MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Reelle algebraische Geometrie

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This conference, under the direction of E. Becker (Dortmund), L. Bröcker (Münster) and M. Knebusch (Regensburg), was the third one on this subject held at Oberwolfach. It is the goal of these meetings to present and to discuss recent developments in real algebraic geometry and related branches of mathematics. As before this conference also found a remarkable international interest. But it was the first time that so many mathematicians from East Europe could participate. Some of the main topics treated during the meeting were geometry over the reals, semi-algebraic geometry, topological aspects of real geometry, sub-analytic geometry, Nash functions, stability theory and computational aspects of algebraic geometry. The talks at the conference showed that there are many areas of mathematics overlapping with real algebraic geometry. This fact will certainly stimulate broader interest in methods and results of real algebraic geometry and will also help to develop this branch of mathematics.

Vortragsauszüge

E. BECKER:

Real Closures without Zorn's Lemma

This talk presents a report on results by Hollkott, PhD thesis, Hamburg 1941; Zassenhaus in "Computational aspects in abstract algebra", Ed. J. Leech, 1967; Lombardi/Roy, to appear in Proc. MEGA 1990 and T. Sander, to appear in J. Pure Appl. Algebra. They all prove that the real closure of an ordered field can already be constructed in ZF, without the axiom of choice. It was also mentioned that in a certain model of ZF the ordered field $\mathbb{Q}$ has two non-isomorphic algebraic closures; in particular there exists an algebraic closure of $\mathbb{Q}$ admitting no real closure in it.

R. BERR:

The 17th problem for sums of 2n-th powers

Let K be a formally real field. Let $V|K$ be an affine, irreducible, smooth variety, with $\dim V \geq 1$. We consider the following property P_n $(n \in \mathbb{N})$: $\quad \forall f \in K[V] : f \in \Sigma K(V)^{2n} \Leftrightarrow f(x) \in \Sigma K(x)^{2n}$ for every algebraic point $x \in V$.

Now define

$$S(V|K) := \{n \in \mathbb{N} \mid V|K \text{ satisfies } P_n\}.$$

The following results were presented:

Thm 1: $S(V|K)$ is a semi-group with 1, generated by a set of primes.

Let $\mathbb{P}_0$ be a set of primes. Denote by $\langle\mathbb{P}_0\rangle$ the multiplicative semi-group with 1 generated by $\mathbb{P}_0$.

Thm 2: For every set $\mathbb{P}_0$ of primes there is a "pair" $V|K$, such that $S(V|K) = \langle\mathbb{P}_0\rangle$.

Thm 3: Let K_0 be a totally archimedean field and $K|K_0$ a function field with tr. d. $K|K_0 \geq 1$. Then

$$S(V|K) = \mathbb{N}.$$

Thm 4: Let K_0 be totally archimedean, $F|K_0$ a function field with tr.d. $F|K_0 \geq 1$ and $K = F(X_1,\ldots,X_e)$ $(e \geq 1)$ a pure transcendental extension. Then for every $n \in \mathbb{N}$ we have
$\forall f \in K[V]$: $f \in \Sigma K(V)^{2n} \Leftrightarrow f(x) \in \Sigma K^{2n}\ \forall x \in V(K)$.

Finally, let K be totally archimedean. If $V|K$ is a smooth projective curve, then we have

$$f \in \Sigma K(V)^{2n} \Leftrightarrow f \in \Sigma K(V)^2 \text{ and } 2n|\mathrm{div}_{real}(f).$$

If dim $V \geq 2$, the following result holds:

Thm 5 (-, Ruiz, Prestel):

$$f \in \Sigma K(V)^{2n} \Leftrightarrow \forall \text{ irreducible curve } C \subset V\text{: } f_{|C} \in \Sigma K(C)^{2n}.$$

joint work with
E. Becker - F. Delon - D. Gondard

J. BOCHNAK:

Elliptic curves and real algebraic morphisms into 2-spheres

Given a compact connected nonsingular oriented real algebraic variety X of dimension n, let $R(X,S^n)$ be the set of (real) regular mappings from X into the unit n-sphere and let

$$\mathrm{Deg}_R(X) = \{k \in \mathbb{Z}\,|\,k = \deg f,\ f \in R(X,S^n)\}$$

be the set of topological degrees of regular mappings from X into S^n. If X is of odd dimension, then $\mathrm{Deg}_R(X)$ is always either $\mathbb{Z}$ or $2\mathbb{Z}$ (however no example of an X with $\mathrm{Deg}_R(X) = 2\mathbb{Z}$ is known). If dim $X = 2$ then $\mathrm{Deg}_R(X)$ is a subgroup of $\mathbb{Z}$ (it is conjectured, but not proved that this is still true for all even n). By using complex elliptic curves we were able to make a significant progress in studying the structure of $\mathrm{Deg}_R(X)$, when dim $X = 2$.

Assume from now on that X is a nonsingular compact connected orientable affine real algebraic surface. Since the set $\mathrm{Deg}_R(X)$ is then a subgroup of $\mathbb{Z}$, one has necessarily $\mathrm{Deg}_R(X) = b(X)\mathbb{Z}$ for some, uniquely determined, nonnegative integer $b(X)$.

Theorem. Let M be a C^∞ compact connected orientable surface and let b be a nonnegative integer. Then there exists an affine nonsingular real algebraic surface X, diffeomorphic to M, such that $b(X) = b$, that is $\mathrm{Deg}_R(X) = b\mathbb{Z}$.

One of the steps in the proof of the above theorem is the study of the invariant $b(C \times D)$ for the product $C \times D$ of nonsingular (connected) cubic curves C and D in $\mathbb{R}P^2$. This study is fully completed. Here we quote only one corollary.

Theorem: There exist (up to isomorphism) exactly 18 unordered pairs (C,D) of nonsingular cubic curves in $\mathbb{R}P^2$, defined over $\mathbb{Q}$, such that the set $R(C \times D, S^2)$ of regular mappings from $C \times D$ into S^2 is dense in $C^\infty(C \times D, S^2)$ (= the set of C^∞ mappings).

The equations of these curves are known explicitly.

The joint paper with W. Kucharz will appear soon.

J.-L. COLLIOT-THÉLÈNE

Etale Kohomologie und klassische Kohomologie reeller algebraischer Varietäten

Sei X eine irreduzible Varietät über einem reell abgeschlossenen Körper R, und sei d die Dimension von X. Sei s die Anzahl der Zusammenhangskomponenten von X(R) (im semi-algebraischen Sinne). Unter $\underline{H}^n$ bezeichnen wir die Zariski Garbe auf X, die zu der Prägarbe $U \to H^n_{\acute{e}t}(U, \mathbb{Z}/2)$ assoziert ist. Für Kurven geht der folgende Satz auf E. Witt zurück:

Satz 1 (C.-T./Parimala, Invent. Math. 101 (1990) 81-99). Sei X/R glatt. Für $n \geq d+1$, ist die Gruppe $H^0(X, \underline{H}^n)$ der globalen Schnitte von $\underline{H}^n$ zu $H^0(\mathrm{Spec}_r(X), \mathbb{Z}/2) = (\mathbb{Z}/2)^s$ isomorph. (Falls $R = \mathbb{R}$, stimmt die letzte Gruppe mit $H^0(X(\mathbb{R}), \mathbb{Z}/2)$ überein.)

