MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 25/1985

Quadratische Formen

2.6. bis 8.6.1985

Die vierte Oberwolfach-Tagung über Quadratische Formen fand unter der Leitung der Herren Knebusch (Regensburg), Pfister (Mainz) und Scharlau (Münster) statt. Im Mittelpunkt des Interesses standen algebraische Fragen, es wurden aber auch Zusammenhänge mit Zahlentheorie und algebraischer Geometrie dargestellt. Die Tagungsteilnehmer kamen überwiegend aus dem Ausland.

Vortragsauszüge

E. BECKER: Reduced Witt rings of higher level

This is a report on joint work with A. Rosenberg. Let $n \in \mathbb{N}$ be given. A character $\chi : K^* \to S^1$ is called a signature of level n of the field K if $\ker \chi$ is additively closed. In particular $\chi(-1) = -1$. Let X_n be the set of all signatures of level n of K. An element $a \in K^*$ induces by evaluation a map on X_n : $\hat{a}(\chi) = \chi(a)$. We impose the weak topology with respect to all these $\hat{a}$'s on X_n and get a compact space. The n-th reduced Witt ring of K is defined as the subring of $C(X_n, \mathcal{O}(2n))$

generated by these functions $\hat{a}$'s , $a \in K^*$ where $\mathcal{O}(2n)$ denotes the ring of integers of the 2n-th cyclotomic field $\mathbb{Q}(\sqrt[2n]{1})$ with the discrete topology. For $n = 1$ we get the ordinary reduced Witt ring $W(K)/\mathrm{Nil}\, W(K)$. Three topics were discussed in the talk: presentation of the n-th reduced Witt ring $W^{(n)}_{red}(K) =: W^{(n)}$, the n-th reduced stability index and the representation theorem for $W^{(n)}$.

<u>Theorem 1</u> $\mathbb{Z}[K^*/(\Sigma K^{2n})^*] / \langle \bar{1}+(-\bar{1}), 1+\bar{a}-\overline{1+a}-\overline{a(1+a^{2n-1})} \mid a, 1+a \neq 0 \rangle = W^{(n)}$.

<u>Theorem 2</u> $C(X_n, \mathcal{O}(2n))/W^{(n)}\mathcal{O}(2n)$ is a torsion group.

Set $st^{(n)}(K)$ = exact exponent of $C(X_n, \mathcal{O}(2n))/W^{(n)}\mathcal{O}(2n)$ or ∞ and let $2m$ = exact exponent of $K^*/(\Sigma K^{2n})^*$ then

<u>Theorem 3</u> i) $p \mid 2m \Longleftrightarrow p \mid 2 . st^{(n)}$ (p prime)

ii) $F \mid R$ function field over a real closed field R of transcendence degree d , F formally real $\Rightarrow st^{(n)}(F) = (2n)^d$

A preordering $T \subset K$ is any subset satisfying $T+T \subset T$, $TT \subset T$, $K^{2n} \subset T$, $-1 \notin T$. Such a preordering is called a fan if every character $\chi : K^* \to S^1$, $\chi_{|T^*} = 1$, $\chi(-1) = -1$ is a signature. Set $X_T = \{\chi \in X^{(n)} \mid \chi(T^*) = 1\}$.

<u>Theorem 4</u> Given $f \in C(X_n, \mathcal{O}(2n))$ then $f \in W^{(n)}$ iff for all $a \in K^*$ and for all fans T of finite index in K we have

$$\frac{1}{\# X_T} \sum_{\chi \in X_T} \overline{\chi(a)} f(\chi) \in \mathbb{Z} \; .$$

R. FITZGERALD: Primary Ideals of Witt Rings

The primary ideals of a Witt ring (of a field of characteristic not 2) are determined. We show every ideal is a finite intersection of primary ideals if and only if the field has finite height and only finitely many orderings. This includes many non-noetherian Witt rings. The primary decomposition is used to find conditions equivalent to unique factorization of odd dimensional forms and an ideal theoretic characterization of SAP .

J.L. YUCAS: Local Factors of Finitely Generated Witt Rings

A local Witt ring is an abstract Witt ring, having a unique non-trivial 2-fold Pfister form. The main result given here is necessary and sufficient conditions for a finitely generated Witt ring to be a product (in the category of Witt rings) of two Witt rings, one of which is local. The basic motivation is to develop a tool for the study of whether every finitely generated Witt ring is of elementary type (that is, can be built from local Witt rings by a succession of products and group ring extensions). This is joint work done with Robert Fitzgerald.

N. SCHWARTZ: Chain signatures and Witt rings of higher level

In general, for a field K with an ordering P of higher level there are nonisomorphic real closures. If one considers chains of higher level orderings instead of single orderings of higher level, the isomorphism type of a real closure is uniquely determined. When dealing with the set of all chains of higher level orderings it is useful to consider chain signatures.

