MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Zyklische Kohomologie und ihre Anwendungen

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The meeting was organized by W. Borho (Wuppertal), A. Connes (Bures-sur-Yvette), J.L. Loday (Straßburg) and F. Waldhausen (Bielefeld). About 40 participants from America and Europe attended the conference. The 16 talks dealt with recent developments in cyclic cohomology and its applications. They covered a wide range of areas in functional analysis, algebra, topology and physics.

Computing the Cyclic Homology of Curves Susan C. Geller (joint work with L. Reid and C. Weibel)

We are only going to compute the cyclic homology up to the cyclic homology of the normalization. By localization we need only consider rings with one singularity. By analytic isomorphisms we need only consider the analytic type of singularity, i.e. graded rings with one singularity.

We thus assume that $R = k \oplus R_2 \oplus R_3 \oplus \ldots = k \oplus \overline{R}$ where k is a field of characteristic 0. The assumption that $R_1 = 0$ is for technical reasons and, since one can regrade the ring, causes no problems. Since $S(HC_*(\overline{R})) = 0$, the SBI sequence splits into short exact sequences and we need only compute $H_*(R)$.

Our procedure, then, computes $H_*(R)$ using the Leray spectral sequence whose E_2^{pq} term is $Rp \otimes H^q(R;k)$. Since the spectral sequence is multiplicative with R along the $q = 0$ axis, we need only find $H^q(R;k)$. This is obtained from the Serre spectral sequence whose E_2^{pq} terms are the same as the Leray but which converges to 0(except for E_∞^{00}). We then explictly compute the d_r of the Leray spectral sequence from the generators of $H^q(R;k)$ and the Hochschild boundary map. $H_i(R)$ is then read from the rows of E_∞^{pq} whose generators have length i. The weight of the generators is used in computing $HC_i(\overline{R})$ from $0 \to HC_{i-1}(\overline{R}) \to H_i(R) \to HC_{i-1}(\overline{R}) \to 0$.

The example $A = \mathbf{Q}[t^2, t^2]$ is worked out and

$$HC_i(A) = HC_i(\mathbf{Q}) \oplus \begin{cases} \overline{R} & i = 0 \\ 2\mathbf{Q} & i \text{ odd} \\ 0 & i \text{ even},\ i \neq 0 \end{cases}$$

The connection to K-theory is explained. In particular, $K_2(A \otimes k)$ maps onto $K_2(k) \oplus k \oplus k \oplus \Omega_k^2$ and $K_3(A \otimes k)$ maps onto $K_3(k) \oplus \Omega_k \oplus \Omega_k$. Other rings are then discussed.

A Topologist's View of Cyclic Homology (T. Goodwillie)

Waldhausen's algebraic K-theory of spaces can be thought of as the algebraic K-theory of a kind of "generalized ring". For a basepointed connected space with loop group G one thinks of

$$A(X) \text{ as } K(R) \text{ where } R = \Omega^\infty \Sigma^\infty (G_+)$$

is the "group ring" of G over the universal "ring" $\Omega^\infty \Sigma^\infty S^0 = QS^0$. This heuristic point of view has been made precise in several ways by several people. In particular Bökstedt has defined a notion of "generalized ring" such that one can define algebraic K-theory and

Hochschild homology in his setting. The "rings" $Q(G_+)$ give one class of examples; their K-theory is Waldhausen's $A(BG)$. Discrete rings give another class, and in this case K-theory is Quillen algebraic K-theory. Bökstedt's "topological Hochschild homology" $THH(R)$ of a generalized ring is a spectrum. In the case $R = Q(G_+)$ this is the unreduced suspension spectrum of the **free** loop space

$$\Lambda(BG) = Map(S^1, BG).$$

In the case of discrete rings it is a new object. There is a map of spectra

$$\underline{K}(R) \to \underline{THH}(R)$$

(for a generalized ring R) which generalizes and refines the Dennis trace map and also generalizes a map

$$A(X) \to Q\Lambda(X)_+$$

defined by Waldhausen. In fact, because $\underline{THH}(R)$ is constructed as a cyclic object it is equipped with an S^1-action and there is (up to weak homotopy) a factorization of the trace map through the homotopy fixed-point spectrum:

$$\underline{K}(R) \to \underline{THH}(R)^{hS^1} \to \underline{THH}(R).$$

This generalizes and refines a factorization of Dennis' trace which has been defined by various people in recent years

$$K_*(R) \to HC_*^-(R) \to HH_*(R)$$

for R discrete (or perhaps simplicial).

