MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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# Kombinatorische und reell-algebraische Geometrie

13.08. bis 19.08.1989

The purpose of this conference, under the direction of E. Becker (Dortmund), A. Dress (Bielefeld) and J. Wills (Siegen), was to bring together mathematicians specialized either in combinatorial geometry or in real algebraic geometry. In combinatorial geometry there is an increasing need for methods, results and efficient algorithms of real algebraic geometry. Real algebraic geometers, on the other hand, should become acquainted with possible applications to combinatorial geometry. Accordingly, several talks covered recent results in real algebraic geometry with special emphasis on algorithms and complexity questions. Topics in combinatorial geometry ranged from matroid theory, incidence polytopes to convex geometry and geometry theorem proving. The talks and many fruitful discussions resulted in finding out the areas of mutual interest and where the one theory can be applied to the other. Evidently, in the next years one will see further and deeper interrelations between combinatorial and real algebraic geometry.

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## Vortragsauszüge

E. BECKER:

### Semi-algebraic geometry and the real spectrum

If A denotes a commutative ring with unit then the real spectrum Sper A ($\text{Spec}_r A$, X(A) or R-Spec A) is, by definition, the set

$$\text{Sper } A = \{\alpha = (\mathfrak{p}, P) \mid \mathfrak{p} \in \text{Spec } A,\ P \text{ an (linear) order of } k(\mathfrak{p}) = \text{quot}(A/\mathfrak{p})\}$$

endowed with a topology given by the subbasis $\{D(f)\}_{f \in A}$ where $D(f) = \{\alpha \mid f(\alpha) = f + \mathfrak{p} > 0$ relative to $P$ in $k(\mathfrak{p})\}$. The notion of the real spectrum was introduced by Coste and Coste-Roy in 1979. Since then, the fundamental importance of this notion for real algebraic geometry has become evident. In this talk several applications of this notion were explained. For a detailed study of real algebraic geometry one should consult the book "Géométrie Algébrique Réelle" by Bochnak, Coste and Roy.

J. BOKOWSKI:

### On the algebraic chirotope variety

New algebraic varieties can be defined in close relationship to the Grassmann variety. They form the basic tool for the following assertions:

- Matroids and oriented matroids differ exactly by the underlying field.
- New basis exchange theorems can be obtained by algebraic methods.
- A simple algebraic proof for Las Vergnas' theorem is possible which says essentially that already 3-term Grassmann-Plücker polynomials are sufficient to define oriented matroids.
- The oriented matroid variety over the reals characterizes oriented matroids also in the nonrealizable case. Its dimension is bounded by the number of points and by the dimension of the Grassmannian.

U. BREHM:

On the topology of the power complex

To each simplicial complex $K$ with n vertices is associated canonically a subcomplex $2^K$ of the cube $[0,1]^n$, called the power complex of $K$. For each simplicial complex $K$ and each $i \geq 0$ holds for the homology groups of $2^K$:

$$H_i(2^K) \cong \oplus \tilde{H}_{i-1}(L),$$

where the sum is taken over all induced subcomplexes $L \subseteq K$. Moreover the fundamental group $\pi_1(2^K)$ is generated by rank $H_1(2^K)$ many elements.

L. BRÖCKER:

Topological types of semialgebraic sets

Let $S \subset \mathbb{R}^n$ be semialgebraic, say $S = \cup(\cap f_{ij}\alpha_{ij})$ where $f_{ij} \in \mathbb{R}[X]$, $X = (X_1,\ldots,X_n)$ and $\alpha_{ij}$ stands for $> 0$, $\geq 0$, $= 0$. Call the formal expression $\Delta = (n,\cup(\cap k_{ij}\alpha_{ij}))$ a diagram for S and write $S \leq \Delta$ where $k_{ij}$ stands for the complexity $k(f_{ij})$, k may be the degree or the additive complexity. More generally, the $f_{ij}$ may be taken from an algebra $A_n$ of analytic functions $f : \mathbb{R}^n \to \mathbb{R}$ with complexity and some Bezout-property. Then S is global semianalytic.