When considering only orderings of 2-power level, a chain signature is a homomorphism $f : K^* = K - \{0\} \to \{\pm 1\} \times \hat{\mathbb{Z}}_2$ (2-adic integers) such that $\ker f$ is a fan. Let $\kappa(K)$ be the set of chain signatures. $\kappa(K)$ carries a natural topology which makes it a Boolean space. This can be used to represent the Witt ring W_n of level 2^{n-1} as a subring of the ring $C(\kappa(K), \mathbb{Z}[\hat{\mathbb{Z}}_2/2^n\hat{\mathbb{Z}}_2])$ of continous functions $\kappa(K) \to \mathbb{Z}[\hat{\mathbb{Z}}_2/2^n\hat{\mathbb{Z}}_2]$, $\mathbb{Z}[\hat{\mathbb{Z}}_2, 2^n\hat{\mathbb{Z}}_2]$ with discrete topology. Using this representation one finds a system

$$\begin{array}{ccccccccccc}
W_{n/n} & \hookrightarrow & W_{n-1/n} & \hookrightarrow & W_{n-2/n} & \hookrightarrow & \cdots & \hookrightarrow & W_{1/n} & \hookrightarrow & W_n \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & W_{n-1/n-1} & \hookrightarrow & W_{n-2/n-1} & \hookrightarrow & \cdots & \hookrightarrow & W_{1/n-1} & \hookrightarrow & W_{n-1} \\
 & & & & \downarrow & & & & \downarrow & & \downarrow \\
 & & & & W_{n-2/n-2} & \hookrightarrow & \cdots & \hookrightarrow & W_{1/n-2} & \hookrightarrow & W_{n-2} \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & \vdots & & \vdots \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & W_{1/1} & \hookrightarrow & W_1 \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & W_0
\end{array}$$

of abstract Witt rings. This system can be used to analyse connections between the prime ideals $p_0, \ldots, p_n \subset W_n$ belonging to a chain signature.

R. WARE: <u>Witt rings and Galois groups</u>

Let $\mathcal{O}$ be a valuation ring with field of fractions F and residue field k of char $\neq 2$. If $\mathcal{O}$ is "2-henselian" then the following are well known:

(1) $W(F) \cong W(k)[\Gamma/\Gamma^2]$, Γ = value group.

(2) If $F(2)$, $k(2)$ are the maximal 2-extensions of F, k (resp.) and $F_{nr}(2)$ is the max. nonramified 2-ext. of F (rel. to $\mathcal{O}$) then $Gal(F(2)/F_{nr}(2))$ is an abelian group and there is a split exact sequence

$$1 \to Gal(F(2)/F_{nr}(2)) \to Gal(F(2)/F) \to Gal(k(2)/k) \to 1 .$$

In this talk (1), (2) were investigated independently of valuation theory and the following theorem was discussed:

Theorem. Assume that F is a field of char $\neq 2$ which is either formally real pythagorean or contains all 2-power roots of unity.

I For any cardinal $\# r > 0$ the following are equivalent

(a) There exists a field k and an elementary abelian 2-gp. Δ such that $W(F) \cong W(k)[\Delta]$ and $\dim_{\mathbb{F}_2} \Delta = r$.

(b) There exists a split exact sequence of pro-2-groups $1 \to A \to Gal(F(2)/F) \to T \to 1$ with A abelian and rank $A = r$.

II If in condition (b) the subgroup A is "maximal" split and F contains an element $a \notin F^2 \cup -F^2$ with $|D_F(<1,a>| > 2$ then there exists a valuation ring $\mathcal{O}$ on F with $1+Max \subseteq F^2$ ($\Rightarrow$ $\mathcal{O}$ is 2-henselian) such that $A = Gal(F(2)/F_{nr}(2))$, where $F_{nr}(2)$ is max. nonramified 2-ext. (rel. to $\mathcal{O}$) .

B. JACOB: Relative Rigidity and Galois Cohomology

Let $v : F \to G$ be a 2-Henselian valuation where char $\bar{F} = 2$. The Witt ring WF of such a field is an ugly mess, depending heavily upon the arithmetic of $\bar{F}$. In particular, there seems to be nice characterization of WF as there is in the non-dyadic case (Springers Theorem).

In this talk I describe a ring theoretic condition called "relative rigidity", which is satisfied in many dyadic valued fields as above. The main theorems are that such conditions "go up" under certain quadratic extensions, and that the zero sequence

$$\ldots \to I^{s-1}F \to I^sF \to I^sF\sqrt{t} \to I^sF \to I^{s+1}F \to \ldots$$

is exact for these extensions. As a consequence we can deduce that whenever WF is "relatively rigid" then

$$e_F^* : GWF \to H^*(Gal(F(2)/F), \mathbb{Z}/2\mathbb{Z})$$

is a well-defined isomorphism. Further we obtain a short exact sequence $0 \to J \to G_F(2) \to M \to 0$ of pro-2-groups, when J is pro-free and M is metabelian.

J.-P. SERRE: <u>On the quadratic forms of type $Tr(x^2)$</u>

Such forms have a long history, which goes back (at least) to Jacobi: See Borchardt's paper in Crelle 53 (1857), 281-283. For a more recent account, see also P. Conner - R. Perlis, <u>A Survey of Trace Forms of Algebraic Number Fields</u>, Singapore, 1984.

The talk was concerned with a formula for the Witt invariant (or should one say "Hasse" invariant?) of the form $Tr(x^2)$; this formula has just been published in <u>Comm</u>. <u>Math</u>. <u>Helv</u>. 59 (1984), 651-676 (and the proof has been "simplified" by many people since).

Applications of this formula have been mentioned to:

a) Galois extensions with Galois group A_S , à la Klein-Kronecker-Hermite - ...

b) Galois extensions of $\mathbb{Q}$ with Galois group the Griess - Fischer simple group, à la Thompson.

E. BAYER: Hermitian forms over reflexive modules

Let $\Lambda = \mathbb{Z}[X]/(\lambda)$ where $\lambda \in \mathbb{Z}[X]$ is a square free polynomial, $\lambda(1) = 1$. We shall consider finitely generated reflexive Λ-modules, and unimodular hermitian forms $h : M \times M \to \Lambda$. These forms play an important role in knot theory. One has the following

Theorem 1 (E.B. - Neal W. Stoltzfus)

Stably hyperbolic forms are hyperbolic.