The basic idea in all of this is to think of QS^0 as the ground ring, a ring more universal than $\mathbf{Z}$. A module over QS^0 is a spectrum. The tensor product over QS^0 is the smash product of spectra. A (generalized) ring is more or less a spectrum $\underline{R}$ together with an associative multiplication $\underline{R} \wedge \underline{R} \to \underline{R}$. Bökstedt goes to much trouble to choose a notion of spectra, and of smash product, such that for any one of his rings he can make a cyclic object in the category of spectra

$$\underline{R} \leftleftarrows \underline{R} \wedge \underline{R} \Lleftarrow \cdots$$

This is the $\underline{THH}(R)$.

Note that there is no reason to expect the Hochschild homology groups of, say, $\mathbf{Z}$, to be the same as the ordinary $HH_*(\mathbf{Z})$ in which $\mathbf{Z}$ is the ground ring. In fact Bökstedt proved:

$$\pi_*\underline{THH}(\mathbf{Z}) = \begin{cases} \mathbf{Z}, & * = 0 \\ \mathbf{Z}/n\mathbf{Z}, & * = 2n-1 \\ 0, & else. \end{cases}$$

He also proved

$$\pi_* \underline{THH}(\mathbf{F}_p) = \begin{cases} \mathbf{F}_p, & * = 2n \geq 0 \\ 0, & else. \end{cases}$$

In fact there is a ring structure because $\mathbf{F}_p$ is a <u>commutative</u> generalized ring, and one has $\pi_* \underline{THH}(\mathbf{F}_p) = \mathbf{F}_p[x], x \in \pi_2$. Bökstedt and I have recently found the "topological HC^-" of $\mathbf{F}_p$:

$$\pi_*(\underline{THH}(\mathbf{F}_p)^{hS^1}) = \mathbf{Z}_p^\wedge[x,T]/xT = p \qquad x \in \pi_2, T \in \pi_{-2}.$$

<u>Cyclic homology of differential forms and the Chern Charcter</u> John D.S. Jones (joint work with Ezra Getzler)

Cyclic homology provides a natural model for differential forms on the (smooth) free loop space of a compact manifold - the Hochschild complex of the differential graded algebra of differential forms on the manifold X. In this model Connes B-operator corresponds to the interior product i_T where T is the vector field generated by the action of the circle on the loop space LX given by rotating loops. Ideas of Atiyah and Witten have led Bismunt to construct an equivariant differential form $CH(E, \nabla)$ on the loop space LX given a vector bundle E with a connection ∇. This equivariant differential form is equivariantly closed and when restricted to the fixed point set of the circle action on LX, that is X regarded as the constant loops, it gives the usual chern character form $Tr\ e^F$ where F is the curvature of the connection ∇.

The main point of this talk is to describe this model for the differential forms on LX and to explain how to construct this equivariant differential form in terms of the model. We hope also to be able to describe the Witten current on the loop space in terms of this model. This current μ is an equivariant current on the loop space of a spin manifold and it has the property that $< \mu, ch(E, \nabla) >$ is the index of the twisted Dirac operator D_E.

<u>Cyclic Homology and the Macdonald conjectures</u> (Phil Haulon)

Let A_k denote the truncated polynomial ring $\mathbf{C}[t]/_t^{k+1}$. We consider the Lie algebra cohomology of $L \otimes A_k$ for L a finite dimensional complex Lie algebra. We grade $L \otimes A_k$ by letting $L \otimes t^i$ be its i-th graded piece.

It is easy to check that this is a Lie algebra grading hence it extends to a grading on the cohomology of $L \otimes A_k$ which we call <u>the weight</u>. Thus $H(L \otimes A_k)$ is bigraded by degree and weight and we are interested in computing $H(L \otimes A_k)$ as a bigraded module.

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A simple deformation-theoretic argument shows that

$$dim(H(L \otimes A_k)) \geq \dim(H(L)^{\otimes k+1}).$$

We say L has property M if the dimensions are equal for all k.

CONJECTURE 1: If L is semisimple, then L has property M.

The importance of CONJECTURE 1 is that it implies (using other known results) the Macdonald Root-System Conjectures.