Es wurde der Beweis von C.-T. und Parimala skizziert. Er beruht auf: 1) die Bloch-Ogus Theorie; 2) Ergebnisse aus der Theorie der quadratischen Formen über reellen Funktionenkörpern, sowie der Galoiskohomologie solcher Körper (Pfister, Lam, Arason, Elman, Jacob); 3) den Satz von Mahé, der besagt, auf X gibt es genug quadratische Räume, um die Zusammenhangskomponenten zu trennen.

Es wurde dann eine neue Methode (Claus Scheiderer, Februar 1990), den Satz 1 zu beweisen, skizziert: bei dieser Methode wird von quadratischen Formen kein Gebrauch gemacht.

Sei $\varphi : \mathrm{Spec}_r X \to X$ die natürliche "Träger" Abbildung. Wie C. Scheiderer bemerkt hat, kann man den Satz 1 als $\underline{H}^n \simeq \varphi_*(\mathbb{Z}/2)$ $(n \geq d+1)$ interpretieren.

<u>Satz 2</u> (C. Scheiderer). Sei X/R beliebig. Für $i \geq 1$ ist $R^i\varphi_*(\mathbb{Z}/2) = 0$. Es folgt sofort die Erweiterung von Satz 1:

<u>Satz 3</u> (C. Scheiderer, Februar 1990). Sei X/R eine glatte irreduzible Varietät. Dann ist für jedes $n \geq d+1$ und jede Zahl $i \geq 0$:

$$H^i(X,\underline{H}^n) \simeq H^i(\mathrm{Spec}_r X,\mathbb{Z}/2) \quad (= H^i(X(\mathbb{R}),\mathbb{Z}/2) \text{ falls } R = \mathbb{R}).$$

Während der Tagung haben L. Mahé und ich versucht, den Satz 1 auf singuläre Varietäten auszudehnen. Wir hatten schon einen vielversprechenden Weg gefunden, der über die "multiplikative" Verlagerung lief, als mir klar geworden ist, daß - wenigstens im Falle $R = \mathbb{R}$ - die Sätze 1 und 3 für <u>beliebige</u> reelle Varietäten aus dem obigen Satz 2 und einem Ansatz von Artin-Verdier (1964), der von D. A. Cox (Proc. A.M.S. 76 (1979), 17-22) bearbeitet wurde, schon zu beweisen waren. Aus der Arbeit von D. A. Cox folgen nämlich für beliebige $X/\mathbb{R}$ Isomorphismen

$$H^m_{\text{ét}}(X,\mathbb{Z}/2) \simeq \oplus_{i=0,\dots,d} H^i(X(\mathbb{R}),\mathbb{Z}/2) \quad (m \geq 2d+1),$$

die zusammen mit Satz 2 Isomorphismen $\underline{H}^m \simeq \varphi_*(\mathbb{Z}/2)$ $(m \geq 2d+1)$ liefern. Daß man $(2d+1)$ auf $(d+1)$ in der letzten Ungleichung reduzieren kann, folgt aus der Tatsache (SGA 4, XIV): für X <u>affin</u> und $m \geq d+1$ ist $H^m_{\text{ét}}(X \times_{\mathbb{R}} \mathbb{C},\mathbb{Z}/2) = 0$.

M. COSTE:

Problems about Nash functions

We consider some open problems concerning Nash functions. Let $\Omega \subset \mathbb{R}^n$ be a semialgebraic open set, $N = \mathcal{N}(\Omega)$ (resp. $A(\Omega)$) be the ring of Nash (resp. analytic) functions on Ω.

P1 (Separation problem)? If ρ is prime in N, then $\rho A(\Omega)$ is prime.
The second problem deals with Nash sheaves. Here we consider only finite covers of open semialgebraic sets, i. e. the sheaves are over the constructible set $\tilde{\Omega}$ corresponding to Ω. Let $\mathcal{N}$ be the sheaf of Nash functions. A sheaf $\mathcal{F}$ of $\mathcal{N}$-modules is said to be A-coherent (following Tognoli) when there is a global exact sequence $0 \to \mathcal{N}^p \to \mathcal{N}^q \to \mathcal{F} \to 0$.

P2? a) Any sheaf of ideals which is locally of finite type is A-coherent.
b) The functor $\Gamma(\tilde{\Omega},-)$ is exact on A-coherent sheaves.
P2 implies a strong version of the extension problem: for any sheaf of ideals I locally of finite type, any global section of $\mathcal{N}/_I$ lifts to a Nash function on Ω.

These problems are quoted in [Shioba], and there it is shown that they are strongly related. Actually, we prove that P1 and P2 are equivalent, and both equivalent to

P3? For any finite covering $(V_i)_{i\in I}$ of Ω by open semialgebraic sets, the diagram $\coprod_{i,j} \operatorname{Spec} \mathcal{N}(V_i \cap V_j) \rightrightarrows \coprod_i \operatorname{Spec} \mathcal{N}(V_i) \to \operatorname{Spec} N$ is a cokernel.

The main ingredients in the proof of the equivalence are, beyond the arguments in [Shioba], a result of R. Huber (Nash functions separate analytic components locally - where locally means, as above, by passing to a finite cover), and also the consideration of a canonical morphism from the topos of sheaves over $\tilde{\Omega}$ to the ètale topos of Spec N.

The problem P1 (and hence P2 and P3) has an affirmative answer for any $\Omega \subset \mathbb{R}^2$. This is in [Efroymson]; the proof there contains serious gaps, but they can be filled.

It may be noted that P2 and P3, translated in the context of regular functions are false.

ref: [Efroymson] Nash rings on planar domains, Trans A.M.S. 249 (1979).
[Shioba] Nash manifolds, Lect. Notes in Math., Springer.

joint work with M. Diop

C. N. DELZELL:

On the Pierce-Birkhoff Conjecture (PBC)

I. Let R = a real closed field, K a subfield, $A \subseteq R^n$ a subset.

Def: $f : A \to R$ is called piecewise-polynomial (PWP) (resp. piecewise-rational, PWR) if (1) $A \subseteq \bigcup_{i=1}^{I} A_i \subseteq R^n$, A_i K-R-semialgebraic (s.a.),

(2) $f|_{A_i} = P_i|_{A_i}$, some $P_i \in K[X]$ (resp. $K(X)$), $(X = (X_1,\ldots,X_n))$.