We use "$\prec$" for "effectively bounded in terms of", $B_i = i^{th}$ Betti number and $B_* = \Sigma B_i$. Then we discuss the following statements:

1. $\Delta$ algebraic, $S \leq \Delta \Rightarrow B_0(S) \prec \Delta$
2. $\Delta$ closed, $S \leq \Delta \Rightarrow B_*(S) \prec \Delta$
3. $S \leq \Delta \Rightarrow B_0(S) \prec \Delta$
4. $S \leq \Delta \Rightarrow B_*(S) \prec \Delta$
5. For given $\Delta$ there are finitely many spaces $S_1,\ldots,S_e$, $e \prec \Delta$ such that for all $S \leq \Delta$ one has $S \simeq S_i$, $i \in 1,\ldots,e$.

1.-5. hold, if k = degree and 1.-4. hold for the additive complexity in the semialgebraic case.

In general, 1. for dimension 0 is the Bezout-property, which implies 1.-5. and also 5. for dim $\leq$ 3.

R. CONNELLY:

Generic global rigidity

Given a realization of a graph in three-space, a fundamental question is to be able to tell when there is no other non-congruent configuration of the vertices with the same edge lengths for the corresponding graph. When this happens we say that the graph is globally rigid. It is natural to conjecture that if the configuration is in generic position, the graph is rigid even with the removal of any single edge, and the graph is vertex 4-connected, then the graph is globally rigid. We show that there is a counterexample to this conjecture.

H. CRAPO:

The isotopy problem for projective lines

Given two indexed families $L = \{L_1,\ldots,L_n\}$, $M = \{M_1,\ldots,M_n\}$, each consisting of n mutually non-intersecting lines in real affine (or projective) 3-space, can the lines $L_i$ be moved simultaneously, in a continuous manner, and in such a way that at no time two of the lines intersect, until they arrive at the positions of the corresponding lines $M_i$, for $i = 1,\ldots,n$? If so, we say the families L,M are affinely (or projectively) isotopic. There are how many distinct configurations of n lines, up to affine or projective isotopy?

We show that the affine problem has a trivial solution: any two families of n lines are affinely isotopic. The projective problem is, however, interesting. The product of exterior products of the three pairs formed from a triple of lines

$$\chi_{ABC} = (A \vee B)(A \vee C)(B \vee C)$$

is a projective invariant which does not change sign under permutation of A,B,C or under re-coordination of any line. Its sign

$$\sigma_{ABC} = \mathrm{sign}(\chi_{ABC}) = +1 \text{ or } -1$$

is isotopy-invariant, symmetric, and orientation-invariant. For a family of n lines, the chiral signature is the assignment $\sigma : ABC \to \sigma_{ABC}$ of signs +1,-1 to the triples of these lines. The only known restriction on the values of these signs is that for any four lines A,B,C,D,

$$\chi_{ABC}\chi_{ABD}\chi_{ACD}\chi_{BCD} = 1.$$

We investigate certain simple configurations of lines, and show how the problem can be modeled as a problem concerning linked circles in 3-space.

L. DANZER:

Some recent results and aspects of quasi-periodicty in $E^2$ and $E^3$

In 1984 SHECHTMAN et al. discovered alloys with a novel kind of structure, intermediate between crystalline and amorphous. These alloys, now called quasicrystals, exhibit long-range orientational order but no translational symmetry. Since fivefold and even icosahedral symmetry is observed, several authors conjectured that the socalled "golden rhombohedra" yield a geometric explanation, in analogy to the PENROSE pieces in the plane. N. G. de BRUIJN and others developed the "strip projection method", using cubic lattices in higher dimensions to obtain quasiperiodic tilings with the golden rhombohedra. Guided by the idea that the long-range order of the quasicrystals must stem from some local conditions, I have sought families of prototiles ("protosets") which force quasiperiodicity when subject to appropriate matching conditions - as do the PENROSE pieces.

In this talk I first shall give a semi-axiomatic approach to the subject, and then describe a family of four tetrahedra that may serve as a protoset and realizes the "axioms". I'll also consider the relationships among the three most important methods: Matching rules - inflation/deflation principle - strip projection.

