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Algebraic and Combinatorial Problems in Multivariate Approximation Theory

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The conference was organized by W. Dahmen (Berlin) and A. Dress (Bielefeld). Its purpose was to discuss recent developments in multivariate approximation theory which appeared to have close connections with commutative algebra and combinatorics. Consequently, about two thirds of the participants were researchers who have worked mainly in the areas of approximation theory and numerical analysis, while one third were algebraists and combinatorialists.

As expected, the interaction turned out to be extremely fruitful and stimulating for both sides. Many problems were clarified, if not solved, by viewing them simultaneously from the different aspects provided by the different backgrounds of the participants.

In particular, it was shown that techniques from homological algebra facilitate the computation of the dimension of various spline spaces and related tasks. Other techniques, relevant for spline theory, draw on combinatorial geometry, in particular matroid theory and rigidity theory.

On the other side, the algebraists were not only happy to see the power of their methods confirmed once again, they were also stimulated to identify or construct further conceptual frameworks to encompass the richness of ideas, presented by spline theory, and they were amazed to see that spline theory can actually be applied to solve interesting problems in linear diophantine analysis.

The generously timed schedule of formal lectures was complemented by several informal presentations which evolved from the discussions.

It was the general consensus that it would be desirable to meet again in 2 or 3 years to share the further insights which are certain to result from this remarkably stimulating encounter of two separate fields.

Abstracts

Some algebraic and combinatorial problems related to multivariate splines

(joint work with A. Dress and C.A. Micchelli)

Wolfgang Dahmen, Berlin, FRG

Several recent developments in the theory of multivariate splines are reviewed which lead to the problem of determining the dimension of the common null space of families of commuting linear operators, typically difference or differential operators. In particular, studying local linear independence of translates of box splines or exponential splines and counting the number of solutions to linear systems of diophantine equations with the aid of discrete splines leads to instances of this problem which are covered by a general setting where the family of operators is determined by a matroid structure. The possibility of extending these results using tools from homological algebra is briefly indicated.

Continuous splines and arrangements of subspaces

Sergey Yuzvinsky, Eugene, OR 97403, USA

In this talk we deal with the algebra $C = C^0(\Delta)$ of continuous splines on a polyhedral complex Δ purely embedded in an affine space $\mathbf{R}^d$. This algebra is also viewed as a module over the coordinate ring of $\mathbf{R}^d$. The talk consists of two parts. In the first part the complex Δ is assumed to be simplicial. Here we generalize a result of L. Rose who gave a characterization of the freeness of C in terms of topology of Δ. We give a non-surprising characterization of the projective dimension of C in terms of topology of Δ.

In the second part we study the freeness of C for the case of general Δ. For that we construct an arrangement of affine subspaces which gives C back as the seminormalization of the coordinate ring of the union of these subspaces. This corresponds to a generic embedding of Δ into a larger affine space. In particular, we obtain a characterization of the freeness of C in terms of sheaf cohomology of the face lattice of Δ.

On the existence of local bases for trivariate spline spaces

Larry Schumaker, Nashville, TN 37240, USA

We consider the space $S_d^r(\Delta)$ of splines of degree d and smoothness r on a tetrahedral partition in $\mathbf{R}^3$. Using Bernstein-Bézier methods, we show how to construct a basis for this space consisting of locally supported cardinal splines. A key ingredient is a detailed analysis of certain partitions formed by a set of tetrahedra surrounding a single edge (which we call an orange). For such Δ we generalize my earlier work on cells to find exact dimension formulae and explicit determining sets of Bézier points. The approach yields a formula for the dimension of $S_d^r(\Delta)$ on arbitrary Δ, except for terms involving the dimension of spline spaces on the 30-cell (or star of a vertex).

Multisets, Matroids and Λ-trees

Robert Simon, Bielefeld, FRG

We consider monomial ideals of $K[X]$, K a field, X a finite set. We define a Λ-tree to a diagram of monomial ideals created by the operation of division by an $x \in X$ and addition by (x). We determine necessary and sufficient conditions on the generating multiset system of an ideal I so that the only primes occuring in a Λ-tree of I are the minimal primes containing I. These conditions are closely related to the axioms for matroids. Also, composition series for R/I are considered, also in which the only primes that occur are the minimal primes containing I.

