MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Zelluläre Automaten

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The workshop has been organized by K.P.Hadeler (Tübingen) and H.-O.Peitgen (Bremen). Cellular automata are widely used to model natural phenomena like spatial spread or pattern formation. They have been studied to answer fundamental mathematical questions (e.g. the existence of a selfreplicating machine), in connection with discrete dynamical systems (e.g. the life game) and problems of geometry and topology (e.g. fractals). Since long the similarities between certain cellular automata and parabolic differential equations have been observed and in some cases these have substantiated into mathematical theorems. There are numerous relations to combinatorics, stochastic processes, formal languages, topology, and geometry. The visual aspects which lend much appeal to cellular automata sometimes obscure the fact that the underlying mathematical problems are rather difficult, due to the discrete nature of the objects. The objective of this small workshop has been the exchange of results, tools, and views of different groups in a rather inhomogeneous field of research.

Vortragsauszüge

Christoph Bandt

L-systems and outward fractal constructions

Two graphical interpretations of L-systems have been suggested which lead to two different self-similar constructions. Dekking uses global coordinates and obtains rather special self-affine HIFS. Prusinkiewicz applied turtle geometry instructions which leads to more general results. In a modified version, all self-similar HIFS can be obtained. Another approach to outward fractal constructions is by set equations containing expansive maps. There is always a continuum of outward constructions of the same type which discribes the geometry near points of the respective inward construction.

Some applications of symbolic dynamics

Full one-sided shifts, Markov subshifts and sofic systems are used to describe self-similar sets symbolically. The study of points with more than one address leads to a complete description of the topological structure. Applications comprise the automorphism group, various topological properties, definition of an interior metric by Hausdorff measure and local matching rules for an appropriate outward construction.

André Barbé

A Cellular Automaton with a Fractal State Space - Fractal Matrices

A quaternary one dimensional CA obtained from the independent superposition of two binary automata obeying rule 102, was presented. One of the automata has the structure of a so-called change propagation net. The other one generates a regular so-called difference field. Their combined evolution is constrained by a statewise conservation of the difference between occupied and nonoccupied cells of the difference field at the places covered by the change propagation net.
The automaton develops along a trial - and error path. Some initial states (seeds) lead to evolutions that abort quickly, others produce potentially unbounded complex state-time patterns. The set of all legal states for the automaton has a fractallike structure that can be constructed in a recursive way. However, it is conjectured that the sets of arbortive and productive seeds, which are also fractallike, are not recursively constructible when taken apart. A few open problems: 1. Is this conjecture true? Can it be decided whether a given seed is abortive or not? 2. Do potentially unbounded solutions actually exists? 3. How many solutions can be obtained from a given productive seed?

In a second part, we presented an extension and generalization of the fractallike set of legal states of the CA discussed above. This leads to a broad class of integer valued matrices obtained by a matrix substitution system where the substitution rule is arithmetical in nature. These matrices exhibit self-similar properties under agglomeration and decimation. A number theoretic interpretation for the elements of matrix was given in terms of proper number-base representations of their coordinates.

Andreas Dress

Simplistic Modelling

Excitable media exhibit various forms of pattern formation some of which can be simulated by very simple cellular automata: each cell has the dynamics of a simple oscillator, any two neighbouring cells are coupled by diffusion. For a large range of diffusion rate parameters such automata produce circular or spiral wave fronts which run over the screen and interact with each other the way chemical wave fronts do. This presents a considerable challenge to mathematics: at present, we do not even have appropriate mathematical concepts to state conjectures which would describe the phenomena in a mathematically satisfying way. So, right now, only suggestions on how to approach this problem can be discussed.

K.P. Hadeler

Accessible stationary states in asynchronous cellular automata

For the class of asynchronous cellular automata described in the abstract of B. Schönfisch the set $\mathcal{E}(Z)$ of all those stationary states is studied, which can be reached from a given initial element z. It is shown that the set $\mathcal{E}(z)$ has a maximal and a minimal element (with respect to the natural ordering of functions from $\mathbb{Z}^d$ to $\{0,1\}$).