Conjecture (Pierce-Birkhoff, 1956): If $A = R^n$, $f : A \to R$ is continuous and PWP, then there exist finitely many $s_{jk} \in K[X]$ s.t. $f = \bigvee_j \bigwedge_k s_{jk}$ on R^n ($\vee, \wedge$ means sup, inf resp.). (We say "f is sup-inf-polynomially definable" or "$f \in$ SIPD".)

Note: (1) Converse obvious. (2) PBC known only for $n \le 2$ (Mahé '84). (3) It's known that we can't replace $A = R^2$ by arbitrary $A \subseteq R^2$ (Mahé '84).

Thm (Delzell, 87): For $n \ge 1$ and $A \subseteq R^n$ arbitrary, if $f : A \to R$ is PWR (cont. or not), then there exist finitely many $s_{jk} \in K(X)$ s.t. $\forall x \in A$ at which each s_{jk} is defined, $f(x) = \bigvee_j \bigwedge_k s_{jk}(x)$ (i.e. f is "SIRD" almost everywhere "o.e.").

Pf: Write $A = Z \cup \bigcup_{e=1}^{E} B_e$ where (a) the sets $B_e := \{x \in A \mid \bigwedge_{m=1}^{M_e} q_{em}(x) > 0\}$

and $Z = \{x \in A | p(x) = 0\}$ ($p, q_{e_m} \in K[X]$) are pairwise disjoint, and (b) $\forall e \exists i := i(e)$ s.t. $B_e \subseteq A_i$. Suffices to prove that the characteristic function $\chi_{B_e} : A \to R$ is (o. e.) SIRD, for then $f = \Sigma \chi_{B_e} p_{i(e)}$ o. e. on A. (Fact: SIRD is a ring, by Henriksen-Isbell 1962). For this, note that $g_e := 0 \vee \bigwedge_m \frac{1}{q_{e_m}} \in$ SIRD is > 0 on B_e and $= 0$ o. e. on $A \backslash B_e$. If B_e is bounded, then $\exists \varepsilon_e \in K$ s.t. $g_e \geq \varepsilon_e > 0$ on B_e. Then $\chi_{B_e} = 1 \wedge g_e/\varepsilon_e \in$ SIRD. If B_e is unbounded we construct $u_e \in K(T)$ (T an indet.) s.t. $g_e(x) \geq u_e(x_1^2 + \ldots + x_n^2) > 0$ o. e. on B_e. Then again $\chi_{B_e} = 1 \wedge g_e(x)/u_e(x_1^2 + \ldots + x_n^2) \in$ SIRD.

II. Let A = comm. ring, $\alpha \in \mathrm{Spec}_r(A)$, $k(\alpha)$ = real closure of $(\mathrm{qf}(A/_{\mathrm{supp}(\alpha)}), \leq_\alpha)$, SA(A) = Schwartz's ring of abstract s.a. functions, $\tilde{A} = \{\tilde{a} | a \in A\} \subseteq SA(A)$, where $\tilde{a}(\alpha) = a + \mathrm{supp}(\alpha) \in k(\alpha)$, $F_r(A)$ = sub-lattice of SA(A) generated by $\tilde{A}$ (= a ring by Henriksen-Isbell), and $PW(A) = \{s \in SA(A) | \forall \alpha \in \mathrm{Spec}_r(A)\ \exists a \in A \text{ s.t. } s(\alpha) = \tilde{a}(\alpha)\}$ (for each $s \in PW(A)$ we need only finitely many $a \in A$, by Tychonoff compactness of $\mathrm{Spec}_r(A)$).

<u>Def.</u> (Madden, Arch. Math., 1989): A called a Pierce-Birkhoff ring if $F_r(A) = PW(A)$. ($\subset$ is automatic).

<u>Note:</u> $PW(R[X]) \approx PWP(R^n)$. So the PBC is equivalent to: "R[X] is a P.B. ring".

<u>Prop.</u> (Madden): (localness of P.B. property): A is P.B. $\Leftrightarrow \forall t \in PW(A)$, $\forall \alpha, \beta \in \mathrm{Spec}_r(A)$, $\exists h \in A$ s.t. $\tilde{h}(\alpha) \geq t(\alpha)$ and $h(\beta) \leq t(\beta)$.

<u>Def.</u> (Madden): For $\alpha, \beta \in \mathrm{Spec}_r(A)$, define the separating ideal $\langle\alpha, \beta\rangle$ of α and β to be the ideal of A generated by all $a \in A$ s.t. $\tilde{a}(\alpha) \geq 0$ and $\tilde{a}(\beta) \leq 0$.

Madden's Main Thm: A is P.B.R. $\Leftrightarrow \forall t \in PW(A)$, $\forall \alpha, \beta \in Spec_r(A)$, $t_\alpha - t_\beta \in \langle \alpha, \beta \rangle$, where t_α is any $a \in A$ s.t. $\tilde{a}(\alpha) = t(\alpha)$ (likewise for t_β).

Cor: Every field A is a P.B.R.

Cor: K(x) is P.B. (This is equivalent to my thm.)

J.-P. FRANCOISE:

Decidability of real algebraic sets

We consider the decidability problem: given $f \in \mathbb{R}[x_1,\ldots,x_n]$, is the algebraic variety $V = \{f(\underline{x}) = 0\}$ empty or not? We propose a solution based on the index formula. This approach allows to give an estimation of the number of operations needed. We use an approximation by a Riemann sum and elementary majoration.

J. M. GAMBOA:

Algebraic images of the real plane

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be an analytic map. Then, the set $X = f(\mathbb{R}^m)$ verifies it is pure dimensional, connected and the ideal I_X of analytic functions vanishing on X is prime. Moreover, if dim X = d, there exists an analytic curve $\varphi : \mathbb{R} \to X$ which meets each connected component of the analytic d-dimensional regular locus X_{reg} of X. A recent result of Shioba proves the following:

Theorem. A semialgebraic, connected, pure dimensional subset X of $\mathbb{R}^n$ is the image of $\mathbb{R}^m$, m = dim X, for some Nash map $f : \mathbb{R}^m \to \mathbb{R}^n$ if and only if there exists an analytic curve $\varphi : \mathbb{R} \to X$ which meets each connected component of X_{reg}.

Concerning regular or polynomial maps, very few is known. We can however state some examples:

1) Each subset $X \subset \mathbb{R}^n$ with $(\mathbb{R}^n \setminus X)$ finite is the image of $\mathbb{R}^n$ by a polynomial map.

2) If $f \in \mathbb{R}[x,y]$ generates a real ideal and $X = (f > 0)$ is a polynomial image of $\mathbb{R}^2$, then the zero set of f is a rational curve.
3) Open half hyperplanes are polynomial images of $\mathbb{R}^n$.