A. DRESS:

## Geometric algebra for combinatorial geometries

Given a combinatorial geometry (or "matroid") M, defined on a finite set E, one can associate with M in a canonical way a certain abelian group $\mathbf{T}_M$ which controls many important properties of M and, in particular, turns out to be the (abelianized) multiplicative group of the coordinatizing (skew-) field if M is a projective space. Cross ratios for appropriate subspace configurations and determinants of automorphisms of M "live" naturally in $\mathbf{T}_M$, leading e.g. to a new construction of the Dieudonné - determinants over skew-fields. A paper with the same title, explaining all this in detail, will appear soon in the "Advances in Mathematics".

J. E. GOODMAN:

## The complexity of order types

There are several natural measures of the "spread" of a configuration S of points in general position in $\mathbb{R}^d$. The one we prefer is the following, which is invariant under affine transformations:

$$\sigma(S) = \frac{\max_{p_0,\ldots,p_d \in S} \mathrm{vol}\langle p_0,\ldots,p_d\rangle}{\min_{p_0,\ldots,p_d \in S} \mathrm{vol}\langle p_0,\ldots,p_d\rangle} .$$

If we consider all configurations $S' \sim S$ having the same order type (i.e. oriented matroid, or chirotype, structure) as S, we define the *instrinsic spread* $\tilde{\sigma}(S)$ by

$$\tilde{\sigma}(S) = \min_{S' \sim S} \sigma(S).$$

The problem we consider is: how large can $\tilde{\sigma}(S)$ be for configurations S of n points in the plane?

We relate this to the problem posed by Bernard Chazelle several years ago of how large a grid is needed to accomodate all n-point planar configurations in general position, up to order type.

The result is the following theorem, proved in joint work with Richard Pollack and Bernd Sturmfels:

THEOREM. Let f(n) be the smallest integer N such that every configuration S of n points in general position in the plane can be realized on the grid $\{(i,j) \mid -N \le i,j \le N\}$, and let $g(n) = \max \tilde{\sigma}(S)$ over all configurations S of n points in general position in the plane. Then there exist constants $c_1, c_2$ such that

$$2^{2^{c_1 n}} \le f(n), g(n) \le 2^{2^{c_2 n}}$$

H. HARBORTH:

Ganzzahlige Abstände in Punktmengen

The diameter D(d,n) is the smallest of all largest distances for point sets of n points (no three in line) of $E^d$ with pairwise integral distances. -D(2,n) = 1,4,8,8,33,56,56 for $3 \le n \le 9$; D(3,n) = 1,3,4,8,13,17,17 for $4 \le n \le 10$; D(d,d+1) = 1, $3 \le D(d,n) \le 4$ for $d+1 \le n \le 2d$, D(d,2d) = 4, D(d,d+2) = 3 for d = 3,6,8.- If no d+1 points are in a hyperplane, and no d+2 points are on a d-sphere, then the diameters are 8,73,174 for n = 4,5,6 (d = 2), and 3,16 for n = 5,6 (d = 3). For n ≥ 6 even the existence of integral point sets is not known. - For special graphs in the plane with all edges of integral length the diameters are determined for all graphs with up to 6 vertices (connected, degrees ≥ 3), and for small regular graphs. - The platonic graphs have diameters 4,1,7,1,8, and in plane drawings (without crossings) 17,2,13,2,159 for tetrahedron, cube, octahedron, dodecahedron, icosahedron. - Special regular graphs with unit distance edges only are also discussed. Does a 4-regular plane graph with edges of unit length only exist with fewer than 52 vertices?

T. F. HAVEL:

## Distance geometry and molecular biology

Some applications of Mengers theory of distance geometry to problems of contemporary interest in molecular biology are presented. These include the determination of protein structure from Nuclear Magnetic Resonance data, and the rationalization of structure/activity relations observed for drug molecules. It is shown how these diverse applications can all be reduced to the problem of determining the three-dimensional realizability of distance and oriented matroid information, and a numerical algorithm which finds such realizations is described.

