairs of distinct lines, the double lines through p and the single point p. The arc A is differentiable at p if there are tangent characteristic curves of A at p, osculating characteristic curves, 4-osculating characteristic curves and a 5-osculating characteristic curve. The non-tangent characteristic curves through p [the non-osculating tangent characteristic curves at p; the osculating characteristic curves which are not 4-osculating; the 4-osculating characteristic curves which are not 5-osculating] all intersect A at p or all support. The tangent characteristic curves are non-degenerate conics or pairs of distinct lines. The osculating characteristic curves are either non-degenerate conics or a pair of lines, one of them the ordinary linear tangent, the other another line through p (Type 1); or all consist of a pair of lines through p, neither of which is the tangent of A at p (Type 2); or a pair of lines, one of which is the tangent of A at p while the other does not pass through p (Type 3). The 4-osculating characteristic curves are either non-degenerate conics or the linear tangent of A at p counted twice (Type 1a) or all consist of the tangent of A at p and another line through p (Type 1b). The 5-osculating conic is non-degenerate or the double tangent at p (Type 1a i) or the point p (Type 1a ii) or the double tangent of A at p (Type 1a iii). There are ten kinds of differentiable points which are not cusps and ten kinds of cusp points. The non-cusp points have the characteristics (1,1,1,1,1;1a i); (1,1,1,1,2;k), k = 1a i, 1a ii, 1a iii; (1,1,1,2,1;1b); (1,1,2,1,1;k), k = 1b, 2, 3; (1,2,1,2,2;2) and (1,1,1,1,2;3). The order of p is never inferior to the sum of the digits of the characteristic of p and, if p is elementary, then the order of p is equal to the sum of the digits of its characteristic.

N. D. LANE:

Foundations of the geometry of orders V. Direct polynomial differentiability

In the polynomial case, the characteristic curves are the (oriented) graphs of polynomials (of degree at most n) in the affine plane, together with certain sets of vertical lines and rays which can be limits of sequences of graphs of polynomials. The arc A is differentiable at p if

$$\lim_{s \to p} K(s,p) = K(p^2), \text{ the tangent line,}$$

$$\lim_{s \to p} K(s,p^2) = K(p^3), \text{ the osculating parabola,}$$

$$\ldots,$$

$$\lim_{s \to p} K(s,p^i) = k(p^{i+1})$$

all exist ($1 \leq i \leq n+1$). We define the index I(p) of p to be the smallest integer i such that $K(p^i)$ is a non-degenerate polynomial and $k(p^{i+1})$ is a line (or a ray of a line) through p parallel to the y-axis. If K_h is the set of all those polynomials which have h-point contact with $K(p^h)$ at p, then the polynomials of $K_h \backslash K_{h+1}$ all intersect A at p or all support A at p ($h = 1, 2, \ldots, n+1$; here $K_{n+2} = \emptyset$). The numerical characteristic of p has the form $(a_0, a_1, \ldots, a_n; I(p))$. There are 6n+2 different kinds of differentiable points. The order of a point p on A is never inferior to the sum of the digits of its characteristic.

N. D. LANE:

Foundations of the geometry of orders VI. Direct parabolic differentiability

Basic definitions and outline of the classical parabolic theory: direct differentiability, Scherk's Lemma, numerical characteristic, relation between the numerical characteristic and the local order. A set of four points determines two parabolas if all four are extremal points of the convex hull of the set, a family of degenerate parabolas if exactly one point is on the boundary of the convex hull but not extremal, none if one of the four points is an interior point. This property makes the parabolic case so pathological that it cannot be included as a special case in the generalization by means of the theory of quasigraphs.

J. M. TURGEON:

Foundations of the geometry of orders VII. Quasigraphs

The book Geometrische Ordnungen by O. Haupt and H. Künneth starts with a certain set of axioms. These axioms require certain adaptations for work in direct differential geometry. In this lecture, we present the notion of characteristic quasigraph, which is meant to replace the Haupt-Künneth notion of characteristic curve in our context. We also discuss the isotopic families of quasigraphs, the notions of mutual support and of mutual intersection of quasigraphs, and the local decomposition of the plane by a finite subset of an isotopic family of quasigraphs.

J. M. TURGEON:

Foundations of the geometry of orders VIII. Ordered geometry and matroids

The dimension of a family of quasigraphs is defined by means of matroid theory. All maximally independent sets have the same cardinality k, the "Grungzahl" of Haupt and Künneth's axioms. If the order of the basic arc B at the point p is finite, then the set S∪{p,s} must be independent, for every s on B\{p} sufficiently close to p. Thus the set of the quasigraphs containing S∪{p,s} will consist of a single quasigraph K(s). In all the classical cases, the pencil {K(s)} is isotopic as s varies along B\{p} in a neighbourhood of p on B.

J. M. TURGEON:

Foundations of the geometry of orders IX. Partial results

Under certain conditions, we obtained a general proof of the basic Lemma that makes possible the definition of the numerical characteristic of a point on the basic arc. A second result, under the same conditions, shows that certain of these characteristics are never realized. These conditions and properties are verified in all classical cases.

Berichterstatter: J. M. Turgeon.

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