A fundamental problem of multivariate splines

Peter Alfeld, Salt Lake City, UT 84112, USA

I discussed the following problem: Let T be a triangulation of $\Omega \subset \mathbf{R}^k$. Let

$$S_d^r = \{S \in C^r(\Omega) : S|_\Delta \in P_d^k \qquad \forall \Delta \in T\}$$

where P_d^k is the space of all k-variate polynomials of degree up to d. A basis of S_d^r is "minimally supported" if the support of each basis function is contained in the star of a vertex in T. For which values of d do minimally supported bases exist? Can the approach used by Morgan and Scott for $k = 2$ be generalized?

The geometry and combinatorics of bivariate and trivariate splines

Walter Whiteley, St. Lambert, Que J4P 3P2, Canada

I describe recent work along two parallel tracks.

(1) I translate the space of C^r_d-splines over a cell complex Δ into the solutions of a matrix equation $\beta M^r_d(\Delta) = 0$. Combinatorial information about this space is obtained by analysing the matrix and its companion row matroid. The deep analogies with the matrix/matroid for rigidity of frameworks in various dimensions permit the transfer of a number of combinatorial techniques and results.

(2) The projective geometry of "singular" (non-generic) realizations of an abstract cell complex (again analogous to rigidity results) is studied. In particular, the central projection of a vertex star gives explicit information on the splines of the star. The geometry of "non embedded realizations" of the abstract structure give insight into the behaviour of higher dimensional splines, such as the connection between "singular vertices" in the plane and special position "univariate structures".

Solvability of systems of linear operator equations

Rong Qing Jia, Sherman Riemenschneider and
Zuowei Shen, Edmonton T6G 2G1, Canada
(presented by Zuowei Shen)

Let G be a semigroup of commuting linear operators on a linear space S with the group operation of composition. The solvability of the system of equations $lif = \phi i$ $i = 1, \ldots, r$, where $li \in G$ and $\phi i \in S$, was considered by Dahmen and Micchelli in their studies of dimension of the kernel space of certain linear operators. The compatibility conditions $lj\phi i = li\phi j$, $i \neq j$ are necessary for the system to have a solution in S. However, in general, they do not provide sufficient conditions. We discuss what kind of easily verifiable conditions on operators will make the compatibility conditions sufficient for such systems to be solvable in S. We apply this result to the system of differential and difference equations and give the corresponding results.

Perturbation of polynomial ideals and its applications to multivariate approximation theory

Rong Qing Jia, Eugene, OR 97403, USA

Following the lead of de Boor and Ron, we consider perturbation of polynomial ideals. It is demonstrated that the technique of perturbation of polynomial ideals can be applied to multivariate polynomial interpolation and box spline theory. For a certain kind of ideals generated by a family of polynomials, we establish a lower bound on their codimension using the perturbation technique together with topological degree theory. This result extends a previous result of de Boor and Ron for the special case where each polynomial generator is a product of polynomials of degree 1.

The least solution of the multivariate interpolation problem

Amos Ron, Madison, WI 53706, USA

This work is joint with Carl de Boor

We consider the following problem. Let $\Pi := \Pi(\mathbf{R}^s)$ be the space of all polynomials in s-variables, Π' its algebraic dual, equipped with the weak-* topology, Λ a subspace of Π', endowed with the induced topology. Find a subspace $P \subset \Pi$, such that $P = \Lambda^*$ in the sense that for every $F \in \Lambda^*$ there exists $p \in P$ such that

$$\lambda p = F\lambda \qquad \forall \lambda \in \Lambda.$$

To solve the problem we represent each $\lambda \in \Pi'$ as the formal power series $\widehat{\lambda} = \sum_{\alpha \in \mathbf{Z}_+^s} \frac{\lambda m_\alpha}{\alpha!} X^\alpha$, where m_α is the monomial $m_\alpha : x \to \chi^\alpha$. Then we write each power series $\widehat{\lambda}$ in a graded form, i.e., $\widehat{\lambda} = \lambda_0 + \lambda_1 + \lambda_2 + \ldots$ where, for each j, λ_j is a homogeneous polynomial of degree j. We then select the initial form $\lambda\downarrow$ of λ defined as:

$$\lambda\downarrow := \{\lambda_j : \lambda_j \neq 0, \lambda_i = 0, \qquad \forall i < j\},$$

and define the internal homogenezation of Λ as:

$$\Lambda\downarrow := span\,\{\lambda\downarrow : \lambda \in \Lambda\}.$$

Theorem: $\Lambda\downarrow$ solves the above interpolation problem, and is of smallest degree among all polynomial solutions to that problem; namely, if P is another solution for the problem, then, for $j = 0, 1, \ldots,$

$$dim\,(\Lambda\downarrow \cap \Pi_j) \geq dim\,(P \cap \Pi_j),$$

where Π_j is the space of all polynomials of degree $\leq j$.