As specific examples of open questions we can state:

1) Is the set $x = \{x^2 + y^2 > 1\}$ a polynomial image of $\mathbb{R}^2$? The answer is YES, for regular maps, as a consequence of 3) above.
2) Is the open quadrant $\{x > 0, y > 0\}$ a regular image of $\mathbb{R}^2$?

D. J. GRIGOR'EV:

Efficient algorithm for testing whether two points can be connected by a curve in a semialgebraic set

The decidability of this problem is due to A. Tarski but the complexity of his algorithm is not elementary. G. Collins has produced an algorithm with expontential time-bound. The author jointly with N. N. Vorobjov (jr.) has designed subexponential-time algorithm for this problem, which allows also to count the number of connected components, relying on the algorithm for solving systems of polynomial inequalities and deciding Tarski algebra, constructed by the authors earlier.

Z. HAJTO:

Some remarks about Graßmann blowing-ups

T. C. Kuo and D. J. A. Trotman introduced in their joint paper "On (w) and $(+^s)$-regular stratifications", Invent. math. 92, 633-643 (1988) the notion of G-blowing-up ("G" for Graßmann). Using the notation from S. Łojasiewicz book "Introduction to Complex Analytic Geometry" we generalize G-blowing up as follows:
Let $S^k(W)$ be $= \{T \in \mathbb{G}_k(\mathbb{K}^n) : T \supset W\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$,
$E^d_{n,k} = \{(x,L) \in \mathbb{K}^n \times S^k(W) : x \in L\}$ - canonical d-plane bundle.
$\Pi = \Pi^d_{n,k} : E^d_{n,k} \to S^k(W)$ for $k \geq \dim W + 1$. The map
$\beta = \beta^d_{n,k} : E^d_{n,k} \ni (x,L) \mapsto x \in \mathbb{K}^n$ we call G-blowing-up of $\mathbb{K}^n$ in W. The restriction $\beta^\Omega : (E^d_{n,k})_\Omega \to \Omega$ to the open neighbourhoods of $0 \in \mathbb{K}^n$ we

call local canonical G-blowing-ups (we can use them as a model for defining a more general process i. e. G-blowing-up of a manifold in a closed submanifold).

Using notion of generalized G-blowing-up we prove the following theorem which generalizes former works of Henry Merle and Navarro Azuat:

Theorem. Let X be an analytically constructible submanifold in an open subset $D \subset \mathbb{C}^N$ such that $d = \dim \dot{X} \geq 2$, $0 \in \overline{X} \cap D$. Let Y be a line through the $0 \in \mathbb{C}^N$, $\beta^D = (\beta^1_{n,k} + 1)^D$ the local G-blowing-up of D in $Y \cap D$. Suppose that there exists $k \geq 1$ such that $S_{\overline{X},0} = 0 \times S^{k+1}(Y)$ (i. e. exceptional fiber over origin is whole $S^{k+1}(Y)$), then for every hyperplane $L \in \mathbb{P}^{N-1}$ which is transverse to Y and $\overline{X}$ at 0 there exists Zariski open and dense subset $U \subset S^{k+1}(Y)'$ with the following property:

(P) for every sequence $(x_n, H_n) \in \beta^{-1}(X)$ such that $(x_n, H_n) \to (0,H) \in 0 \times U$ and there $\exists \lim_{m\to\infty}(T_{x_n}X \cap L) = \tau \cap L$ we have that $\tau \cap L$ intersects H transversally.

The Lemma plays a crucial role in the theory of polar varieties.

U. HELMKE:

Waring's Problem for Binary Forms

Waring's problem for binary forms $\phi(X,Y) = \sum_{j=0}^{d} \binom{d}{j} a_{j+1} X^{d-j} Y^j$ with coefficients a_j in a field k of characteristic zero is this:

(i) When does there exist a decomposition of Φ as a sum of powers of linear forms $\phi(X,Y) = \sum_{i=1}^{N} (a_i X + b_i Y)^d$, $a_i, b_i \in k$? What is the minimal number $s(\phi)$ of summands needed? Parametrize all minimal decompositions!

(ii) When does there exist a weighted sum decomposition $\phi(X,Y) = \sum_{j=1}^{N} c_j (a_j X + b_j Y)^d$, $a_j, b_j, c_j \neq 0 \in k$? What is the minimal number of summands $s_w(\phi)$ needed? Parametrize all minimal sum decompositions.

The problem is classical (for $k = \mathbb{C}$) and only <u>generic</u> solutions are known (Sylvester, Gandelfinger; Kung, Rota, Lascoux). Let $H(\phi)$ denote the <u>Hankel</u> matrix associated with ϕ:

$$H(\phi) = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & & \cdot^{\cdot^{\cdot}} & \vdots \\ a_3 & & \cdot^{\cdot^{\cdot}} & & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & & \vdots \\ a_n & \cdots & \cdots & \cdots & a_{2n-1} \end{bmatrix} \qquad H(\phi) := \begin{bmatrix} a_1 & a_2 & \cdots & a_n & a_{n+1} \\ a_2 & & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots & \vdots \\ a_n & \cdots & \cdots & a_{2n-1} & a_{2n} \end{bmatrix}$$

for $d = 2n - 2$ even $\qquad$ for $d = 2n - 1$ odd.

<u>Thm A</u> Let k arbitrary of characteristic 0. Then
$\operatorname{rank} H(\phi) \le s_w(\phi) \le \deg \phi + 1.$

<u>Thm B</u> Let $k = \bar{k}$ algebraically closed.

1) $d = 2n - 2$. Then $s(\phi) = \operatorname{rank} H(\phi)$ or $s(\phi) = n + \operatorname{corank} H(\phi)$, where $s(\phi) = \operatorname{rank} H(\phi)$ holds generically. For $s(\phi) < n$ there is essentially only one decomposition, while for $n \le s(\phi) \le 2n - 1$ there exists a $2(s(\phi) - n)$-parameter family of solutions.

2) $d = 2n - 1$. $s(\phi) = \operatorname{rank} H(\phi)$ or $s(\phi) = n + 1 + \operatorname{corank} H(\phi)$ where the first case holds generically. For $s(\phi) \le n$ there is essentially only 1 decomposition while for $n < s(\phi) \le 2n$ there exists a $2(s(\phi) - n)$ parameter family of solutions.

<u>Thm C</u> k real closed, $d = 2n - 2$. Waring's problem (i) is solvable over $k \iff H(\phi) \ge 0$. $s(\phi) = \operatorname{rank} H(\phi)$. For $\operatorname{rank} H(\phi) < n$ there exists a unique solution, for $\operatorname{rank} H(\phi) = n$ there exist a 1-parameter family of decompositions.

The proofs use techniques from control theory: partial realization theory resp. Padé approximations.