In trying to prove Conjecture 1 for $L = gl_n(\mathbf{C})$ by induction on n one is naturally led to consider the case $L = H_n$ where H_n is the $(2n+1)$- dimensional Heisenberg Lie algebra.

CONJECTURE 2: The Heisenberg Lie algebras have property M.

One might ask whether every finite-dimensional Lie algebra L has property M. The answer is no. Let $L_\alpha = < e, f, x >$ be the three dimensional Lie algebra whose nonzero bracketts

$$[x, e] = -[e, x] = e, [x, f] = -[f, x] = \alpha f.$$

L_α has property M so long as α is not a negative rational. However, there are examples of negative rationals α where $L\alpha$ does not have property M.

KK-theory and cyclic homology (J. Cuntz)

We associate with every algebra A an algebra qA as the kernel of the natural map $id * id$ from the free product $A * A$ to A. This algebra has the following properties:

1. qA is a classifying space for KK-theory.

2. The operation of associating qA to A is "dual" to the one of associating $M_2(A)$ to A.

3. qA consists of "K-theory differential forms" over A.

The first two points throw a completely new light on Kasparov's KK-theory. The third one gives the natural link between K-homology and cyclic cohomology.

Differential Algebras in Field Theory (R. Stora)

The Feynman algorithm, which describes perturbative expansions in quantized local field theory, accomodates a description of exact or broken symmetries. This description goes through the construction of some differential algebras whose "local" cohomologies provide "anomalies" which are due to the breaking of the symmetry at the quantum level. This happens specifically in the case of gauge symmetries e.g. in Yang Mills theories, 1st quantized string theory.

The lectures were derived into 4 sections:

1) The Feynman algorithm
2) Current algebras
3) Quantized gauge theories
4) 1st quantized strings.

Unfortunately, by lack of time, 4) was not covered.

Algebraic K-theory of spaces (F. Waldhausen)

The main purpose of this talk was to motivate the construction of the algebraic K-theory of spaces from the point of view of the topology of manifolds, namely (1) the h-cobordism theorem and (2) the study of parametrized families of h- cobordisms, to which the so-called pseudo-isotopy theory may be reduced. Now, if one does not just want to study an individual h-cobordism (as in the h- cobordism theorem) but a parametrized family of such, it will not be enough anymore to keep track of the attaching map of handles by (say) their homotopy classes only; rather it is necessary to keep track of such data in a more direct way. This leads to a modification of Quillen's algebraic K-theory where algebraic data (i.e. modules and isomorphisms) are replaced by more geometric data (i.e. spaces and weak homotopy equivalences), the point being that not just homotopy classes of the map in question are used, but whole spaces of such [For accounts for the construction of $A(X)$ from related points of view cf. (1) Proc. Conf. Alg. Top. London (Ontario) 1981, Contemp. Math. Series, AMS, and (2) Proc. Conf. Alg. Top. Durham 1985, Lond. Math. Soc. Publ.].

An offshoot of the theory is a re-interpretation of $A(X)$ as the algebraic K- theory of the "ring" $\Omega^\infty S^\infty(\Omega X_+)$. One has to cope with "rings" here which are "multiplicative spectra" in the sense of algebraic topology. Recently Bökstedt has found a satisfactory solution to the problems that his point of view entails. For such "rings" R (which include the usual ones) he has also constructed a Hochschild homology over the "ground ring" $\Omega^\infty S^\infty$; this is called $THH(R)$, the 'topological Hochschild homology'. $THH(R)$ is a cyclic object in the usual way, and I am happy to report that the cyclic structure has proved useful in Bökstedt's computation of $THH(R)$ in certain cases. Such computations, together with Bökstedt's trace map $K(R) \rightarrow THH(R)$ and its factorization (up to homotopy) through the homotopy fixed point set $THH(R)^{hS^1}$, are undoubtedly among the most promising tools in algebraic K- theory at this time.

Cohomology of current Lie algebras and applications to deformations

Claude Roger (joint work with P. Lecante)

We consider the Lie algebra of sections of the associated Lie algebra bundle to a principal bundle; following Faddeev, we shall call it the current algebra of type G, if G is the classical Lie algebra corresponding to the bundle, and denote it by G_p. We compute the cohomology of G_p with coefficients in the adjoint representation in low degrees. For G simple, the results are as follows

$H^2(G_p, G_p) \equiv 0$ if G is not $SL(n)$ for $n \geq 3$.