$D(X)$

Carl de Boor, Madison, WI 53706, USA

This work is joint with Amos Ron

The basic facts about $D(X) := \bigcap_{Z \in Y(X)} ker\, D_Z$ are surveyed. Here: $X \in \mathbf{R}^{s \times n}$ easily ran $X = \mathbf{R}^s$ and

$$0 \notin \{\xi \in X\}, \mathsf{Y} := \{Z \subseteq X : Z \cap B \neq \emptyset \qquad \forall B \in \mathsf{B}(X)\},$$

and $\mathsf{B}(X) := \{Z \subseteq X : Z \text{ is a basis for } \mathbf{R}^s\}$. In particular, Dahmen & Micchelli's surprising result that $dim\, D(X) = \#\mathsf{B}(X)$ is reproved by exhibiting an easily constructible basis for $D(X)$.

The symbolic computation of spline interpolants

Mike Stillman, Ithaca, NY 14853-7901, USA

The space $S^r(\Delta)$ of piecewise polynomial functions smooth of order r, over a polyhedral complex $\Delta \subset \mathbf{R}^d$ is an $R = \mathbf{R}[x_1, \ldots, x_d]$-module. We first recall how this module can be computed using Gröbner bases and syzygies. We introduce an operation, called "modulo", which produces a refinement of the syzygy module of the columns of a matrix, and which finds $S^r(\Delta)$ in a more efficient manner. This operation has many applications in commutative algebra and algebraic geometry. In particular, almost every "finite" construction in the homological algebra of modules can be computed using "modulo", together with a small number of other operations. These operations have been implemented in our computer algebra system Macaulay (developed jointly with D. Bayer). After indicating some applications, including intersections, $Hom_R(M, N)$ and $Ext^i_R(M, N)$, we describe a method for finding an interpolating basis of the subspace $S^r_k(\Delta)$ of splines of degree $\leq k$, if such a basis exists. If such a basis does not exist, one can find symbolically those vertices $v \in \Delta$ such that there does not exist $F \in S^r_k(\Delta)$ which vanishes at every vertex of Δ except v.

Algebraic properties of multivariate splines

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For a finite polyhedral subdivision Δ of a region in $\mathbf{R}^d$, we consider $C^r(\Delta)$, the space of piecewise polynomial functions (splines) on Δ which are smooth of order r. $C^r(\Delta)$ is an $\mathbf{R}$-algebra and a module over $R = \mathbf{R}[x_1, \ldots, x_d]$. Our aim is to determine the algebraic structure of $C^r(\Delta)$, especially those structures which are determined by the combinatorial structure of Δ. Let $C_k^r(\Delta)$ denote the $\mathbf{R}$-subspace of $C^r(\Delta)$ consisting of splines of degree at most k. We can obtain information about dimensions and $\mathbf{R}$-bases of the $C_k^r(\Delta)$'s by studying $C^r(\Delta)$.

It turns out that the series $\Sigma dim_{\mathbf{R}} C_k^r(\Delta) t^k$ is the Hilbert series of a module related to $C^r(\Delta)$ and thus has the form of a rational function. We obtain further conditions on the numerator of this function. When $C^r(\Delta)$ is a **free** R-module, we can find a basis which immediately corresponds to vector space bases for the $C_k^r(\Delta)$'s, for all k. We then discuss necessary and sufficient conditions for $C^r(\Delta)$ to be a free module. A complete characterization is known for certain d, Δ, and r, but not in general. Finally, we note that $C^r(\Delta)$ can be viewed as the kernel of a map between free R-modules. With this representation, we can use the technique of Gröbner bases to find bases, generating sets and Hilbert series.

Surfaces in *CAGD* and multivariate splines

Xi-Quan Shi, Berlin, FRG

A set Δ of vertices, segments and triangles is called a triangular mesh if it satisfies the following conditions:

i). If a triangle belongs to Δ, then its vertices and edges also belong to Δ.

ii). If $S_1, S_2 \in \Delta$, then $S_1 \cap S_2 \in \Delta$.

iii). Any one of the vertices and segments in Δ belongs to a triangle in Δ.

iv). The number of triangles with a common segment as an edge is not more than two in Δ.

v). There is not a vertex in the inside of a segment in Δ.

For a triangular mesh Δ, the authors have constructed a GC^1 surface which passes the given points, and its restriction to a triangle in Δ is a quintic polynomial, and the tangent planes of the given points can be taken arbitrarily, respectively.