R. HUBER:

<u>Two questions about open mappings</u>

In this talk we prove the following two theorems:

1) Let $f : X \to Y$ be a morphism of complex analytic spaces and let x be a point of X. Then f is open at x if and only if the mapping $\mathrm{Spec}\ \mathcal{O}_{X,x} \to \mathrm{Spec}\ \mathcal{O}_{Y,f(x)}$ is surjective.
2) Let $f : X \to Y$ be a morphism of schemes of finite type over a field k. Put $L := \{x \in X | f$ is universally open at $x\}$. Then L is constructible in X.

The main arguments in the proof of theorem 1 are: Hilbert's Nullstellensatz; Gabrielov's Theorem about the complement of subanalytic sets; curve selection lemma for subanalytic sets.

To prove theorem 2 one can use the real spectrum or the valuation spectrum.

From theorem 1 one can deduce the following corollary: Let $f : X \to Y$ be a morphism of complex analytic varieties, $x \in X$. Then f is universally open at x (in the Zariski topology) if and only if $f_{\mathbb{C}} : X(\mathbb{C}) \to Y(\mathbb{C})$ is open at x (in the strong topology).

J. HUISMAN:

The underlying real algebraic structure on complex elliptic curves

The underlying real algebraic structure $X_{\mathbb{R}}$ on a complex abelian variety X is a real algebraic group. It turns out that, for complex abelian varieties X and Y, $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ are birationally isomorphic if and only if they are isomorphic as real algebraic groups. Using this remarkable property, we are able to classify the underlying real algebraic structure on complex elliptic curves. Here is a short summary of the main results.

Let X and Y be complex elliptic curves. Suppose that X has complex multiplication, i. e. the ring of endomorphisms $\mathrm{End}\ X$ is not $\mathbb{Z}$. We prove that then $Y_{\mathbb{R}}$ is isomorphic to $X_{\mathbb{R}}$ if and only if $\mathrm{End}\ Y$ and $\mathrm{End}\ X$ are isomorphic. Let, for a real algebraic variety M, $p(M)$ be the number of non-isomorphic complex algebraic varieties Y such that $Y_{\mathbb{R}}$ is isomorphic to M. It follows that for an elliptic curve X with complex multiplication, $p(X_{\mathbb{R}})$ is equal to the class number of the ring End X. Furthermore, we prove that whenever X is without complex multiplication, $p(X_{\mathbb{R}}) = 1,2$ or 4. We conclude that $p(M)$ is finite for every real algebraic torus M.

F. ISCHEBECK:

Projective modules and SK_1 over real regular rings

For $X \subset \mathbb{R}^n$ let $R(X) = S_X^{-1}\,\mathbb{R}[X_1,\ldots,X_n]/I_X$ be the ring of real regula[r] functions and $C(X)$ be that of the continuous ones. Define $P(A) =$ $= \{$iso-classes of f. g. projective A-modules$\}$, $SK_1A = SL/E(A)$. According to Swan one has for compact X:

$$P(R(X)) \hookrightarrow P(C(X)) \text{ injective and}$$
$$SK_1(R(X)) \xrightarrow{\sim} SK_1(C(X)) \text{ bijective.}$$

For non compact X there are easy counterexamples. On the other hand one has $P(R(\mathbb{R}^n)) = \mathbb{N}$ for $n \leq 4$, $SK_1R(\mathbb{R}^n) = 0$ for $n \leq 2$. For a convex set X in $\mathbb{R}^3$ (resp. $\mathbb{R}^2$) one has $P(R(X)) = \mathbb{N}$ (resp. $SK_1R(X) = 0$). The same holds, if $X = C \setminus M$, where $C \subset \mathbb{R}^3$ (resp. $\subset \mathbb{R}^2$) is compact contractible with C^2-boundary and $M \subset \partial C$ any subset. The proof in the "P-case" requires Gabbers theorem on projective modules over a localized local ring in a slightly generalized form.

P. JAWORSKI:

The upper bound for number of squares necessary to represent a positively defined germ of an analytic function

Let $f : (R^n,0) \to R$ be a germ of an analytic function. It is well-known that if f is positively defined (i. e. $f(x) \geq 0$) then f is a sum of squares of meromorphic functions. But, in general case, the number of squares P_f necessary to represent f is unknown. There are known only the following estimates:

$$n = 1 \qquad P_f = 1;$$
$$n = 2 \qquad P_f \leq 2.$$

Our recent result is the following: Let $f_{\mathbb{C}} : (\mathbb{C}^n,0) \to \mathbb{C}$ be the complexification of f; $f_{\mathbb{C}}|_{R^n} = f$.

Theorem. If f is positively defined and $\dim_{\mathbb{C}} \mathrm{Sing}\, f_{\mathbb{C}} \leq 1$ then $P_f \leq 2^n$.

From this we obtain:

Corollary. For $n = 3$ 8 squares is enough ($P_f \leq 8$).

V. KHARLAMOV:

On the topology of intersections of real quadrics

Not only intrinsic topology but also algebraic, called rigid, isotopies were considered. Special attention was given to intersections of 3 quadrics. The real variant of classical theorem asserts that strong regular intersections of 3 real quadrics are in one-to-one correspondence with real non-singular plane curves supplied by a real even nondegenerated θ-characteristic. According to Agrachev' results to reread homological type of the intersection one needs to calculate some data: I call them index code and monodromy code. The interpretation of the index code as a spin-orientation and the monodromy code as a spin-weight was given. Relation with the Rokhlin spin-structure was discussed.

The results about intersections were used in apposite direction. They were applied to prove that there is no real plane M-curve of odd degree such that all ovals except one lie outside another and the exceptional one contains all other.

Other results are about classification of low dimensional intersections up to rigid isotopies. The full answer is obtained for intersections in projective spaces of dimension < 4. Some partial results are proved for 3 quadrics in 4- and 5-dimensional spaces. Non-amphiceiral intersections of 3 quadrics in the 5-dimensional space are found. In some cases their amphiceirality is connected with amphiceirality of some associated configurations of projective subspaces.

W. KUCHARZ:

Algebraic cycles of codimension 2

Given an affine nonsingular compact real algebraic variety X, denote by $H^k_{alg}(X,\mathbb{Z}/2)$ the subgroup of $H^k(X,\mathbb{Z}/2)$ generated by the cohomology classes of the algebraic subvarieties of X of codimension k.