A. IVIC-WEISS:

## Chiral polytopes

In the past ten years much work has been done on the construction and classification of abstract regular polytopes $P$ (incidence polytopes). Here regularity means that the automorphism group of $P$ is transitive on the flags of $P$. Abstract regular polytopes are symmetrical by reflection and thus may be called reflexible. In recent years the term "chiral" has been used for geometrical figures which are symmetrical by rotation but not by (hyperplane) reflection. We discuss, and characterize, the groups of abstract chiral polytopes. In particular, we give a canonical construction of a chiral polytope from its group. Various problems on universal chiral polytopes are discussed.

W. KÜHNEL:

Manifold structures on abstract regular polytopes

This reports about joint work with U. Brehm and E. Schulte on the following problem: How can we associate a (d-1)-dimensional manifold with a given regular d-incidence-polytope $P \in \langle P_1, P_2 \rangle$?

In the classical case of reflexible regular honeycombs $\{p,q,r\}_t$ the manifold structure is clear because the facets are 3-balls and the vertex figures are 2-spheres. For examples such as $2^{\{3,4\}}$ and $2^{\{3,5\}}$ (Danzer's construction) compare U. Brehm's talk at this conference. Here we deal with the particular case of 4-incidence-polytopes where $P_1$ is a regular map on an orientable surface and $P_2$ is one of the Platonic solids. In a paper in 1977 Coxeter and Shephard have found that $P = \{\{4,4\}_{3,0},\{4,3\}\}$ leads to a 3-sphere if the facets are interpreted as toroids in a certain way. We find that $\{\{6,3\}_{1,1},\{3,4\}\}$ becomes a 3-sphere or real projective 3-space depending on the choice of the toroids. Similarly $\{\{4,4\}_{2,0},\{4,3\}\}$ and $\{\{6,3\}_{1,1},\{3,4\}\}$ become connected sums of orientable handles $S^1 \times S^2$. In other cases it is possible to compute the fundamental group or the homology of the manifold.

H. M. MÖLLER:

## Gröbner bases as a tool for solving algebraic equations

I consider a system of equations and inequations

$$f_i(x_1,\ldots,x_n) = 0,\ i=1,\ldots,m,$$
$$g_j(x_1,\ldots,x_n) \neq 0,\ j=1,\ldots,s,$$

with n-variate polynomials $f_i$ and $g_j$. In the talk, I will describe two methods which decompose this system into a finite number of systems

$$F_{i1}(x_1) = 0,\ F_{i2}(x_1,x_2) = 0,\ldots,F_{in}(x_1,\ldots,x_n) = 0,$$ with

polynomials $F_{ij}$ provided, the given system has only a finite number of solutions. The methods use symbolic manipulations, esp. Gröbner basis techniques. One of these methods is already installed in the Computer Algebra System REDUCE.

B. MONSON:

## Finite incidence polytopes of type {3,3,p}

Recently, McMullen has described, for each odd prime p, a regular map of type {3,p}, whose automorphism group has order $p(p^2-1)$. Using a related but quite different construction, we have discovered a regular 4-dimensional incidence polytope $\mathcal{P}$ of type {3,3,p}. The tetrahedral cells of $\mathcal{P}$ are circumscribed to the null cone in a certain 4-dimensional orthogonal space over GF(p). For $p \equiv \pm 1 \pmod 4$, the automorphism group of $\mathcal{P}$ has order $p^2(p^2-1)(p^2\mp 1)$. When $p \equiv -1 \pmod 4$, the rotation group is $PSL_2(p^2)$. When $p \equiv 1 \pmod 4$, certain edges of the polytope form a kind of finite Hopf fibration.

T. RECIO:

Some computational aspects in real algebraic geometry

I consider, as a measure for the complexity of a semialgebraic set $S \subseteq \mathbb{R}^N$, the minimum height h of any algebraic computation tree (a.c.t) (in the sense of Ben-Or) describing S. Given a semialgebraic set there is an algorithm to compute its complexity; moreover one can show that the number of topological types of semialgebraic sets in $\mathbb{R}^N$ with bounded complexity is also bounded by a function depending on N and the complexity; similar results hold for the complexity of the images by semialgebraic functions, triangulations, etc. Concrete computations of complexity can be performed using as an intermediate step an extension of Ben-Or techniques for non-scalar complexity plus a generalized notion of degree-intersecting of a semialgebraic set with a quadratic hyperplane. In this way we show that the Largest Empty Circle problem has $\Omega(N \log N)$ as a lower bound in a.c.t. model.