$H^2(G_p, G_p) = \Lambda_2(V)$ the space of contravariant antisymmetric tensors on V, the base manifold. An explicit formula can be given for cocycles using the symmetric bracket

$$I : G \times G \rightarrow G \text{ defined by } I(A, B) = AB + BA - \frac{2}{n} hABI_n$$

has an algebraic invariant. We deduce from that computation that G_p admits a lot of infinitesimal deformations, but computation of the Richardson- Nijenhuis bracket classical in deformation theory, implies that none of those deformations admit prolongations, so that G_p is always rigid for G simple.

The computations can be made also for G reductive and then the space deformation is much bigger; for example there exist deformations linked with local Lie algebra structures on the base of manifold V, or with cyclic cohomology of functions on the manifold. One can get also formal deformations related with star products on V (here the Morita invariance for Hochschild cohomology is used).

One can extend those computations to the case of deformations of modules over G_p and deformations of the associated gauge group; one of the motivations are physical applications: one could try to carry over the Flato-Lichnerowicz program of deformations, which has turned out to be successful for quantum mechanics, to the case of quantum gauge theories.

<u>**Positivity in cyclic cohomology**</u> (A. Connes and J. Cuntz)

In functional analysis one of the most important tools is positivity, for instance given an involutive algebra A over C a positive linear form Φ over A such that $\Phi(a^* a) \geq 0$ $\forall\, a \in A$ and it readily defines a Hilbert space with inner product $< x, y >= \Phi(y^* x)$. When one develops cyclic homology over C instead of an arbitrary field the cochains are $n+1$ linear forms and one has the following notion of positivity:

$\tau \geq 0 \Leftrightarrow$ the following inner product on $\bigotimes_0^m A$ is positive:

$$< a^0 \otimes a^1 \otimes \ldots \otimes a^m \;,\; b^0 \otimes b^1 \otimes \ldots \otimes b^m >= \tau(b^{0^*} a^0, a^1, \ldots, a^m, b^{m^*}, \ldots, b^{1^*})$$

Here n=2m is even.

Recall that the algebra qA of J. Cuntz is constructed (as the universal differential graded algebra) as $\sum a^0 qa^1 \ldots qa^n$ with the rule

$$q(ab) = (qa)b + a(qb) - (qa)(qb).$$

Moreover if A is a *-algebra, then so is qA with $(qa^*) = (qa)^*$ $\forall\, a$. Given a functional T on qA one defines the components:

$$T^{(m)}(a^0, \ldots, a^m) = T(a^0 qa^1 \ldots qa^m) \qquad \forall\, a^i \in A.$$

<u>Theorem:</u>

1) A functional T on qA is a trace iff
   a) for m even one has $bT^{(m)} = 0, B_0 T^m = (B_0 T^m)^\lambda$ (cyclic invariance),
   b) for m odd one has $bT^{(m)} = T^{(m+1)}$, $B_0 T^{(m)} = T^{(m-1)}$ where $\Phi' = \Phi - \frac{1}{2} b B_0 \Phi$.

2) For any positive trace T on qA and any even $n = 2m$ the component $T^{(n)} = T^{(2m)}$ is a positive <u>cocycle</u>.

3) For any positive trace T on qA there exists a Fredholm module ϵ relative to a semifinite van Neumann algebra N such that $T(qx^0 \ldots qx^n) = Chain_N(\epsilon)$.

We then show that the Dirichlet integral is a basic positive 2-cocycle yielding the conformal structure of a Riemann surface, which allows to define what is a non-commutative elliptic curve. We finally explain the work of J. Bellisard on the Quantum Hall effect as an application of the integrality of Chern classes of $C_{q,\Theta}$ the basic non-commutative elliptic curve.

Homotopy of Lie algebras, cyclic homology, crossed simplicial groups...

(Jean-Louis Loday)

For a commutative unitary algebra A over a characteristic zero field k the following is known for $H_\bullet(gl_n(A),k)$ (joint work with D. Quillen)

Theorem 1:

$$H_i(gl_{n-1}(A)) \xrightarrow{\sim} H_i(gl_n(A)) \qquad \text{if } i < n$$

$$H_n(gl_{n-1}(A)) \to H_n(gl_n(A)) \to \Omega^{n-1}_{A/k}/d\Omega^{k-2}_{a/k} \to 0$$

is exact (A commutative).