Theorem. Let M be a compact connected orientable C^∞ submanifold of $\mathbb{R}^n$ of dimension m with $m \geq 5$ and $2m + 1 \leq n$. Let G be a subgroup of $H^2(M,\mathbb{Z}/2)$. Then the following conditions are equivalent:

a) There exists a C^∞ embedding $e : M \to \mathbb{R}^n$, arbitrarily close in the C^∞ topology to the inclusion mapping $M \hookrightarrow \mathbb{R}^n$, such that $X = e(M)$ is a nonsingular algebraic subset of $\mathbb{R}^n$ and $\varphi^*(G) = H^2_{alg}(X,\mathbb{Z}/2)$, where $\varphi : X \to M$ is the diffeomorphism defined by $\varphi(e(m)) = m$ for m in M;

b) There exists an affine nonsingular real algebraic variety Y and a homeomorphism $h : Y \to M$ such that $h^*(G) = H^2_{alg}(Y,\mathbb{Z}/2)$;

c) $w_2(M) \in G$ and G is contained in the subgroup

$$W^2(M) = \{w_2(\xi) \in H^2(M,\mathbb{Z}/2) \mid \xi \text{ is a vector bundle on } M\}$$

of $H^2(M,\mathbb{Z}/2)$ ($w_2(\cdot)$ stands for the second Stiefel-Whitney class, $w_2(M) = w_2(TM)$).

This result is a part of a joint work with Jacek Bochnak.

S. ŁOJASIEWICZ:

On the closure of partially semi-algebraic sets

R. Thom has posed the following question: given an analytic vector field on an analytic manifold M, is its limit direction set in $\mathbb{P}(T_aM)$ a semi-algebraic set? This follows from the following theorem: The closure of an N-semi-algebraic subset of $M \times N$ (were M is an analytic manifold and $N = \mathbb{R}^n$; it means the set is described, locally over M, by polynomials with analytic coefficients) is always N-semi-algebraic. An idea of a proof has been given.

A. PARUSINSKI:

Regular projections theorem and applications

A Regular Projections Theorem was introduced by Mostowski in his proof of the existence of Lipschitz stratifications for complex analytic varieties. We present a generalization of this theorem which works in the case of projections of any codimension and estimate the number of needed projections.

Let $X \subseteq R^n$ be semi-algebraic or compact subanalytic. Fix $d \in \{1,\dots,n-1\}$.

Definition: We call an orthogonal projection $\pi = \pi_V : R^n \to V^\perp$, where $V \in G(n,d)$, ε-regular at $x \in R^n$, for $\varepsilon > 0$, if:

(a) $\pi|_X$ is finite,

(b) $S = X \cap \{x + v;\ v \in V' - \{0\},\ d(V',V) < \varepsilon\}$ is a nonsingular analytic set of pure dimension $n - d$,

(c) for each $x \in S$ and $v \in T_xX$ $\|\pi_{V'}(v)\| \geq \varepsilon\|v\|$.

Theorem: If $\dim X \leq n - d$, then for generic $V_1,V_2,\dots,V_{n+1}$, there exists $\varepsilon > 0$ such that for each $x \in R^n$ we can find $V \in \{V_1,V_2,\dots,V_{n+1}\}$ such that π_V is ε-regular at x.

From this theorem follow some results concerning the function of distance to the sets in question, for example:

Corollary 1: If $\dim X < n - d$, then there exists $C > 0$ such that for each $x \in R^n$

$$\mathrm{dist}(x,X) \leq C \max_i \{\mathrm{dist}(\pi_{V_i}(x),\pi_{V_i}(X))\},$$

where V_i, $i = 1,\dots,n+1$, are given by the Theorem.

Corollary 2: If $\dim X = n - d$, then there exists $C > 0$ and $Y \subseteq R^n$, such that $\dim Y < n - d$ and

$$C\ \mathrm{dist}(x,X) \geq \min(\min_i\{\mathrm{dist}_{V_i}(x,X)\},\ \mathrm{dist}(x,Y)),$$

where V_i are given by the Theorem and dist_{V_i} denotes the distance in the fibres of π_{V_i}.

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W. PAWLUCKI:

On Gabrielov's regularity condition for analytic mappings

This talk concerns geometry of subanalytic sets, although the results are formulated in terms of analytic relations among analytic functions.

Let X and Y denote (real or complex) analytic connected manifolds. Let $\mathcal{O}_X$ and $\mathcal{O}_Y$ denote the sheaves of germs of analytic functions on X and Y, respectively. Suppose that $\varphi : X \to Y$ is an analytic mapping.

Let $x \in X$. Then φ induces a homomorphism of local algebras $\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$, and thereby a homomorphism of sheaves of local algebras $\varphi^* : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, where $\varphi^{-1}\mathcal{O}_Y$ denotes the inverse image of $\mathcal{O}_Y$ by φ. Let k denote the generic rank of φ. (Then k is the topological dimension of $\varphi(X)$.)

We say that φ is regular at x (in the sense of Gabrielov), if there exists a neighborhood U of x and a locally analytic subset S of Y such that dim S = k and $\varphi(U) \subset S$.

Main results of this talk are the following:

Theorem 1: Put $nR(\varphi) = \{x \in X : \varphi$ is not regular at $x\}$. Then $nR(\varphi)$ is an analytic subset of X.

Theorem 2: (a 'semi-coherence' theorem) Let $a \in X$. There is a neighborhood W of a in X and a filtration of W by closed analytic subsets;

$$W = X_0 \supset X_1 \supset \dots \supset X_{s+1} = nR(\varphi) \cap W \qquad (s \in \mathbb{N}),$$

such that, for each $j = 0,\dots,s$, the sheaf of ideals Ker φ^* is finitely generated over $X_j \setminus X_{j+1}$; i. e., there exists a finite set $\{G^{jk}\}$ $(k = 1,\dots,m_j)$ of sections of Ker φ^* over $X_j \setminus X_{j+1}$ such that for each $x \in X_j \setminus X_{j+1}$, $\{G_x^{jk}\}$ $(k = 1,\dots,m_j)$ generate the ideal Ker φ_x^*.

It is not in general possible to extend the filtration $\{X_j\}$ beyond $nR(\varphi)$ which may be shown by an example.

A. PRESTEL:

Continuous representations of real polynomials as sums of 2^m-th powers

Let $f(C,X) = C_d X^d + \ldots + C_0 \in \mathbb{Z}[C,X]$ be the general polynomial of degree d and let $2^n | d$. Denote by B the ring of continuous functions $\alpha : \mathbb{R}^{d+1} \to \mathbb{R}$ s.t. $\alpha = \max_i \min_j q_{ij}$ with $q_{ij} \in \mathbb{Z}[C]$ and $\{q_{ij}\}$ finite. We have proved the following

Theorem: To all $N,M \in \mathbb{N}$ there exist $k,\sigma',\sigma'' \in \mathbb{N}$ and $\alpha_i,\beta_j \in B$ and $g_i,h_j \in B[X]$ for $1 \le i \le \sigma'$, $1 \le j \le \sigma''$ s.t.

$$f \cdot \sum_{i=1}^{\sigma'} \alpha_i g_i^{2m} = f^{2m\cdot k} + \sum_{j=1}^{\sigma''} \beta_j h_j^{2m}$$

and $\alpha_i(a),\beta_j(a) \ge 0$ whenever $f(a,X)$ satisfies $(a \in \mathbb{R}^{d+1})$:

i) $|a_i| \le N$ $(0 \le i \le d)$ and $\frac{1}{N} \le a_d$;

ii) $\frac{1}{M} \le$ |imaginary part of non-real zeros of f| and

2^m |multiplicity of real zeros of f.

