MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 38/1992

Web Geometry and Related Fields

23.8.–29.8.1992

The meeting was organized by V.V. Goldberg (Newark, NJ, USA) and K. Strambach (Erlangen). The program focused on questions of the differential geometry of webs and touched on related topics and their applications to other branches of mathematics. Special attention was given to developments in the field following discussions at the previous MFO meeting in 1984. In web theory itself, topics such as closed G-structures associated with webs, proving algebraizability theorems for some geometric objects using web-theoretic techniques, an algebraic interpretation of higher order tensors of a three-web, three-webs in homogeneous spaces, rank problems for different kinds of webs and their applications to physics gave a deeper insight on web geometry and its connection with algebraic geometry, the theory of differential-geometric structures and physics.

It is well-known that web geometry is closely connected with different algebraic structures (local loops, algebras etc.). Interesting developments in the theory of homogeneous left Lie loops and tangent algebras, developments of algebraic nature in the theory of comtrans algebras (which have appeared as local algebras for $(n+1)$-webs), applications of first degree loops to quadratic systems of ordinary differential equations as well as global aspects of loop theory, (loops as invariant sections in groups and collineation groups generated by Bol reflections) were presented. In addition, the Lie theory of semigroups and its application to chronogeometry and semigroups in topological planes and their relation with the foundation of geometry were discussed.

At the end of the meeting a problem session took place.

The session was devoted to the memory of Professor Dr. Gerrit Bol (1906–1989), one of the founders of web geometry. The director of MFO Professor Dr. Martin Barner gave an informal talk on the life and the career of G. Bol.

Abstracts of contributed lectures

M.A. AKIVIS

Three-webs and closed G-structures

A three-web W given on a differentiable manifold X, $\dim X = 2r$, defines a G_W-structure on X with the structure group $G = \mathbf{GL}(r)$. In each point $x \in X$ the fundamental tensors $c_1, \ldots, c_k$ of this G_W-structure generate local algebras $W_k(x)$ of orders $k = 2, 3, \ldots, n, \ldots$ A G_W-structure is called closed of order k if $c_s = f(c_2, \ldots, c_k), s > k$.

It is proved that

1. If a G_W-structure is closed of order k, then the local algebra $W_k(x_0)$ given at the point x_0 defines a three-web W on X.

2. G_W-structures connected with three-webs, on which one of the classical closure conditions are satisfied, are closed. In particular, for webs (T): $k = 1$ and the W_2-algebra is trivial; for webs (R): $k = 2$ and the W_2-algebra is a Lie algebra; for webs $(B_l \& B_r)$: $k = 2$ and the W_2-algebra is a Mal'cev algebra; for webs (B_l), (B_r) and (B_m): $k = 3$ and the W_3-algebras are Bol algebras; for webs (H): $k = 4$ and the W_4-algebra is called the hexagonal algebra.

M.A. AKIVIS :

Problems of algebraizability of submanifolds in projective differential geometry

The algebraizability problem for a system of smooth submanifolds V_α, $\alpha = 1, \dots, d$, of dimension r of a projective space P^n is to find conditions under which these submanifolds belong to an algebraic submanifold of the same dimension. This kind of conditions was known for a system of curves in a projective plane (theorems of Abel and Reiss).

We prove the theorems generalizing the theorems of Abel and Reiss for a system of hypersurfaces and submanifolds of codimension $p > 1$ in the space P^n. We show different methods of proofs of this kind of theorems, in particular, the method using the techniques of the theory of webs.

D. GERBER :

Applications of first degree loops to quadratic systems

The first degree loop $L(F^{-1})$, with product $x \circ y = F^{-1}(y)x + y$, is admissible (for the quadratic system $\dot{x} = x * x, x(0) = \xi$) iff $x(t,\xi) = -t^{-1}(t\xi)^\lambda$. Here $Q(*)$ is a real commutative algebra and x^λ is the left inverse of x in L. We have shown earlier that L is admissible if it is an LIP loop and that an admissible loop exists for every quadratic system. Because L is assumed to be analytic, $x \circ y = x + y - xy + \dots$, where xy is multiplication in $\mathcal{L}$, the related algebra of L. In this talk we define strongly admissible first degree loops (which are admissible). We then show that a strongly admissible loop exists for every $\mathcal{L}$, that it is uniquely determined by $\mathcal{L}$, and that it is an RIP loop. These results are then extended to quadratic systems with a linear term $\dot{x} = x * x + Bx, x(0) = \xi$.