Theorem 2:

$$PrimH_n(gl(A)) = HC_{n-1}(A) \qquad (\text{ cyclic homology also denoted } H^\lambda_n(A)).$$

In fact $H^\lambda_\bullet$ and $HC_\bullet$ agree only in char.0.

The first one, that I call Connes' homology is the homology of the complex

$\ldots \to A^{\otimes n+1}/(1-t) \xrightarrow{b} A^{\otimes n}/(1-t) \to \ldots, b =$ Hochschild boundary, $t =$ cyclic operator.

Cyclic homology is the homology of the (B,b)-complex

$$\begin{tikzcd}
\vdots \arrow[d, dashed, "b"'] & \cdot \arrow[l, "B"'] \arrow[d, dashed] & \cdot \arrow[l] \\
A^{\otimes 2} \arrow[d, dashed, "b"'] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

where B is essentially given by

$$B(a_o,\ldots,a_n) = \sum_{i=0}^{n} (-1)^i (1, a_i, \ldots, a_n, a_0, \ldots., a_{n-1}).$$

It seems to me that it is important to think of $H^\lambda_\bullet$ and $HC_\bullet$ as two different theories.

Theorem 1 is analogous to a theorem of Suslin concerning $H_\bullet(GL_n(G)), F =$ infinite field which has recently been generalized to (non-commutative) local rings by D. Guin. Theorem 2 has only been proved in the GL-case in low dimension, $n = 1,2$. The analogy leads to a dictionnary:

multiplicative	*additive* (*or infinitesimal*)
$\mathbf{C}_m$	$\mathbf{C}_a$
GL_n	gl_n
det	$trace$
K_n	H^λ_n
K^M_n	$\Omega^{n-1} H\Omega^{n-2}$
motivic cohomology	?

Motivic cohomology is a conjectural cohomology theory defined for schemes over *Spec***Z** (hopefully) and universal for Chern classes. Beilinson derived some axioms for this theory involving $\mathbf{C}_m$, K-theory, Milnor K-theory. Similar axioms can be envisioned in the additive case (use the dictionary). My belief is that in the additive case the answer is cyclic homology. There is a lot of evidences for this in the affine case.

The following topological point has been dealt with in the lectures. The simplest simplicial model for S^1 has one non degenerate 0-all and one non degenerate 1- cell, all the others are degenerate. Then there are $(n+1)$ simplices in dimension n and there is a natural bijection with $\mathbf{Z}/(n+1)\mathbf{Z}$. Though this is not a simplicial group it is almost the case. A face map (for instance) is not a group homomorphism but is a crossed group homomorphism:$d_i(xy) = d_i(x)(x \cdot d_i)(y)$ where x is acting on d_i by $x\dot{d}_i = d_{x(i)}$ (think of $x \in \mathbf{Z}/(n+1)\mathbf{Z}$ acting cyclically on $\{0, 1, .., n\}$. This leads to a definition of a crossed simplicial group (joint work with Z. Fiedorowicz). Other examples are the family of dihedral groups, quaternionic groups, symmetric groups, braid groups, hyperoctahedral groups. A classification of crossed simplicial groups was given. This lead to other theories mimiking cyclic homology: dihedral homology (related to $H_*(o(A))$ and $H_*(sp(A))$, joint work with Prusi) and symmetric homology.

Cyclic Homology of Envelopping Algebras (Christian Kassel)

For any Poisson manifold equipped with a Poisson bracket $\{,\}$ Brylinski constructed a degree -1 differential δ on the differential forms anticommuting with the exterior derivative

$$\delta(f_o df_1 \dots df_p) = \sum_{i=1}^{p} (-1)^{i+1} \{f_o, f_i\} df_1 \dots \widehat{df}_i \dots df_p$$

$$+ \sum_{1 \le i < j \le p} (-1)^{i+j} f_o d\{f_i, f_j\} df_1 \dots \widehat{df}_i \dots \widehat{df}_j \dots df_p$$

Let us consider the canonical Lie-Poisson structure on the dual of a Lie algebra $\mathcal{G}$ or a field k of characteristic zero. We prove

Theorem:

a) $H_*(\Omega^{*}_{s(\mathcal{G})}, \delta)$ is isomorphic to the Hochschild homology of the envelopping algebra $V(\mathcal{G})$.