Let D be the maximal order of $\mathbb{Q}[X]/(\lambda)$. We have

Theorem 2 Two unimodular hermitian forms (L,h) and (L',h') are stably isomorphic if and only if $(L,h) \boxplus H(D) \cong (L',h') \boxplus H(D)$.

Using results of D. Coray and F. Michel one can also give a complete criterion for a hermitian form (L,h) such that $(L,h) \boxplus (L,h)$ is hyperbolic to be stably isomorphic to some hermitian form (L',h') with $(L',h') \cong (L',-h')$.

The proof uses results of Quebbemann, R. Scharlau, W. Scharlau and Schulte as well as the strong approximation theorem for SU .

H.G. QUEBBEMANN: Some definite modular lattices

An even lattice in n-dimensional Euclidean space is called $\sqrt{l}$-modular, $l \in \mathbb{N}$, if it is the image of its dual lattice under a similarity transformation of norm l (determinant $\sqrt{l}^n$). The theta-function of such a lattice L is a modular form for Hecke's group $G(\sqrt{l}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1/\sqrt{l} \\ -\sqrt{l} & 0 \end{pmatrix} \right\rangle$. For $l = 1,2,3$ this property leads to the bound

$$\min\{v \cdot v \mid 0 \neq v \in L\} \leq 2 + 2[(1+l)n/48] ,$$

a result due to Siegel and Mallows-Odlyzko-Sloane for $l = 1$. The bound is known to be attained in the following cases (there may be more):

$l = 1$; $n = \underline{8},16$ (min 2), $\underline{24},32,40$ (min 4), $\underline{48},64$ (min 6),
$l = 2$; $n = \underline{4},8,12$ ("), $\underline{16},20,24,28$ ("), $\underline{32},36,40,44$ ("),
$l = 3$; $n = \underline{2},4,\ldots$ ("), $\underline{12},16,18,\ldots$ (").

Except for $l = 1$, $n = 48$, the min 6 lattices appear to be new. A uniform construction is given which utilizes codes in the spirit of Leech-Sloane (however rather trivial ones). In the underlined cases the lattices give rise to the densest known sphere-packing of $\mathbb{R}^n$.

M. MARSHALL: Exponentials and Logarithms on Witt Rings

Let p be a prime, G an abelian group of exponent p, and let $R = \mathbb{Z}[G]/K$ be an abstract Witt ring for G in the terminology of Knebusch, Rosenberg, and Ware. Thus R_{tor} := the torsion part of R is p-primary torsion. Let $\overline{M} := M/K$ where M is the ideal in $\mathbb{Z}[G]$ generated by p and all elements $1-g$, $g \in G$. Assume K satisfies (*) $x \in K \Rightarrow \frac{x^p}{p} \in K$. [Main example: Suppose $p = 2$ and K is generated by an element $1+e$, $e \in G$ together with certain elements of the form $(1-g)(1-h)$, $g,h \in G$]. Then there are divided powers $\gamma_n : (\overline{M})_{tor} \to R$, $n \geq 0$, and $\gamma_n(x) = 0$ for n sufficiently large if $x \in (\overline{M}^2)_{tor}$. This also holds for $x \in (\mathbb{Z}_p + \overline{M}^2)_{tor}$ if p is odd. Thus there are induced inverse isomorphisms $(\overline{M}^2)_{tor} \rightleftarrows 1 + (\overline{M}^2)_{tor}$ [resp. $(\mathbb{Z}_p + \overline{M}^2)_{tor} \rightleftarrows 1 + (\mathbb{Z}_p + \overline{M}^2)_{tor}$ if p is odd]. From this one can derive a more or less complete description of the group of units of finite order in R in terms of the additive structure on $(\overline{M}^2)_{tor}$ [resp. $(\mathbb{Z}_p + \overline{M}^2)_{tor}$ if p is odd].

A. MICALI: Classification theorems for quadratic forms in characteristic two

(joint paper with O.E. Villamayor)

Si φ est une forme quadratique sur un K - espace vectoriel V où K est un corps commutatif de caractéristique 2 , on peut décomposer V en des sous-K-espaces vectoriels V_1, V_2, V_3 tels que: (i) la restriction de φ à V est non singulière; (ii) la restriction de φ à V_2 est additive; (iii) la restriction de φ à V_3 est nulle. De plus, ces trois sous-espaces sont orthogonaux. Hipps, Mount et Villamayor ont généralisé la notion classique de polarisation de polynômes homogènes au cas des corps de caractéristique quelconque et ont observé que la polarisation d'ordre zéro dans le cas des formes quadratiques est exactement la forme bilinéaire symétrique associé. Donc $V_2 \oplus V_3$ est le noyau d'une telle polarisation et V_3 est le noyau de la polarisation d'ordre 1 . La première partie de l'article est l'étude des formes quadratiques additives via polarisation.

Dans la seconde partie nous démontrons un théorème 2 en représentant la forme quadratique comme une suite de valeurs définies sur une base convenable et en décidant quand deux telles suites correspondent à des espaces isométriques.

La généralisation au cas où la caractéristique de K est $p \geq 3$ est indiquée.