M. F. ROY:

Algorithms in real algebraic geometry

The following problems are known to have an algorithmic answer:

a) decide whether a semi-algebraic set is empty,
b) elimination of quantifiers in a formula of ordered fields,
c) computation of connected components.

In the few last years a lot of progress has been made in the definition of new algorithms leading to simple exponential complexity bounds instead of doubly exponential ones. (Grigor'ev, Vorobjov, Canny and others). The situation is not completely clarified for topological problems.

For the moment these very general algorithms do not give concrete answer to problems arising from applications.

M. F. ROY:

## Complexity of semi-algebraic sets and Tarski-Seidenberg principle

The following results have been obtained in collaboration with Joos Heintz, M. F. Roy and P. Solerno:

a) Complexity of semi-algebraic sets: a new algorithm based on linear algebra, and thus well parallelizable, for testing whether a semi-algebraic set is empty or not in sequential simple exponential time.

b) An algorithm for quantifier elimination in the real case, doubly exponential only in the number of alternations of quantifiers, and also well parallelizable.

E. SCHULTE:

## Hermitian forms and locally toroidal polytopes

Abstract regular polytopes are combinatorial structures resembling the classical regular polytopes. Traditionally polytopes are locally and globally of spherical type.

We discuss the classification of the finite universal locally toroidal regular polytopes. For example, the universal polytope $\{\{6,3\}_{m,0},\{3,3\}\}$ is finite if and only if $m \leq 4$. The basic technique is to associate with the polytopes a Hermitian form which governs the structure. The polytopes are finite if and only if the Hermitian forms are positive definite.

A. M. VERSHIK:

Classification in projective and convex geometry and universality of the stratifications

Classification of the convex polytopes as geometrical objects, and configurations as finite representation of matroids.
Universalities in categories of algebraic varieties and others.
Spaces of polytopes, linear programs, configurations etc.
Representation theory of posets and the scale of classifications.
Problem of the complexity of optimization problems and singularity theory.

N. L. WHITE:

Cayley (or Grassmann) factorization

An important problem in computer-aided geometric reasoning is to automatically find geometric interpretations for algebraic expressions. For projective geometry, this question can be reduced to the Grassmann factorization problem. A Grassmann factorization of a homogeneous bracket polynomial P is a Grassmann algebra formula (using only the join and meet operations) which evaluates to P. The main result of this paper is an algorithm which solves the Grassmann factorization problem in the special case that P is multilinear. The straightening algorithm of classical invariant theory is used repeatedly as a subroutine, and we describe several variations which are relatively more efficient (although still exponential in complexity) and hence more useful in the Grassmann factorization algorithm.

J. M. WILLS:

Lattice polytopes in convex geometry

For a convex body K in Euclidean d-space $E^d$ let $V_i(K)$, $i=0,1,\dots,d$ be its intrinsic volumes (or normed quermassintegrals). We investigate algebraic and geometric properties of the polynomial $W(\lambda K) = \sum_{i=0}^{d} V_i(K)\lambda^i$, $\lambda \in \mathbb{C}$ and its zeros $\lambda_1,\dots,\lambda_d$. Various relations between W, the Steiner-polynomial and the lattice-point enumerator $G(K) = \operatorname{card}(K \cap \mathbb{Z}^d)$ are considered.

In particular we show that a Fenchel-Alexandrov-type-inequality implies the existence of complex zeros of $W(\lambda K)$; and we show inequalities between successive minima and the zeros of W.

In the analogous problem for the Steiner-polynomial (first investigated by B. Teissier) it might be helpful to introduce the series of inradii $r_i$ of largest i-dimensional balls inscribed in K and the corresponding series of outradii $R_i$ of smallest circumscribed cylinders.

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