b) The homology of the double complex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega^2_{S(\mathcal{G})} & \xleftarrow{d} & \Omega^1_{S(\mathcal{G})} & \xleftarrow{d} & \Omega^0_{S(\mathcal{G})} & \longleftarrow & 0 & \longleftarrow & \cdots \\
\delta\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega^1_{S(\mathcal{G})} & \xleftarrow{d} & \Omega^0_{S(\mathcal{G})} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
\delta\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega^0_{S(\mathcal{G})} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array}$$

is isomorphic to Connes cyclic homology of $V(\mathcal{G})$.

c) If $dim G < \infty$, then the periodic cyclic homology of $V(\mathcal{G})$ is the same as the one for the ground field k.

This result allows the computation of the cyclic homology of many $V(\mathcal{G})$. It shows also that for all semi-simple Lie algebras, Connes spectral sequence for the cyclic homology of $V(\mathcal{G})$ never collapses at E^2 (it collapses at E^3 or beyond). In the talk, we illustrate this new and interesting phenomenon with the case of $sl(2)$.

<u>Relative K-theory and Cyclic Homology</u> (C. Ogle and C. Weibel)

If R is a $\mathbf{Q}$-algebra, there are two relationships between $K(R)$ and $HC(R)$. One is the chern character $K_n(R) \xrightarrow{ch} HP_n(R)$ of Connes-Karoubi, where HP_n fits into

$$0 \to \varprojlim_k{}^1 HC_{n+2k-1}(R) \to HP_n(R) \to \varprojlim_k HC_{n+2k}(R) \to 0;$$

ch factors through the Karoubi-Villmayor theory $KV_n(R)$. If $K_n^{rel}(R)$ denotes the third term of the usual long exact sequence of $K_nR \to KV_nR$, the second relationship is a secondary chern character $\nu : K_n^{rel}(R) \to HC_{n-1}(R)$. Thus we have:

$$\begin{array}{ccccccc}
KV_{n+1}(R) & \longrightarrow & K_n^{rel}(R) & \longrightarrow & K_n(R) & \longrightarrow & KV_n(R) \\
\downarrow ch & & \downarrow \nu & & \downarrow & & \downarrow ch \\
HP_{n+1}(R) & \xrightarrow{S} & HC_{n-1}(R) & \xrightarrow{B} & HC_n^-(R) & \xrightarrow{I} & HP_n(R)
\end{array}$$

The map ν is not an isomorphism on rings, but in relative situations we have isomorphisms!

<u>Theorem 1</u>: (Goodwillie) If I is a nilpotent ideal in R, then ν is an isomorphism

$$K_n(R,I) \xleftarrow{\cong} K_n^{rel}(R,I) \xrightarrow[\cong]{\nu} HC_{n-1}(R,I)$$

Theorem 2: (Ogle-Weibel) If I, J are ideals in R so that $I \cap J = 0$, then

$$K_n(R,I,J) \xleftarrow{\cong} K_n^{rel}(R,I,J) \xrightarrow[\cong]{\nu} HC_{n-1}(R,I,J)$$

Conjecture: If $A \subset B$ has conductor ideal I, then the map

$$K_n(A,B,I) \xleftarrow{\cong} K_n^{rel}(A,B,I) \xrightarrow{\nu} HC_{n-1}(A,B,I)$$

should be an isomorphism.

These recent theorems have given new calculations of algebraic K-groups. For example, let $R = k[x,y]/(xy = 0)$ for k a field of char $k = 0$. Then (for $\tilde{R} = R/k$)

n	0	1	2
$\widetilde{HC}^{k}_{n-1}(R)$	0	$\tilde{R}$	k
$HC^{\mathbf{Q}}_{n-1}(R)$	0	$\tilde{R}$	$k \oplus (\tilde{R} \otimes \Omega_k)$
$K_n(R)$	$\mathbf{Z}$	0	k

previously known

n	3	4	5	etc.
$\widetilde{HC}^{k}_{n-1}(R)$	0	k	0	
$HC^{\mathbf{Q}}_{n-1}(R)$	$\Omega_k \oplus (\tilde{R} \otimes \Omega_k^2)$	$k \oplus \Omega_k^2 \oplus (\tilde{R} \otimes \Omega_k^3)$	$\Omega_k \oplus \Omega_k^3 \oplus (\tilde{R} \otimes \Omega_k^4)$	
$K_n(R)$	Ω_k	$k \oplus \Omega_k^2$	$\Omega_k \oplus \Omega_k^3$	

only known via above theorems.