D.B. SHAPIRO: Sums of Squares in Division Algebras

(work of Leep, Shapiro, Wadsworth and Lewis)

Let D = division algebra finite dimensional over its center F .
Jacobson Question: If $-1 \notin \Sigma D^2$ then must D be commutative

(i.e. $D = F$)? This can be motivated by Albert's Theorem (1940): If D has an ordering then D is commutative. Let $T_D : D \times D \to F$ be the trace form: $T_D(x,y) = tr(xy)$ where tr = reduced trace. We see easily that if $-1 \in \Sigma D^2$ then T_D is weakly isotropic as a quadratic form /F (i.e. $T_D \perp \ldots \perp T_D$ isotropic). So, counterexamples to Jacobson's question can be found using algebras D where T_D is strongly anisotropic. For instance if $D = \left(\frac{a,b}{F}\right)$ is quaternion then $T_D \cong \langle 2\rangle\langle 1,a,b,-ab\rangle$. If F is any field (formally real) which is not SAP, then $\exists\, a,b \in F$ where $\langle 1,a,b,-ab\rangle$ is strongly anisotropic and the example is found. It turns out that examining the trace form does characterize the problem.

<u>Theorem 1</u>. Suppose D,F as above and $\operatorname{char} F \neq 2$. The following are equivalent.

(1) T_D is weakly isotropic over F; (2) $0 \in \Sigma \dot{D}^2$; (3) $D = \Sigma D^2$.

<u>Corollary</u>. If $\operatorname{char} F \neq 2$ and $\dim D \neq$ 2-power then $D = \Sigma D^2$. If $F \subseteq K \subseteq D$ where K = field and $\exists\, \sigma \in \operatorname{Aut}(K/F)$ of order > 2, then $D = \Sigma D^2$.

<u>Theorem 2</u>. Suppose A = central simple algebra over F, where $\operatorname{char} F = 2$. Then $A = \Sigma A^2 \iff F$ = perfect. In general, $\Sigma A^2 = \{a \in A : tr(a) \in F^2\}$.

The central simple case over F for char $\neq 2$ involves no difficulty, due to a result of Griffin & Krusemeyer (1977) that if R is a ring with $\frac{1}{2}$ and $n \geq 2$ then every element of $M_n(R)$ is a sum of 3 squares. The proofs of Theorems 1 and 2 involve analysis of linear subspaces of A which are invariant under all inner automorphisms.

R. BAEZA: Sums of squares of linear forms

Let F be a field and $\Sigma^1(n,F) = \{\sum_1^r L_i(X_1,\dots,X_n)^2 \mid L_i$ n-ary F-linear forms, $r \geq 1\}$. Define $g_F(n) = \mathrm{Min}\{r \mid \forall\, Q \in \Sigma^1(n,F)$ $\exists\, L_1,\dots,L_r$ n-ary F-linear forms s.t. $Q = L_1^2 + \dots + L_r^2\}$. If $F/\mathbb{Q}$ is a global field, we prove $g_F(n) = n+3$ $\forall\, n \geq 3$, $g_F(2) = 4$ if $p(F) \leq 3$, $g_F(2) = 5$ if $p(F) = 4$, where $p(F) = g_F(1)$ = Pythagoras number of F. This generalizes a result of Mordell for $F = \mathbb{Q}$ (M.Z. 35, 1932). Define $l(F) = \mathrm{Min}\,\{r \mid$ every totally positive quadratic form of dim. r represents all totally pos. elements of $F\}$. If F is non real, $l = u$, usual u-invariant of F. If $l(F) < \infty$, then $g_F(n) = n + l(F) - 1$ $\forall\, n \geq l(F)-1$. (If F is global field, the local-global theorem implies $g_F(n) = n+3$, $\forall\, n \geq 3$). One can give some estimates of $l(F)$ in function of other invariants of F, e.g. u-invariant, $h(F)$ = height of F. For example:

i) If $u(F) < \infty$, then $l \leq 1+(h-1)u$.

ii) If $I \subset W(F)$ is the ideal of even dimensional forms of the Witt ring, then $I_t^{n+1} = 0$ implies $l \leq 2^{n-1}+1+(2^{n-1}-1)u$.

iii) If $a \in F$ is totally positive, $u(F(\sqrt{-1})) \leq 2^n$, then $l \leq 1+(2^n-1)^2$ and if $u(F(\sqrt{-a})) \leq 4$, then $l \leq 6$. For example it follows $l(\mathbb{R}(X,Y)) \in \{4,6\}$.

T.Y. LAM: Cyclic Algebras and Quartic Extensions

(Report of joint work with David Leep, with contributions from J.-P. Tignol)

A well-known theorem of Albert states that central simple algebras A of degree 4 with exponent ≤ 2 are precisely the "biquaternion" algebras $B \otimes C$ (B,C = quat. algebras). Albert constructed examples of such algebras which are not cyclic; on the other hand, Risman, Tignol and others have constructed such algebras which are cyclic.

In this work, we obtain two criteria for biquaternion algebras to be cyclic: one criterion is quadratic form theoretic, the other is purely algebra-theoretic. These criteria can be used to make concrete computations in determining whether certain specific biquaternion algebras are cyclic or otherwise.

On the quadratic form side, we have completely determined the Witt ring kernel W(E/F) for a quartic extension E/F which admits an intermediate quadratic extension. It turns out that W(E/F) is always a {1,2} Pfister ideal (i.e. it is generated by 1-fold and 2-fold Pfister forms), and that the 2-folds in W(E/F) can be written down very simply in a 1-parameter family. In the case when E/F is biquadratic, this computation also retrieves the Elman-Lam-Wadsworth result that W(E/F) is simply 1-Pfister.