Splines, matroids and the *Ext*-Functor

Andreas W.M. Dress, Bielefeld, FRG

It is shown that the computation of the dimension of the space of functions/ polynomials which are annihilated by certain products of differential operators can be facilitated by the use of homological algebra. A specific and particularly interesting example, studied in the theory of box splines, arises when these products correspond to the cocircuits of a matroid defined on a set X in which case an analysis of the composition factors of a certain module over the polynomial ring $\mathbf{R}[X]$, associated with the matroid, reduces the computation to the very simple case where this matroid consists of one basis, only, provided that certain *Ext*-groups vanish which in turn follows through partial integration.

Louis J. Billera, Ithaca, NY 14853-7901, USA

Bernstein-Bezier Subalgebras of Face Algebras

For a d-dimensional simplicial complex $\Delta \subset \mathbf{R}^d$, it is known that the ring $C^0(\Delta)$ of continuous piecewise polynomial functions on Δ is isomorphic to $A_\Delta / < x_1 + \ldots + x_n - 1 >$, where A_Δ is the face ring of Δ, via the isomorphism $x_i \mapsto X_i$. Here X_i is the unique piecewise linear function satisfying $X_i(X_j) = \delta_{ij}$. We seek a description of those $F \in A_\Delta$ such that $F(X_1, \ldots, X_n) \in C^r(\Delta)$. To this end we define the differential operators, for $\sigma \in \Delta_d$,

$$\partial_\sigma = X_1|_\sigma^{(z)} \frac{\partial}{\partial x_1} + \ldots + X_n|_\sigma^{(z)} \frac{\partial}{\partial x_n},$$

where $z \in \mathbf{R}^d$ is a suitably chosen point. For $\tau \in \Delta$, let $I_\tau = < x_i | i \notin \tau >$, an ideal in A_Δ.

Theorem: If

$$A_\Delta^r = \{F \in A_\Delta \mid \forall \sigma, \sigma' \in \Delta_d, (\partial_\sigma^s - \partial_{\sigma'}^s) F \in I_{\sigma \cap \sigma'}, s = 1, \ldots, r\}$$

then $C^r(\Delta) \cong A_\Delta^r / < x_1 + \ldots + x_n - 1 > A_\Delta^r$.

Fiber Zonotopes

For $P \xrightarrow{\pi} Q$ an affine projection of polytopes, we define the fiber polytope $\Sigma(P, Q) = \frac{1}{vol\, Q} \int_Q \pi^{-1}(x) dx$, where the integral is defined as the set of all integrals of measurable sections of π. $\Sigma(P, Q)$ is always a convex polytope of dimension $dim\, P - dim\, Q$, whose face lattice is antiisomorphic to a certain lattice of subdivisions of Q. When $P = \Delta_{n-1}$, the $(n-1)$-simplex, and $Q = conv(A)$, $A = \{a_1, \ldots, a_n\} \subset \mathbf{R}^d$, then $\Sigma(P, Q)$ is the secondary polytope of Gel'fand, Kapranov and Zelevinsky. When $P = I^n$, the n-cube, and Q is a zonotope (with n zones), then $\Sigma(P, Q)$ is a zonotope, whose vertices correspond to "realizable" zonotopal tilings of Q.

Streat Homology and $\dim C_k^1(\Delta)$ for $k \neq 3$

For $\Delta \subset \mathbf{R}^2$, we analyze the dimension of $C_k^r(\Delta)$, the r-fold smooth piecewise polynomials of degree at most k, by means of the homology of $(\Delta, \partial\Delta)$ with coefficients in the sheaves $\mathcal{A}$ and $\mathcal{I}$ on Δ defined by $\mathcal{I}(\sigma) = I_\sigma^{r+1}$, $I_\sigma = \{f \in P_m \mid f|_\sigma = 0\}$, P_m = polynomials of degree at most m, and $\mathcal{A}(\sigma) = P_m/\mathcal{I}(\sigma)$. We establish the following results:

(i) for $k \geq 2$,

$$d_k + s \leq \dim C_k^1(\Delta) \leqq d_k + f_0^0,$$

where d_k is the generic dimension of $C_k^1(\Delta)$, s is the number of singular interior vertices and f_0^0 is the total number of interior vertices; and

(ii) $\dim C_k^1(\Delta) = d_k + s$ for $k \geqq 4$.

The proof of (ii) leads one to believe that this will no longer be true for $k = 3$.

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