On Bott-Chern Secondary Characteristic Classes (C. Soulé)

If X is a complex manifold, E a holomorphic vector bundle on X and h a hermitean metric on E, denote by $ch(E,h) \in A(X) = \underset{p \geq 0}{\otimes} A^{pp}(X, \mathbf{C})$ the Chern form defined using the connection ∇ attached to $h : ch(E,h) = h\, exp(\frac{i}{2\pi}\nabla^2)$.

When $0 \to S \to E \to Q \to 0$ is an exact sequence of bundles on X, give arbitrary metrics $h', h, h"$ on S, E, Q respectively. Bott and Chern defined classes $\widetilde{ch} \in A(X)/(Im\partial + Im\bar{\partial})$ which can be characterized by the following properties:

i) $dd^c \widetilde{ch} = ch(E,h) - ch(S,h') - ch(Q,h"); d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$.

ii) $\widetilde{ch}$ depends functionally of the exact sequence and its metric.

iii) $\widetilde{ch} = 0$ when $(E,h) = (S \oplus Q, h' \overset{\perp}{\oplus} h")$.

We give new constructions of these $\widetilde{ch}$ (joint work with Bismunt and Gillet). In the talk we explain how $\widetilde{ch}$ enters in defining the regulator on $K_1(X)$ (joint with Gillet) and in expressing the variation of determinant of Laplace operators with the metrics (joint with Bismunt and Gillet).

Cyclic homology of commutative algebras (Michel Vigue)

For any chain commutative differential graded algebra (A,d) over a characteristic zero field k, we give an explicit formula that permits us to compute the cyclic homology $HC_\infty(A,d)$ from the construction of a free model of (A,d). Explicit calculations are done for the ring of coordinates of some hypersurface. For example, if A is the ring of a projective hypersurface $P=0$ in $\mathbf{C}P^{r-1}$ with only zero an isolated singularity, then we have, for $n \geq r$,
$HC_n(A) = HC_n(\mathbf{C})$ if $n \equiv r(2)$, and $HC_n(A) = HC_n(\mathbf{C}) \oplus \mathbf{C}^\mu$ if $n \neq r(2)$ with

$$\mu = dim_{\mathbf{C}} \mathbf{C}[x_1, \ldots, x_r]/(\frac{\partial P}{\partial x_1}, \ldots, \frac{\partial P}{\partial x_r}).$$

If A is the ring of an irreducible affine plane curve defined in $k[x_1,x_2]$ by an equation $P = x_1^p - \lambda x_2^q = 0$, then we have

$$HC_{2n}(A) = k, HC_{2n+1}(A) = k[x_1,x_2](\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}).$$

Moreover, the formula described above gives a natural decomposition theorem for $HC_.(A,d)$. We have

$$HC_.(A,d) = HC_.(k) \oplus \bigoplus_{p \geq 1} HC_.(A,d)^{(p)}$$

with $HC_.(A)^{(1)} = H_.(A,d)/k$. The map S sends $HC_.(A,d)^{(p)}$ into $HC_{*-2}(A,d)^{(p-1)}$. Finally, for a commutative algebra A, we prove that $HC_.(A)^{(2)} = T_{.-1}(A/k)$ for $* > 2$, where $T_.(A/k)$ is the Andre-Quillen homology of the inclusion $k \hookrightarrow A$.

Introduction to K-homology and cyclic homology (A. Connes)

The homology theory dual to ordinary K-cohomology is best described (by work of Atiyah-Brown-Douglas-Fillmore-Kasparov) by means of homotopy classes of K-cycles, which are called Fredholm modules. Given the algebra A (it is $C(X)$ for a compact space X) a Fredholm module over A is $\mathbf{Z}/_2$ graded Hilbert space $h = h^+ \oplus h^-$ which is a left A-module, together with an operator F in h such that $F^2 = 1, F\varepsilon = -\varepsilon F$, where ε is the $\mathbf{Z}/_2$ grading $\varepsilon = [\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]$ and that every commutator $[F,a], a \in A$, is a compact operator. We explain how first to introduce a dimension of a Fredholm module by the condition