Fritz v. Haeseler

Cellular Automata and Limit Objects

1. Cellular Automata (CA)

Let $V = \{v_0, ..., v_N\}, N \geq 1$, be a set with distinguished element v_0. Let $\sum(V) = \{\underline{a} : \mathbf{Z} \to V\}$ be equipped with the product topology induced by the discrete topology on V. The *shift* $\sigma : \sum(V) \to \sum(V)$ is defined as $\sigma(\underline{a})(i) = \underline{a}(i-1)$. A continuous map $A : \sum(V) \to \sum(V)$ is called a *cellular automaton* if $\sigma A(\underline{a}) = A\sigma(\underline{a})$ holds for all $\underline{a} \in \sum(V)$. Due to the theorem of Hedlund any CA is defined by a generating function and therefore it sufficies to study automata A which are given by

$$A_\phi(\underline{a})(i) = \phi(\underline{a}(i-d+1), ..., \underline{a}(i)),$$

where $\phi : V^d \to V$ is a map and d is a natural number.

2. Graphical Representation

Let A be a CA which preserves the set $\sum_c(V) = \{\underline{a} \in \sum(V) \mid$ there is a K such that $\underline{a}(i) = v_0$ for all $|i| \geq K\}$. Let $I = [0,1]^2$ denote the closed unit square in $\mathbf{R}^2$. The map $G : \sum_c(V) \to \mathcal{H}(\mathbf{R})^2 \cup \{\emptyset\}$ defined by

$$G(\underline{a}) = \bigcup_{\{i \mid a(i) \neq v_0\}} (I + (i, 0))$$

is called *graphical representation*. The compact set

$$X(A, \underline{a}, m) = \bigcup_{j=0}^{m-1} (G(A^j(\underline{a})) + (0, j))$$

is called *orbit representation*.

3. Existence of Limit Objects

Scaling Problem: Let A be CA which preserves $\sum_c(V)$. What are proper scaling sequences $(n_j)_{j\in\mathbf{N}} \subset \mathbf{N}$, i.e. when is the sequence

$$\frac{1}{n_j}X(A,\underline{a},n_j)$$

a Cauchy sequence in $(\mathcal{H}(\mathbf{R}^2),h)$?
Let $p \geq 2$ a natural number, define

$$\pi_p : \sum(V) \to \sum(V) \quad \text{by} \quad \pi_p(\underline{a})(i) = \begin{cases} \underline{a}(j) & \text{if } i = pj, \\ v_0 & \text{otherwise.} \end{cases}$$

For $v \in V$ we denote the sequence $\underline{v} \in \sum(V)$ such that $\underline{v}(0) = v$ and $\underline{v}(i) = v_0$ for $i \neq 0$. With this notation one has:
Theorem [v.Haeseler, Peitgen, Skordev] Let $p \geq 2$ be a natural number and let $A : \sum(V) \to \sum(V)$ be a cellular automaton of one of the following types
a) *weak Fermat* (in $v \in V$ w.r.t. p), i.e. $GA^{sp}(\underline{v}) = G\pi_p A^s(\underline{v})$ holds for all $s \in \mathbf{N}$,
b) *strong Fermat* (in $v \in V$ w.r.t. p), i.e. $A^{sp}(\underline{v}) = \pi_p A^s(\underline{v})$ holds for all $s \in \mathbf{N}$,
c) *strongly scaling* (w.r.t. p), i.e. $A^p\pi_p(\underline{a}) = \pi_p A(\underline{a})$ holds for all $a \in \sum(V)$.
Then the scaling sequence $(p^{-j})_{j\in\mathbf{N}}$ is a solution of the scaling problem, i.e., the sequence $(p^{-j}X(A,\underline{v},p^j))$ is a Cauchy sequence with respect to the Hausdorff distance.

Hierarchical Iterated Function Systems

1. Iterated Function Systems (IFS)
Let $\mathbf{R}^2$ be equipped with the euclidian distance $\|\ \|_2$. Let $\mathcal{H}(\mathbf{R}^2)$ denote the set of all non-empty compact subsets. Then $\|\ \|_2$ induces a metric, h the Hausdorff distance, on $\mathcal{H}(\mathbf{R}^2)$. The Hausdorff distance of two $A, B \in \mathcal{H}(\mathbf{R}^2)$ is given by

$$h(A,B) = \inf \epsilon > 0 \mid A \subset (B)_\epsilon, B \subset (A)_\epsilon\},$$

where $(A)_\epsilon, (B)_\epsilon$ denotes the ϵ−dilatation of A and B, respectively. Moreover, $(\mathcal{H}(\mathbf{R}^2),h)$ is a complete metric space.
Let $\{f_i : \mathbf{R}^2 \to \mathbf{R}^2 \mid i = 1,...N\}$ be a finite set of contractions (w.r.t. the euclidian distance). Then $\{f_i\}$ induces a map $F : \mathcal{H}(\mathbf{R}^2) \to \mathcal{H}(\mathbf{R}^2)$ defined by