Thus the polynomials $f(a,X)$ satisfying i) and ii) have a representation as sums of 2^m-th powers of rational real functions in X whose coefficients continuously depend on a. It is impossible to have such a continuous representation for all $f(a,X)$ which admit some representation as a sum of 2^m-th powers of rational functions in X. This follows from the authors result that $X^4 + nX^2 + 1 = \sum_{i=1}^{\sigma(n)} \frac{g_i^{(n)}(X)^4}{h^{(n)}(X)^4}$ and $\sigma(n) \to \infty$ or $\deg h^{(n)} \to \infty$ as $n \to \infty$ for every such representation. [Soc. Math. France, 2° série, mémoire 16, 1984].

A. PRESTEL:

Report on the work of Denef and van den Dries on subanalytic sets

We reported on the result of Denef and van den Dries [Am. Math. 128 (1988), 79-138] that the L_{an}^D-theory of the Intervall $[-1,1] = I$ admits elimination of quantifiers. The language L_{an}^D contains symbols for the inequality < on I, for each power series $f(X_1,\ldots,X_N)$ $(N \in \mathbb{N})$

convergent in a neighborhood of I^N s.t. $f(I^N) \subset I$ and for the function $D : I^2 \to I$ defined by $D(x,y) = \frac{x}{y}$ if $|x| \leq |y|$ and $y \neq 0$ and $D(x,y) = 0$ if $|x| > |y|$ or $y = 0$. As a consequence of this result one obtains Gabrielov's Theorem that the complement of a subanalytic set is again subanalytic.

J. RUIZ:

Real stability indices of rings of analytic functions

We sketch the proof of the following

Theorem (Andradas-Bröcker-Ruiz). Let M be a compact real analytic manifold of dimension m. Then every set $S = \{x \in M | f_1(x) > 0,\dots,f_s(x) > 0\}$, where $f_i : M \to \mathbb{R}$ are analytic functions, can be described with $s \leq m$ inequalities.

That proof goes by induction on m and requires:

a) The main results concerning stability indices of fields, due to Bröcker and involving Fans and Valuation Theory.
b) The abstract geometric setting provided by Coste-Roy's notion of Real Spectrum: Positivstellensatz and Łojasiewicz' inequality. This leads to the so-called Global Stability Formula (Scheiderer-Bröcker), which allows computations through residue fields of a ring instead of the ring itself).
c) The actual computation using 1), 2) and Rotthaus' Approximation Theorem, of the stability index of the quotient field of an excellent henselian local domain.

K. RUSEK:

Polynomial automorphisms of $\mathbb{C}^n$ and $\mathbb{R}^n$

The objects of principal interest of this talk are polynomial transformations of $\mathbb{k}^n$ ($\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$) which are bijective and polynomially invertible. We called them polynomial automorphisms of $\mathbb{k}^n$.

Our aim is to present some information concerning the question how to recognize if a given polynomial transformation of $\mathbb{k}^n$ is a polynomial automorphism, namely:

- the statements of famous Keller's Jacobian Problem in complex two-dimensional case,
- some examples of partial solutions of the Jacobian Problem in complex two-dimensional case (Moh, J. Reine Angew. Math. 340 (1983); Heitman, J. Pure Appl. Alg. 64 (1990),
- the Drizkowski reduction of the Jacobian Problem to the "cubic-linear" case (Math. Ann. 264 (1983)),
- some partial solutions of the Jacobian Problem in the cubic-linear case (Drizkowski, Math. Ann. 264; Drizkowski, Rusek, Ann. Polon. Math. 46 (1985); Drizkowski (1989), unpublished).

One can also ask how to recognize polynomial automorphisms of $\mathbb{R}^n$ among polynomial bijections of $\mathbb{R}^n$. This seems to be possible without positive solution of the Jacobian Problem by proving:

<u>Conjecture 1:</u> A bijective polynomial mapping $F=(f_1,\dots,f_n):\mathbb{R}^n\to\mathbb{R}^n$ is a polynomial automorphism iff $\det(\frac{\partial f_i}{\partial x_j})$ is a non-zero constant.

The proof of this conjecture given in the article by Bass, Cornell, Wright (Bull. AMS, Vol. 7, No. 2 (1982)) seems to be uncomplete.

To prove Conjecture 1 it would suffice to prove

<u>Conjecture 2:</u> Let $f:\mathbb{R}^n\to\mathbb{R}$ be an analytic function with algebraic graph and suppose that there exists a non-degenerate polynomial mapping $F:\mathbb{R}^n\to\mathbb{R}^n$ such that for every $\alpha\in\mathbb{N}^n$, $(D^\alpha f)\circ F$ is a polynomial. Then f is a polynomial (or, at least, an element integral over the ring $\mathbb{R}[X_1,\dots,X_n]$).

C. SCHEIDERER:

<u>Real valuations, cohomology and homotopy</u>

Let A be a (comm.) ring. We associate to A a simplicial topological (boolean) space VA. The elements of the n^{th} component V_nA are equivalence classes of homomorphisms $\varphi : A \to K$ into real closed fields K with additional data consisting of convex subrings $C_n \subset \ldots \subset C_1 \subset K$ such that $\varphi(A) \subset C_n$. The construction globalizes to schemes and, more generally, to real closed spaces, giving a functor $X \mapsto VX$ to (locally boolean) simplicial spaces. Let $\varepsilon : V_0X \to X$ be the identity, and let γ_n (resp. σ_n) $: V_nX \to X$ be the composition of ε with the map $V_nX \to V_0X$ corresponding to $0 \mapsto 0$ (resp. $0 \mapsto n$). There are natural functors $a^* : Ab(X) \to Ab(VX)$ (exact) and $a_* : Ab(VX) \to Ab(X)$ (left exact), together with a natural morphism $id \to a_*a^*$. The first main result states that one has cohomological descent, i.e. that $id \to Ra_*a^*$ is an isomorphism (of functors $D^+(X) \to D^+(X)$). Next consider the category of continuous simplicial maps $VX \to E$ into discrete simplicial sets E, and the pro-simplicial set $\{E\}$ indexed by this category. Regarding it as a pro-object in the homotopy category $\underline{H}$ of simplicial sets, one obtains a functor $X \mapsto \Pi X$ from real closed spaces to pro-$\underline{H}$. The second main result states that this functor is homotopy invariant, i.e. that $\Pi f = \Pi g : \Pi X \to \Pi Y$ for homotopic maps $f,g : X \rightrightarrows Y$.