$[F, a] \in L^p$ where L^p is the Schatten class $\{T \in L^p \Leftrightarrow \sum \mu_n(T)^p < \infty\}$ where μ_n is the n^{th} characteristic value of T. Then we quantize the ordinary calculus of forms on a manifold by the following formulae:

$$\begin{aligned} df &= i[F, f] \\ \Omega^k &= \{\textstyle\sum a^0 da^1 \dots da^k\} \\ \int \omega &= Trace(\varepsilon\omega). \end{aligned}$$

Given these laws one gets a differential graded algebra $\Omega = \oplus\Omega^k$ with $\Omega^0 = A$ and differential $d, d^2 = 0$, together with a closed graded trace $\int$ on Ω^n, n <u>even large enough</u>. We then explain how the character:

$$\tau(a^0, a^1, \dots, a^n) = \int a_0 da^1 \dots da^n = Trace(\varepsilon a^0 i[F, a^1] \dots i[F, a^n])$$

is a cocycle as follows:

1) $\tau^\lambda = \varepsilon(\lambda)\tau$ where λ is cyclic permutation

2) $b\tau = 0$ where b is the Hochschild coboundary.

It is these properties which give the definition of cyclic cohomology (see Oberwolfach meeting Sept. 1981), and the possibility of replacing n by $n+2$ which yield the operator:

$$S : H^n_\lambda(A) \to H^{n+2}_\lambda(A).$$

We then explain how the long exact sequence with I, B, S comes out of the above considerations. We end up by discussing the meaning of the above construction when $A = C_o(\mathbf{R}^3)$ and F is the phase of the Dirac Hamiltonian of mass m, and relate it to the Dirac electron theory.

<u>**Splitting Theorems for A(X) and Related Functors**</u> (Chrichton Ogle)

We extend the result due to Carlsson, Cohen, Goodwillie, Hsiang and (independently) the author, which proves that there is a weak equivalence $\Omega\overline{A}(\Sigma X) \simeq Q(\bigvee_{q\geq 1} \tilde{D}_q(x))$ for connected X, where $Q(-) = \Omega^\infty S^\infty(-)$ and $\tilde{D}_q(X) = E\mathbf{Z}_q \ltimes_{Z_q} X^{[q]}$. These results used Goodwillie's calculus of functors and Goodwillie's result identifying the n^{th} derivative as $D_n\Omega\overline{A}(Y) \simeq \Sigma^n Q(\tilde{D}_n(Y))(Y\,1\text{-conn.})$. We prove:

<u>**Theorem 1**</u>: For a connected space Y and any integer $m \geq 1$, the Goodwillie Taylor series of the functors

$$Y \to \Omega\overline{A}(\Omega^{m-1}\sum^m Y) \text{ and } Y \to Q(ES^1 x_{S^1}(\Omega^{m-1}\sum^m Y)/BS^1) = B(Y) \quad \text{split.}$$

A consequence of this via Goodwillie's result above on $D_n \Omega \overline{A}(-)$ is:

Cor. 2 $\exists$ a weak equivalence $\Omega \overline{A}(\Omega^{m-1} \Sigma^m Y) \simeq B(\Omega^{m-1} \Sigma^m Y), Y \text{conn.} m \geq 1$, which is natural in Y.

The techniques of proof involve constructing a weight filtration on the appropriate functors which splits the Goodwillie Taylor series. This has an interpretation in terms of Bökstedt's topological cyclic homology:

Cor. 2' $\exists$ a weak equivalence $\Omega \overline{A}(X) \simeq THH(Q(\Omega X_+))_{hS^1}$ for $X = \Omega^{m-1} \Sigma^m Y$, $()_{hS^1}$ = homotopy orbit space.

A similar theorem exists for n-relative Waldhausen K-theory:

Theorem 3: If $I_1, ..., I_n$ are a family of ideals in a $\mathbb{Q}$-algebra R, R simplicial and $\pi_0(I_1 \cap ... \cap I_n)$ is nilpotent in $\pi_0(R)$, then there exists a weak equivalence of n-relative spaces

$$K^w(R, \{I_j\}) \simeq THH(R, \{I_j\})_{hS^1}.$$

For $n = 1$ this is due to Goodwillie, and for R discrete is joint with C. Weibel.(c.f. Weibel's talk).

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