V.V. GOLDBERG :

Rank problems for webs

Let $W(d,n,r)$ be a d-web given by d foliations of codimention r on (nr)-dimensional differentiable manifold X. The definition of the q-rank R_q, $1 \le q \le r$, of such a web is due to P.A. Griffiths. There are two fundamental rank problems; a) To find an upper bound for R_q (if it exists), and b) To describe webs of maximum q-rank. The following results are presented:

1. The connection between the rank problems and the Grassmannization and algebraization problems for webs (Chern–Griffiths, Akivis, Wood, Goldberg).
2. The Chern–Griffiths results on rank problems for webs $W(d,n,1)$.
3. The relationship between almost Grassmannizable webs and webs of maximum rank (in particular, Little's result: a web $W(d,n,r)$, $d > r(n-1)+2$, of maximum r-rank is almost Grassmannizable).
4. The author's results on both r-rank problems for webs $W(d,2,r)$ and $W(6,3,2)$.
5. The author's results on both 1-rank problems for webs $W(d,2,r)$.
6. The Damiano results on $(n-1)$-rank for webs of curves in X^n, generalizing the famous Bol example of a web $W(5,2,1)$ of maximum 1-rank which is not algebraizable.
7. Other results in rank problems (Blaschke, Bompiani, Nazirov, Indrupskaya).
8. Application of rank problems to physics (Ferapontov, Balk).

In conclusion, some open problems for rank of webs are formulated.

A. HÉNAUT:
Linearization problem for webs in $\mathbf{C}^2$

Let W be a d-web in $(\mathbf{C}^2,0)$. We assume that the leaves of W are integral curves of the following vector fields: $X_i = \partial_x + b_i\partial_y$ for $1 \le i \le d$, where $b_i \in \mathbf{C}\{x,y\}$ with $b_i(0) \neq b_j(0)$ for $1 \le i < j \le d$. It follows that there exists a unique polynomial $P_W(x,y;b_i) = \sum_{k=0}^{d-1} P_k b^k \in \mathbf{C}\{x,y\}[b]$ such that $\deg P_W \le d-1$ and $P_W(x,y;b_i) = X_i(b_i)$ for $1 \le i \le d$. Each leaf of W satisfies the second-order differential equation: $y'' = P_W(x,y;y')$. We prove that: W is linearizable iff $\deg P_W \le 3$ and (P_0,P_1,P_2,P_3) satisfies an explicit non-linear differential system. Moreover, up to an automorphism of $\mathbf{P}^2$, every local isomorphism $\phi : (\mathbf{C}^2,0) \to (\mathbf{P}^2,\phi(0))$ giving a linearization of W is given. This result extends works by Liouville, Tresse and Arnold. Using the abelian relation space, we generalize the classical result on the 4-web of maximal rank 3 in $(\mathbf{C}^2, 0)$ due to Poincaré : let W be a d-web in $(\mathbf{C}^2, 0)$ with maximal rank and $d \geq 4$, then W is linearizable (therefore, algebraic) iff $\deg P_W \leq 3$. This result gives the answer to a problem by Chern.

K.H. HOFMANN:
A Lie theory for semigroups

We consider subsemigroups of Lie groups. If S is a subset of a Lie group G with Lie algebra $\mathcal{G}$, we set $L(S) = \{X \in \mathcal{G} : \exp \mathbf{R}^+ \cdot X \subseteq S\}$. If S is a closed semigroup, then $L(S)$ is a Lie wedge, where a Lie wedge W in a Lie algebra $\mathcal{G}$ is a closed convex (not necessarily pointed) cone such that $e^{\mathrm{ad}\, X} W = W$ for all $X \in W \bigcap -W$. A semigroup $S \subseteq G$ is called locally divisible iff there is an open identity neighborhood V in G such that for all $n \in \mathbf{N}$, all $s \in S \bigcap V$ there is an $x \in S \bigcap V$ such that $x^n = s$. A locally divisible semigroup has a Lie wedge $W = L(S)$, a so called Lie semialgebra: There is a Campbell–Hausdorff neighborhood B of O in $\mathcal{G}$ such that $(W \bigcap B) * (W \bigcap B) \subseteq W$. Lie semialgebras have been classified by Eggert (1990). It was shown by Lawson and the speaker that a closed subsemigroup S of a Lie group G is divisible iff $S = \exp L(S)$. These authors also showed that a divisible semigroup is locally divisible if it has no invertible element other than 1. It has been an open problem of long standing whether this is true in general. Recently, Wolfgang A.F. Ruppert and the speaker were able to prove quite generally that the Lie wedge of a divisible semigroup is a Lie semialgebra. The lecture begins with an illustration from chronogeometry as a motivation for a Lie theory of semigroups. It proceeds through an indication of the basic theory and concludes with the results on divisibility mentioned above.