The cyclicity criterion is proved by first working with a single quaternion algebra. In this simple case, our result states that a quaternion algebra Q has a cyclic quartic splitting field (i.e. $\mathbb{M}_2(Q)$ is cyclic) iff the norm form of Q represents a sum of 2 squares which is not a square. Surprisingly, even this simple special case seems to be new. After solving the problem in this case, the case of a biquaternion algebra is now solved by replacing a quaternionic form with an Albert form, and trying to find cyclic quartic extensions to split the Albert form.

M. OJANGUREN: Splitting of quadratic spaces

This is a report on recent work of Inta Bertuccioni. Her main result is as follows: let R be a commutative noetherian ring of Krull dimension d, A a finite R-algebra with an R-linear involution and $\varepsilon = \pm 1$; let (P,f) be an ε-hermitian space over A ; assume that for every maximal ideal m of R $(P,f)_m$ contains a hyperbolic orthogonal summand $H(A_m^n)$ of fixed index $n > d$; then (P,f) contains a hyperbolic orthogonal summand $H(A^{n-d})$.

J.J. SANSUC: Intersections de deux quadriques sur un corps de nombres

(Travail en commun avec J.-L. Colliot-Thélène et P. Swinnerton-Dyer)

Soient k un corps de nombres, Φ_1 et Φ_2 deux formes quadratiques en $m = n+1$ variables à coefficients dans k et $X = X_{2,2} \subset \mathbb{P}_k^n$ l'intersection de deux quadriques définie par $\Phi_1 = \Phi_2 = 0$. On considère la condition suivante:

(*) X est pure, géometriquement intègre et n'est pas un cône.

Théorème A. Si k est totalement imaginaire et si $m \geq 9$, alors le système $\Phi_1 = \Phi_2 = 0$ a une solution non triviale dans k .

Théorème B. Soient k,Φ_1,Φ_2,X comme au début et $m \geq 9$. On suppose la condition (*) vérifiée. On suppose en outre:

(i) aucune forme du pinceau $\{\lambda\Phi_1+\mu\Phi_2 \mid \lambda,\mu \in k, \lambda\mu \neq 0\}$ n'est de rang ≤ 4 ;

(ii) pour chaque complété réel k_v de k , aucune forme du pinceau réel $\lambda\Phi_1+\mu\Phi_2$ n'est semi-définie.

Alors, le système $\Phi_1 = \Phi_2 = 0$ a une solution non triviale dans k .

Théorème C. Soient k et $X = X_{2,2} \subset \mathbb{P}_k^n$ comme au début. On suppose $n \geq 8$. Si, pour chaque complété k_v de k , la variété X a un point lisse dans k_v , alors X a un point lisse dans k et X vérifie l'approximation faible: $X(k)$ est dense dans $\prod_v X(k_v)$.

Les théorèmes A et B dérivent de C. La démonstration du théorème C se fait par réductions successives à des théorèmes analogues pour divers types de variétés, selon le schéma ci-après:

Théorème C

$\Uparrow$

Théorème D_6 = "théorème C" pour $X_{2,2} \subset \mathbb{P}_k^6$, lorsque $X_{2,2}$ a deux points singuliers conjugués

$\Uparrow$

Théorème E_5 = "théorème C" pour $y^2 - az^2 = P(x_1, x_2, x_3) \subset \mathbb{A}_k^5$ avec P irréductible de degré 4

$\Uparrow$

Théorème E_3 = "théorème C" pour $y^2 - az^2 = P(x) \subset \mathbb{A}_k^3$ avec P irréductible de degré 4

$\Uparrow$

Théorème F_7 = "théorème C" pour $X_{2,2} \subset \mathbb{P}_k^7$, lorsque $X_{2,2}$ contient un quadrilatère gauche dont les côtés opposés sont globalement k-rationnels et les sommets non singuliers

$\Uparrow$

Théorème F_5 = "théorème F_7" pour $X_{2,2} \subset \mathbb{P}_k^5$, lorsque $X_{2,2}$ vérifie la condition ci-dessus et qu'il n'existe pas 2 formes de rang 4 globalement k-rationnelles dans le pinceau

$\Uparrow$

Théorème F_4 = "théorème F_7" pour $X_{2,2} \subset \mathbb{P}_k^4$.

Le cas crucial est le théorème F_5 . Les réductions utilisent une méthode de sections hyperplans bien choisies, plus délicates dans le pas final, la théorie de la descente, le théorème d'irréductibilité de Hilbert avec approximations, le corps de classes usuel. Le dernier théorème F_4 est dêjâ connu et dû à Manin: il achève la preuve.

M. KULA: Uniform Linkage Property

An abstract Witt ring is said to be real if its characteristic is 0, and non-real otherwise. We say that an abstract Witt ring satisfies Uniform Linkage Property if there is a 1-fold Pfister form σ such that $I^2 = \sigma I$ (I denotes the fundamental ideal in the Witt ring). In a non-real Witt ring satisfying ULP every 2-fold Pfister form is universal. In the real case every 2-fold Pfister form represents all totally positive elements. It turns out that the additive structure of Witt rings satisfying ULP is completely determined by some numerical invariants.

Conjecture: Every finitely generated Witt ring satisfying ULP is a product of certain standard Witt rings of fields.

The conjecture has been verified in some cases.