$$F(A) = \bigcup_{i=1}^{N} f_i(A).$$

The map f is called *Hutchinson operator*. The Hutchinson operator is a contraction (w.r.t. h) and therefore F has a unique attractive fixed point A_∞ (Banach's Fixed Point Theorem). The fixed point A_∞ is called the attractor of the IFS $\{f_i\}$.
Dimension Formula: Let $\{f_i\}$ be an IFS consisting of similarity contraction with contraction ratio c_i. If there exists an open set $O \subset A_\infty$ such that

a) $f_i(O) \cap f_i(O) = \emptyset$ for all $i \neq j$,
and
b) $\cup_{i=1}^N f_i(O) \subset O$
then the Hausdorff dimension of A_∞ is the unique solution s_0 of

$$\sum_{i=0}^{N} c_i^s = 1.$$

2. Hierarchical Iterated Function Systems (HIFS)
Let M be a natural number. The set $\mathcal{H}(\mathbf{R}^2)^M$ equipped with the metric d defined by

$$d(\underline{A}, \underline{B}) = \max\{h(A_i, B_i) \mid i = 1, ..., M\},$$

where $\underline{A} = (A_1, ..., A_M)$ and $\underline{B} = (B_1, ..., B_M)$, becomes a complete metric space. A map $\mathcal{F} : \mathcal{H}(\mathbf{R}^2)^M \to \mathcal{H}(\mathbf{R}^2)^M$ defined by

$$\mathcal{F}(\underline{A})_i = \bigcup_{j=1}^{M} F_{i,j}(A_j), \qquad (1)$$

or in matrix notation $\mathcal{F}(\underline{A}) = (F_{i,j})\underline{A}$ (the multiplication is defined by 1) is called a HIFS if
a) each $F_{i,j}$ is either a Hutchinson operator or $F_{i,j} = \emptyset$ (the empty set map),
b) for all $i \in \{1, ..., M\}$ there exists a $j \in \{1, ..., M\}$ such that $F_{i,j} \neq \emptyset$, and
c) $(F_{i,j})^n \neq (\emptyset_{i,j})$. The condition c) may be read as: The matrix $C = (c_{i,j})$, where $c_{i,j} = \begin{cases} 1 & F_{i,j} \neq \emptyset \\ 0 & F_{i,j} = \emptyset \end{cases}$ is not nilpotent.
Existence of an attractor: Let $\mathcal{F}$ be an HIFS. The $\mathcal{F}$ is a contraction (w.r.t. d). Then there exists a unique attractive fixed point $\underline{A}_\infty$, the attractor of the HIFS.

Ehler Lange

Self-similar Structures of Classical Number Sequences mod p^ν

Let $\mathbf{Z}_{p^\nu}$ be the factor ring of the integers mod a prime power. Let $r(X) \in \mathbf{Z}_{p^\nu}$, then a linear cellular automata (LCA) is a mapping $A = A(r) : \mathbf{Z}_{p^\nu} \to \mathbf{Z}_{p^\nu}$ given by $s(X) \mapsto r(X)s(X)$. Define a graphical representation for elements in $\mathbf{Z}_{p^\nu}$ by

$$G_f\left(\sum a_i X^i\right) = \bigcup_{a_i \neq 0} I + (i, f(i)) \subset \mathbf{R}^2,$$

where $f : \mathbf{Z} \to \mathbf{Z}$ is a map and $I = [0,1]^2$. Let $A = A(r)$ be a LCA and $s(X) \in \mathbf{Z}_{p^\nu}, p \in \mathbf{N}$ are fixed, then we also define an orbit representation by

$$X(A, s(X), m, f, \rho) = \bigcup_{t=0}^{m-1} G_f(A^t(s(X)) + (0, \rho \cdot t),$$

where $m \in \mathbb{N}$. If f and A satisfy certain conditions then $\lim_{m\to\infty} \frac{1}{p^m} X(A, 1, p^m, f, \rho)$ exists.
Consider now the binominals, gaussian binominals and Stirling numbers of 1st and 2nd kind mod a prime power (Stirling numbers of 2nd kind are only considered with respect to mod p). The recursion formulas of these numbers mod p^ν can be interpreted as local rules of new types of cellular auomata. The so called time-resp. place-dependent automata which model these sequences exactly also induce limit sets. These limit sets are affine homeomorphic to the fractal set $\mathbf{S}_{p^\nu} := \lim_{m\to\infty} \frac{1}{p^m} X(A, 1, p^m, f, 1), f \equiv 0$, which can be seen as the limit set induced by the binominals. $\mathbf{S}_{p^\nu}$ is structurally similar to the Sierpinski gasket. This set is in some sense universal for this sequences.