M. KIKKAWA :
Projectivity of homogeneous left Lie loops and tangent algebras

It is well known that connected and simply connected geodesic homogeneous left Lie loops are characterized by their tangent Lie triple algebras. As for projectivity relation among them, we have:

THEOREM. Let (G, μ) and (G, μ^*) be geodesic homogeneous left Lie loops with the same identity element e, on a connected and simply connected analytic manifold G. Denote by: $g := (T_e(G); XY, [X,Y,Z])$ (resp. $g^* := (T_e(G); X * Y, < X,Y,Z >)$) the tangent Lie triple algebra of (G,μ) (resp. (G,μ^*)), and set $[X,Y] := XY - X * Y$.

Then (G,μ) and (G,μ^*) are in projective relation iff the binary system $L := (T_e(G); [X,Y])$ forms a Lie algebra with the following properties:

(1) $\mathrm{ad}\, L \subset \mathrm{Der}\, g$,

(2) $D(g,g) \subset \mathrm{Der}\, L$,

where Der denotes the derivation algebra and $D(X,Y)$ the inner derivations of the Lie triple algebra g.

V. LAZAREVA :
Three-webs in homogeneous spaces

Let G be a Lie group of dimension $2r+\rho$. Suppose that there are three $(r+\rho)$-dimensional subgroups $G_\alpha, \alpha = 1,2,3$, on G having a common subgroup H. Then a three-web is defined on the homogeneous space $M = G/H$. The leaves of this web are the subsets $g(G_\alpha/H)$. Such a web is called a G-web. The properties of G-webs are considered. The method is given to find a subclass of G-webs in a given web class. All G-webs are found for which G is a simple group. Some open problems are discussed.

P. MIKHEEV :
The Chern–Akivis–Shelekhov problem on hexagonal three-webs

We present the results on the infinitesimal theory of hexagonal three-webs (on "hexagonal algebra") within the context of the general theory of smooth algebraic systems.

P. NAGY :
Collineation groups generated by Bol reflections

The Bol condition for three-nets can be formulated by the existence of involutory collineations with axes in a pencil of parallel lines. We prove that the collineation group generated by these reflections has an epimorphism onto the (left) translation group of the coordinate loop. If the (left) nucleus of this loop is trivial, then this is an isomorphism. Consequently, this collineation group of the classical three-net over the octonion loop P^7 is **PSO**(8). This collineation group has a faithful representation in the group of projectivities if (1) the left nucleus of the loop is trivial, or (2) the loop has automorphic inverse property. If the three-net is embedded into a projective plane in a suitable way, then the Bol reflections can be continued to collineations of the plane.

A. SHELEKHOV :
Three-webs and identities in loops

It is well-known that all classical multidimensional three-webs can be defined by some simple identities fulfilled in every coordinate loop of the web. First, we consider an arbitrary regular identity S and discuss the following problem: which identities give new classes of webs ? In particular, the E-webs defined by the elastisity identity $(xy)x = x(yx)$ are investigated. Secondly, the so called the kth order identities with one variable, generalizing the monoassociativity identity $x^2 \cdot x = x \cdot x^2$, are introduced and described.

A. SHELEKHOV :
An algebraic interpretation of higher order tensors of a three-web

The sequence of the basic tensor fields are connected with a multidimensional three-web W: torsion and curvature tensors a and b, and the covariant derivations (with respect to Chern connection) of the tensor b. It is well-known that the tensors a and b are the obstructions to the commutativity and associativity, respectively, of the coordinate loops of W. We give an interpretation of other basic tensors of a three-web in terms of coordinate loops of this web. Several applications are given and some new problems are discussed.

J.D.H. SMITH :
Comtrans algebras

Comtrans algebras are part of the tangential algebraic structure locally co-ordinatizing a web. Recent progress in the study of comtrans algebras is described, including simple algebras, representation theory, and some interesting examples.

K. STRAMBACH :
Loops as invariant sections in groups

In a joint paper with P. Nagy (Szeged) we identify loops with the set L of its left translations in the group G generated by L and study properties of loops which can be expressed as invariant properties of L with respect to actions of G. For example, we introduce the class of left conjugacy closed loops; these are loops such that L is invariant with respect to conjugations by elements of G. We characterize in this class the subclass closed with respect to isotopisms by a new configurational condition. We show that for differentiable left conjugacy closed loops L the group G is a Lie group; moreover, G contains a sharply transitive normal subgroup N.

M. STROPPEL :
Semigroups and the "Foundations of Geometry"

By the "Foundations of Geometry" one means the study of geometries that share important properties of Euclidean Geometry. Our focus of interest are topological planes : incidence structures (P, l) where two members of P always are (continuously) joined by a unique member of l, while intersections of lines need not to exist. However, we require that the intersection map has an open domain of definition (in fact, a "planarity condition"). For these planes, we study endomorphisms and show : The semigroup of all endomorphisms of a (locally) compact connected projective (or affine) plane of finite topological dimension coincides with its group of automorphisms. Dropping either the assumption of connectedness or the parallel axiom, one obtains a cancellative semigroup that may be compatively large (a main example : the hyperbolic plane).