A. PRESTEL: On the size of solutions of quadratic forms over $\mathbb{R}[x,y]$

Let k be a field and $g_1,\dots,g_m \in k[t_1,\dots,t_d]$. We denote by $\deg\{g_1,\dots,g_m\}$ the maximal total degree of the polynomials $g_1,\dots,g_m$. The following theorems have been stated (and proved):

THEOREM 1. Let $g(X_1,\dots,X_n) = \sum_{1\le i,j\le n} f_{ij}X_iX_j$ with $f_{ij} = f_{ji} \in k[t]$ (i.e. d = 1) and $n \geq 3$. Then

$$\deg\{g_i\} \leq \deg\{f_{ij}\}^{(n-1)/2}$$

for a solution $(g_1,\dots,g_n)$ of $g = 0$ of minimal degree (if g is isotropic).

The proof of Theorem 1 follows Cassels' proof on the size of solutions of quadratic forms over $\mathbb{Z}$ (see his book on 'Rational Quadratic Forms').

THEOREM 2. The form $\sum_{i=1}^{4} [(x+1)X_{1i}^2 + (z+1)X_{2i}^2 + xzX_{3i}^2 - X_{4i}^2]$ with $z = x^2+my+y^2$ is isotropic over $\mathbb{R}[x,y]$ for all $m \in \mathbb{N}$. But the degree of a minimal solution converges to ∞ for $m \to \infty$.

The proof of Theorem 2 uses the Local-Global-Principle of Bröcker-Prestel and an ultra power argument.

P. REVOY: Formes trilinéaires alternées de rang 7 et formes quadratiques

Soit E un espace vectoriel de dimension 7 sur le corps K. L'étude des orbites dans $\Lambda^3 E$ de l'action du groupe linéaire est faite à l'aide

1°) de la classification, dûe a J.A. Schouten, des 3-vecteurs sur un corps algêbriquement clos (5 orbites).

2°) de la cohomologie galoisienne (déduite des groupes d'automorphismes Aut w_i

3°) de la notion de 3-vecteurs scindés.

L'orbite ouverte donne lieu à une famille d'orbites paramètrie par l'ensemble des classes d'isomorphismes de K-algèbres d'octonions modulo une homothétie. On donne aussi la construction pour $n = 6$ et 7 d'un invariant relatif polynômial déduit de la construction analogue de T. Kimura et M. Sato.

K. SZYMICZEK: Generalized rigid elements in fields

We generalize rigid elements to the context of n-fold Pfister forms and powers of the fundamental ideal of Witt ring in the following two ways:

Gefördert durch DFG Deutsche Forschungsgemeinschaft

(a) $x \in \dot{F}$ is n-rigid iff every n-fold Pfister form annihilating $\langle 1,x\rangle$ is divisible by $\langle 1,-x\rangle$ in WF,

(b) $x \in \dot{F}$ is super n-rigid iff every form in I^nF annihilating $\langle 1,x\rangle$ lies in $\langle 1,-x\rangle \cdot I^{n-1}F$.

We study the relations between n-rigid and super n-rigid elements in various classes of fields including global fields, and in abstract Witt rings. In special cases the existence of higher rigidities turns out to be equivalent to important properties of fields known in the literature. Among these are the property A_n, torsion freeness of I^nF and finite stability index of WF.

M.-A. KNUS: Composition of quaternary quadratic forms

(Joint work with M. Kneser, M. Ojanguren, R. Parimala and R. Sridharan)

The first deep results on composition of quaternary forms are due to Brandt. In particular, Brandt gave necessary and sufficient conditions for the existence of composition of integral quadratic forms.

Composition of binary forms over arbitrary commutative rings was considered recently by M. Kneser, J. Number Theory 15 (1982), 406-413. We study composition of quaternary forms in the same generality and present Brandt's results in this generality.

In a first part, relations between quaternion algebras and composition of quaternary forms are presented. Then we give a necessary and sufficient condition for the existence of composition in terms of the Clifford algebra. Finally we show that the Brandt condition is necessary but is in general not sufficient. We give conditions on the ring under which the Brandt condition is sufficient.

J.L. COLLIOT-THELENE: $P(\mathbb{Q}(x,y)) \leq 8$

In $\mathbb{Q}(x,y)$ läßt sich jede Summe von Quadraten als Summe von höchstens acht Quadraten schreiben. Dies folgt leicht aus den drei Sätzen:

Satz 1 (Pfister): Sei k ein Körper, und sei n eine ganze Zahl. Wenn (-1) sich als Summe von 2^n Quadraten in jeder endlichen nicht formal - reellen Erweiterung von k schreiben läßt, dann läßt sich jede Summe von Quadraten im rationalen Funktionenkörper in einer Variablen als Summe von 2^{n+1} Quadraten schreiben.

Satz 2 (Merkurjev-Suslin): Sei k ein Körper, Char. $k \neq 2$, sei D eine Quaternionenalgebra. Ein Element a in k^* ist genau dann eine reduzierte Norm von D, wenn das Cup-Produkt $[a] \cup [D] \in H^3(k,\mu_2^{\otimes 2})$ - wobei $[a]$ die Klasse von a in $k^*/k^{*2} \cong H^1(k,\mu_2)$ und $[D]$ die Klasse von D in der bekannterweise zur 2-Torsion der Brauergruppe von k isomorphen Gruppe $H^2(k,\mu_2)$ bezeichnet - verschwindet.

Satz 3 (Kato): Sei K ein Funktionenkörper in einer Variablen über einem Zahlkörper k. Sei k in K algebraisch abgeschlossen, und bezeichne K_v den Quotientenkörper von $K \otimes_k k_v$, wobei k_v die Komplettierung von k an der Stelle v ist. Die natürliche Abbildung (n eine natürliche Zahl):

$$H^3(K,\mu_n^{\otimes 2}) \to \prod_{\text{alle Stellen } v} H^3(K_v,\mu_n^{\otimes 2})$$

ist eine Injektion.