Birgitt Schönfisch

Lyapunov functions for cellular automata

For a class of cellular automata which we call symmetric, totalistic and monotone (and which, in a loose sense, mimick a heat equation) a concept of Lyapunov functions is introduced. Although the synchronous automata may have nontrivial periodic orbits, these Lyapunov functions are nevertheless usefull tools in the investigation of the corresponding asynchronous automata. For these it can be shown that trajectories approach stationary states.

Guentcho Skordev

Self-smilarity structure of the limit set of a class of cellular automata

Report on a joint work with H.-O. Peitgen, F.v.Haeseler. In the talk of F.v. Haeseler two classes of cellular automata (CA) were defined: p-strong Fermat (p.-SF) and p-weak Fermat (p.-WF). The p.-WF CA have a limit set. The limit set of a p.-SF CA is described via the attractor of some matrix substitution system (MSS). MSS are special HIFS (defined in the first talk of F.v.Haeseler). An algorithm is presented which produces an MSS for any p.-SF CA. The limit set of an explicitely given symbol (w.r.to this MSS) coincides with the limit set of the CA.

Burton Voorhees

Graph theoretic techniques for computing pre-images of finite sequences

By defining certain graphs associated to any given cellular automata rule it is possible to compute all preimages of any given finite sequence of numbers under that rule. It is also possible to derive a set of non-linear Diophantine equations whose solution set determines all automata rules having a given set of finite sequences with no pre-image (Garden of Eden sequences).

Berichterstatterin: Birgitt Schönfisch

Tagungsteilnehmer

Dr. Christoph Bandt
Fachrichtung Mathematik/Informatik
Universität Greifswald
Ludwig-Jahn-Str. 15a

O-2200 Greifswald
GERMANY

Dr. Fritz von Haeseler
Institut für Dynamische Systeme
Universität Bremen
Postfach 33 04 40

W-2800 Bremen 33
GERMANY

Prof.Dr. André M. Barbé
Depart. of Electrical Engineering
Kard. Mercierlaan 94
Kath. Universiteit Leuven

B-3001 Leuven

Ehler Lange
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr.1, PF 33 04 40

W-2800 Bremen 33
GERMANY

Dr. Joachim von Below
Lehrstuhl für Biomathematik
Universität Tübingen
Auf der Morgenstelle 10

W-7400 Tübingen 1
GERMANY

Martin Lüneburg
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31

W-4800 Bielefeld 1
GERMANY

Prof.Dr. Andreas Dress
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31

W-4800 Bielefeld 1
GERMANY

Prof.Dr. Heinz-Otto Peitgen
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr.1, PF 33 04 40

W-2800 Bremen 33
GERMANY

Prof.Dr. Karl Peter Hadeler
Lehrstuhl für Biomathematik
Universität Tübingen
Auf der Morgenstelle 10

W-7400 Tübingen 1
GERMANY

Birgitt Schönfisch
Lehrstuhl für Biomathematik
Universität Tübingen
Auf der Morgenstelle 10

W-7400 Tübingen 1
GERMANY

Prof.Dr. Guentcho Skordev
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr.1, PF 33 04 40

W-2800 Bremen 33
GERMANY

Prof.Dr. Burton Voorhees
Faculty of Arts and Science
Athabasca University
Box 10000

Athabasca Alberta TOG 2RO
CANADA

e-mail-Adressen

Barbé, André M.	barbe@esat.kuleuven.ac.be
v.Below, Joachim	bcmbe01@convex.zdv.uni-tuebingen.de
Dress, Andreas	jordan@math25.mathematik.uni-bielefeld.de
Hadeler, Karl Peter	hadeler@convex.zdv.uni-tuebingen.de
v.Haeseler, Fritz	fritz@mathematik.uni-bremen.de
Lange, Ehler	lange@mathematik.uni-bremen.de
Lüneburg, Martin	jordan@math25.mathematik.uni-bielefeld.de
Peitgen, Heinz-Otto	peitgen@mathematik.uni-bremen.de
Schönfisch, Birgitt	bcmsc01@convex.zdv.uni-tuebingen.de
Skordev, Guentcho	skordev@mathematik.uni-bremen.de
Voorhees, Burton	burt@aupair.ct.athabascau.ca

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