N. SCHWARTZ:

<u>Invertible ideals in real closed rings</u>

The investigation of semi-algebraic line bundles over the real spectrum of a ring can be reduced in many geometrically important cases to studying invertible ideals in real closed rings. If M is a nonsingular connected s.a. (= semi-algebraic) space of dimension n over a real closed field and A is its ring of semi-algebraic functions then geometrically the equivalence classes of invertible ideals in A correspond to $(n - 1)$th homology classes in the Borel-Moore homology of M: Every invertible ideal is assigned a "sign change divisor", i.e., the locus of points at which a local generator of the ideal changes its sign. This sign change divisor is defined on the Picard group of A. It can be expressed as the composition of the first Stiefel-Whitney class $w_1 : Pic(A) \to H^1(M,\mathbb{Z}/2)$ and the Poincaré-duality isomorphism $H^1(M,\mathbb{Z}/2) \to H_{n-1}(M,\mathbb{Z}/2)$. Using this the minimal number of generators of invertible ideals can be computed in some cases: The invertible ideal $K \subset \Gamma(\mathbb{P}^n(R_0))$ (R_0: real algebraic numbers) corresponding to the canonical line bundle is generated by not less than n + 1 elements. For every M as above every invertible ideal can be generated by n + 1 elements. So, K is generated by the maximal number of elements possible. If $I \subset A$ is invertible then I is generated by n + 1 elements if and only if $w_1(I)^n \neq 0$.

J. STASICA:

Subanalytic stratification and triangulation with equisingularity conditions

There exists general method (due to Łojasiewicz) of construction subanalytic stratification with additional condition on strata in case of generic conditions.

Let (γ) be a condition for strata (Λ,Γ) $(\Lambda \subset \partial\Gamma)$ in points P of Λ s. th.

1) (γ) depends only on the germs Λ_p, Γ_p,
2) $\exists \Lambda_0 \subset \Lambda$ open, dense in Λ subanalytic: $\forall p \in \Lambda_0$ (Λ,Γ) fulfils (γ) in p,

then for each locally finite family $\{E_v\}$ of subanalytic subsets of M (an analytic manifold) there exist subanalytic (γ)-stratification of M compatible with $\{E_v\}$.

Open question: Does there exist subanalytic triangulation with Withney condition?

G. STENGLE:

Recent results on V-C classes

This is a serious application of real geometry to the theory of probability, specifically to the theory of empirical processes. The basic problem is to infer a probability law from repeated independent observations. For a single event A (call it "success") the Bernulli law of large numbers ensures that $P_n(A) = \frac{\#\text{successes}}{n} \sim P(A)$ if n, the

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number of observations, is large. The same holds trivially for any finite class of sets but fails in general for infinite classes of sets. However in 1971 Vapnik and Chervonenkis identified certain infinite classes C which satisfy a uniform law of large numbers in the precise sense that $\sup_P \sup_{C\in C} P\{|P(C) - P_n(C)| > \varepsilon\} \to 0$ as $n \to \infty$, where the outer sup is over all probability laws P. These classes have the following characterization.

Definition. Given a set X, a collection of subsets $C \subset 2^X$ shatters a subset $Y \subset X$ if every subset of Y can be realized as $Y \cap C$ for some $C \in C$. C is a V-C class if there is a natural number n such that C shatters no finite subset of size n and the least such n is the V-C dimension of C.

Examples of V-C classes are

1) intervals in $\mathbb{R}$ (obviously intervals shatter no 3-point set),
2) all discs in $\mathbb{R}^2$ (less obvious),
3) positivity sets in $\mathbb{R}^n$ of any finite dimensional vector space of real valued functions (Dudley 1984).

In 1989 Stengle and Yukich obtained a major enlargement of the known body of V-C classes using elimination of quantifiers to reduce the V-C character of semialgebraic families of semialgebraic positivity sets to Dudley's result (example 3 above). They also showed, using methods of Denef and van den Dries, that certain one-parameter families of semianalytic sets are VC. In 1990 Christopher Laskowski has greatly improved these results using methods and ideas due to Shelah to explain the V-C property in terms of the failure of the independence property in model theory. Combining this fundamental insight with results of van den Dries on order-minimal structures, he obtains that a wide class of families of sets defined by first order statements involving semianalytic functions and semianalytic sets are V-C classes.

These recent results differ from older, more constructive, results in their purely existential character and give no estimates for the VC dimension. This number is of some consequence in estimating finer properties of the resulting law of large numbers. It seems an interesting open question to assess this dimension, especially to find qualitative complexity estimate.

R. SUJATHA:

Witt groups of real projective surfaces

It is known (Ayout, Fernández-Carmena) that the Witt group of a smooth projective surface, is finitely generated over the fields $\mathbb{R}, \mathbb{C}$. We compute the structure of the Witt group of a smooth projective real surface in terms of birational invariants associated to X. In the explicit case when X is rational (i.e. $X \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$ is birational to $\mathbb{P}^2_{\mathbb{C}}$) we prove that $W(X) \simeq \mathbb{Z}^s \oplus (\mathbb{Z}/2)^{s+1}$, where s is the cardinality of the connected components of $X(\mathbb{R})$, the set of $\mathbb{R}$-rational points with the Euclidean topology, (if $s \geq 1$). If $X(\mathbb{R}) = \emptyset$, i.e. $s = 0$, we prove that $W(X) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/4$. The proof in this case uses the fact that the level of non-formally real function fields of real rational surfaces is 2. This is proved jointly with Parimala.

Ref: Witt groups of smooth projective real surfaces - To appear in Math. Ann.

Levels of formally real function fields of real rational surfaces (with R. Parimala), To appear in Am. Journal of Math.

Z. SZAFRANIEC:

On the number of singular points of real projective hypersurfaces

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial. Let $X = \{x \in \mathbb{RP}^{n-1} \mid f(x) = 0\}$, and let X_{sing} be the set of singular points in X.

In this talk we show how to verify numerically whether the number $\# X_{sing}$ is finite and how to calculate this number in terms of topological degrees of a finite family of maps $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ defined explicitely by f.

A. TOGNOLI:

On the extension problem for Nash functions

In the theory of Nash functions two main problems are open:

1) when a Nash function defined on a closed Nash subvariety can be extended to a Nash function defined on the whole space,
2) when a Nash subvariety has global equations.

We give an example of a 3-dimensional Nash variety X such that for any Nash embedding $X \xrightarrow{i} \mathbb{R}^n$, i(X) has not global equation in any neighborhood $U \supset i(X)$. We prove after some complex version of Efroymson' extension theorem for Nash function and from this we deduce some criterion (in the normal complex case and in the normal coherent real case) to ensure that X has global equations.

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