Es wurde versucht, einen Einblick in die Methoden von Kato zu geben.

R. ELMAN: Graded Witt Ring and Galois Cohomology

(joint work with Arason and Jacob)

Let S be the collection of abstract Witt rings R satisfying $I^3R = 2I^2R$ is torsion-free, e.g., WF for F real-closed, quadratically closed, finite, local, global, or transcendence degree ≤ 2 over a real closed field. Let C be the smallest subcategory of abstract Witt rings containing S and closed under the operations of finite group ring formation and fiber product over $\mathbb{Z}/2\mathbb{Z}$.

Theorem. If $WF \in C$ then

$$\begin{array}{ccc} & k \times F & \\ \swarrow & & \searrow \\ GWF & \rightarrow & H^*(\mathrm{Gal}(F_q/F), \mathbb{Z}/2\mathbb{Z}) \end{array}$$

are isomorphisms where F_q = quadratic closure of F.

When the proof is restricted to the category of Witt rings of elementary type this proof can be made completely elementary, i.e., is independent of Merkurjev's Theorem and Arason's result on the existence of e^3.

A. WADSWORTH: Totally ramified division algebras

Let D be a division algebra finite dimensional over its center F, and suppose F has a Henselian valuation v with value group Γ_F. Then v extends uniquely to a valuation w on D, with value group Γ_D. If w is totally and tamely ramified over v, there is a nondegenerate symplectic pairing defined on the relative value group Γ_D/Γ_F (a finite group), which carries considerable information about the structure of D, and about what fields can occur as subfields of D. The use of this pairing has clarified past constructions of non-crossed products;

it has also led to the construction of new examples of non-crossed product division algebras over fields with very nice Brauer groups.

D. LEEP: Powers of the Fundamental Ideal in Algebraic Extensions

Let $[K:F] = r$, char $F \neq 2$. We prove that if $I^n(F) = 0$ then $I^m(K) = 0$ if $m \geq n + [\log_2(\frac{r}{3})] + 1$. This improves the Elman-Lam bound. It is conjectured that $I^n(F) = 0$ implies $I^{n+1}(K) = 0$ for K algebraic over F .

P.E. CONNER: Additive Characters on the Witt Ring

For K an algebraic number field there is at each prime an additive character $\gamma_p : W(K) \to \mathbb{C}^*$ (Weil, Scharlau, Knebusch-Scharlau). If $\rho(\mathrm{dis}(X)) : G(K) \to Z^* = O(1,\mathbb{R})$ is the degree 1 representation of the absolute Galois group associated to the square class of the discriminant of the Witt class X then

$\gamma_p(X) = (-2,\mathrm{dis}(X))_p c_p(X) W_p(K,\rho(\mathrm{dis}(X)))$, $\mathrm{rk}(X) \equiv 0 \pmod 2$

$\gamma_p(X) = (-2,\mathrm{dis}(X))_p c_p(X) W_p(K,\rho(\mathrm{dis}(X)))\gamma_p(1_K)$, $\mathrm{rk}(X) \equiv 1 \pmod 2$

where $c_p(X)$ is the stable Hasse-Witt invariant and $W_p(K,\rho(\mathrm{dis}(X)))$ the local root number of the representation (Deligne, Langlands, Tate). If E/K is a relative extension of degree n then $\langle E \rangle \in W(K)$ will be Witt class of the relative trace form while $\rho(E) : G(K) \to S_n$ is the associated transitive permutation representation. Using Serre's results on trace forms it follows that

$$\gamma_p\langle E \rangle W_p(K,\rho(E)) = \gamma_p(1_K)^n$$

at each prime in K .

L. MAHE: Reduced Witt rings of R-varieties

We prove the following theorem:

There exists a function $w : W \to \mathbb{N}$ such that for every affine R-variety (R real closed field), we have $2^{w(d)}\mathrm{Cont}(V(R),\mathbb{Z}) \simeq 2^{w(d)}\,\mathbb{Z}^s \subset W_{red}(R[V])$ (s is the number of connected components of V(R)).

This is obtained with the help of

1°) A previous work which shows that for a certain integer t we have $2^t\,\mathbb{Z}^s \subset W_{red}(R[V])$.

2°) A theorem of L. Bröcker on the bounds of inequalities for the description of an open semi-algebraic set in a d-dimensional R-variety.

3°) A new theorem, so-called "Pfister's theorem for varieties" which tells us that the level of R[V] (V d-dimensional R-variety with $V(R) = \emptyset$) is bounded by $d-1+2^{d+1}$.

D. CORAY: Arithmetic properties of certain intersections of of two quadrics

In this talk I described some methods, used in joint work with Michael Tsfasman, for studying singular intersections of two quadrics in $\mathbb{P}^4_k$, where k is an arbitrary field. This conceptual approach, which is based on the theory of rational singularities of surfaces, yields in particular the following result:

Let $\mathfrak{D}$ be the family of all singular intersections of two quadrics $V \subset \mathbb{P}^4_k$ having only isolated rational double points. Let $\mathcal{I}$ be the subfamily consisting of what we call Iskovskih surfaces (i.e. those having precisely two conjugate double points, and such that the line joining these points does <u>not</u> lie on the surface). Then

(i) $V \in \mathfrak{D}$, $V_{ns}(k) \neq \emptyset \Rightarrow V$ is k-unirational;

(ii) if k is a number field, the clean Hasse Principle holds for the class $\mathfrak{D} \setminus \mathfrak{I}$. In particular,

$V \in \mathfrak{D} \setminus \mathfrak{I}$, $V_{ns}(k_v) \neq \emptyset \Rightarrow V(k) \neq \emptyset$.

(This result does not hold for the class $\mathfrak{I}$, which contains the original counterexample of Iskovskih.)

Berichterstatter: M. Peters

Tagungsteilnehmer

Prof. J.K. Arason
Science Inst.
University of Iceland
Dunhagen 3
IS-107 Reykjavik
Island

Prof. Ricardo Baeza
Depto. de Matematicas
Faculdad de Ciencias
Univ. de Chile
Casilla 653
Santiago, Chile

Dr. Eva Bayer
Section de Mathématiques
Université de Genève
2-4, rue du Lièvre
CH-1211 Genève 24
Schweiz

Prof. Dr. E. Becker
Institut für Mathematik
Postfach 500 500
D-4600 Dortmund 50

Dr. R. Bos
T. Univ. Eindhoven
Dept. Maths. & Comp. Sc.
P O Box 513
Eindhoven
Niederlande

Prof. L. Bröcker
Mathematisches Institut
Einsteinstraße 62
D-4400 Münster

Prof. J.W.S. Cassels
DPMMS
Univ. of Cambridge
16, Mill Lane
Cambridge CB2 1SB
England

Prof. J.L. Colliot-Thélène
CNRS Mathématique
Batiment 425
Université de Paris-Sud
F-91405 Orsay Cedex
Frankreich

Prof. P.E. Conner
Dept. of Math.
Louisiana State Univ.
Baton Rouge, LA 70803
USA

Prof. D. Coray
Université de Genève
Section de Mathématiques
2-4, rue du Lièvre
CH-1211 Genève 24
Schweiz

Prof. A.G. Earnest
Dept. of Math.
Southern Ill. Univ.
Carbondale, IL 62901
USA

Prof. R.S. Elman
Dept. of Math.
University of California
University Park
Los Angeles, CA 90024

Dr. Angelika Faltings
Dept. of Math.
Princeton University
Princeton, NJ 08544
USA

Prof. R. Fitzgerald
Dept. of Math.
Southern Ill. Univ.
Carbondale, IL 62901
USA

Prof. T.Y. Lam
Dept. of Math.
University of California
Berkeley, CA 94720
USA

Prof. D.K. Harrison
Dept. of Math.
University of Oregon
Eugene, OR 97403
USA

Prof. D. Leep
Dept. of Math.
University of Kentucky
Lexington, KY 40506
USA

Prof. B. Jacob
Dept. of Math.
Oregon State University
Corvallis, Oregon 97333
USA

Prof. L. Mahé
IRMAR
Campus de Beaulieu
F-35042 Rennes Cedex
Frankreich

Prof. M. Knebusch
Fachbereich Mathematik
Universitätsstr. 31
D-8400 Regensburg

Prof. M. Marshall
Dept. of Math.
University of Saskatchewan
Saskatoon, Saskatchewan S7N OWO
Canada

Prof. M. Kneser
Mathematisches Institut
Bunsenstr. 3-5
D-3400 Göttingen

Prof. A. Micali
Université de Montpellier II
Place E. Bataillon
F-34060 Montpellier Cedex
Frankreich

Prof. M.A. Knus
Mathematisches Institut
Eidgen. Techn. Hochschule
CH-8092 Zürich
Schweiz

Prof. M. Ojanguren
Institut de mathématiques
Université de Lausanne
CH-1015 Lausanne-Dorigny
Schweiz

Prof. M. Kula
Math. Inst.
Universität Katowice
Bankowa 14
40-007 Katowice
Polen

Prof. M. Peters
Mathematisches Institut
Einsteinstraße 62
D-4400 Münster

Prof. A. Pfister
Fachbereich Mathematik
Saarstr. 21

D-6500 Mainz

Prof. A. Prestel
Fakultät für Mathematik
Postfach 5560

D-7750 Konstanz

Dr. H.G. Quebbemann
Mathematisches Institut
Einsteinstraße 62

D-4400 Münster

Prof. P. Revoy
Université de Montpellier II
Place E. Bataillon

F-34060 Montpellier Cedex

Frankreich

Prof. C. Riehm
Dept. of Math.
Mc Master University

Hamilton, Ontario

Kanada

Prof. J.J. Sansuc
E.N.S.
45 rue d'Ulm

F-75230 Paris Cedex 05

Frankreich

Prof. W. Scharlau
Mathematisches Institut
Einsteinstraße 62

D-4400 Münster

Prof. N. Schwartz
Mathematisches Institut
Theresienstr. 39

D-8000 München

Prof. J.-P. Serre
Collège de France
place Marcelin-Berthelot

F-75005 Paris

Frankreich

Prof. D.B. Shapiro
Dept. of Math.
Ohio State University

Columbus, Ohio 43210

USA

Prof. K. Szymiczek
Math. Inst.
Universität Katowice
Bankowa 14

40-007 Katowice

Polen

Prof. A. Wadsworth
Math. Dept.
U.C.S.D.

La Jolla, Calif. 92093

USA

Prof. R. Ware
Math. Dept.
Pennsylvania State University

University Park, Pennsylv. 16802

USA

Prof. J.L. Yucas
Dept. of Math.
Southern Ill. University

Carbondale, IL 62901

USA