Open problems

1. Find all d-webs of curves on n-dimensional manifold of maximum $(n-1)$-rank. (S.S. Chern)

2. Is the Chern-Griffiths sufficient condition of algebraizability of webs $W(d,n,1)$ true without normality ? (S.S. Chern)

3. Are there other webs $W(5,2,1)$ of maximum rank 6 besides the algebraic ones and Bol's counterexample ? (S.S. Chern)

4. Is the exceptional $(n+3)$-web of curves in X^n, $n > 2$, found by D. Damiano, the only non-linearizable $(n+3)$-web of dimension 1 and maximum $(n-1)$-rank $(n+1)(n+2)/2$? (S.S. Chern)

5. Determine all webs $W(d,n,r), n > 2$, of maximum r-rank. (S.S. Chern)

6. Find examples of non-algebraizable webs $W(d,n,r)$, $n > 2$, of maximum r-rank other than Goldberg's and Little's examples. (V.V. Goldberg)

7. Are all webs $W(r(n-1)+2,n,r)$ of maximum r-rank almost Grassmannizable? If the answer is no,

   a) Find under what condition this is the case; and

   b) Give examples of webs $W(r(n-1)+2,n,r)$ of maximum r-rank which are not almost Grassmannizable. (V.V. Goldberg)

8. Find the upper bound for the maximum q-rank for webs $W(d,n,r)$ and describe webs $W(d,n,r)$ of maximum q-rank for arbitrary or some particular values of q,d,n and r. (V.V. Goldberg)

9. Find canonical expansion for the Bol loops. For this : express all coefficients of this expansion in terms of the curvature and torsion tensors defining the operations in the Bol algebra and use the main identities (of the Bol algebra) which these tensors satisfy. (M.A. Akivis)

10. Is the following theorem valid: if the multiplication operation in a Bol loop is thrice differentiable, then it is analytical ? (M.A. Akivis)

11. Study the structural theory of W-algebras, and in particular, the Bol algebras. (M.A. Akivis)

12. Construct the theory of differentiable quasigroups and multidimensional three-webs by means of differential operators. (This kind of exposition will be more clear for physicists who is interested in this theory). (M.A. Akivis)

13. Find or construct new geometric realization of three-webs

    a) on which one of classical closure conditions holds;

    b) which are isoclinic or transversally geodesic but not Grassmannizable. (M.A. Akivis)

14. Given an Akivis algebra $A = (A, [\ ,\ ], (\ ,\ ,\ ))$, a comtrans algebra $CT(A)$ is obtained with $[x,y,z] = < x,y,z >= [[x,y],z] + (x,y,z) - (y,x,z)$. If $CT(A)$ is simple, does it follow that A is simple? (The answer is "yes" for A a Lie algebra.) (J.D.H. Smith).

15. Let (G,m) be a Lie group. We know that the Akivis loops (resp. Akivis local loops) $m_p, p \in \mathbf{Z}$ (resp. $p \in \mathbf{R}$) on G are in projective relation with (G,m). Find all geodesic homogeneous left Lie loops (resp. local Lie loops) which are in projective relation with (G,m). (M. Kikkawa)

16. There are known few examples of identities on local analytic loops that have infinitesimal theories (Lie groups, Moufang loops, Bol loops, homogeneous (Kikkawa) loops, hyporeductive loops, diassociative loops). New examples are of interest. (P.O. Mikheev)

17. Characterize the set of closed configurations on a multidimensional three-web defined by a regular (non-universal) identity, which is fulfilled in a coordinate loop of this web. (A.M. Shelekhov)

18. Is the following statement true or false: if a regular identity holds in every coordinate loop of a multidimensional three-web W, then the G-structure associated with this web is closed ? (A.M. Shelekhov)

19. Consider the class of such multidimensional three-webs in coordinate loops of which a regular identity with two variables holds. Which identities leads to new classes of webs ? (A.M. Shelekhov)

20. Consider the class of non-hexagonal three-webs defined by the kth order identity with one variable, where $k > 2$.

a) Give an example of such webs.

b) Find the order of the corresponding G-structure. (A.M. Shelekhov)

21. Find the minimal integer N such that the identity $v(x) = w(x)$ of length N with one variable has the order 5. (A.M. Shelekhov)

22. Describe the algebraic properties identities of order k with one variable. Give an algebraic construction of such identities. (A.M. Shelekhov)

23. Find a complete system of tensorial conditions characterizing the class of multidimensional elastic webs. (A.M. Shelekhov)

24. Every smooth multidimensional three-web with elastic coordinate loops is necessary a Bol web. Is this statement true for abstract three-webs? (A.M. Shelekhov)

25. Find the tensorial characterization of the class of multidimensional G-webs. (V.B. Lazareva, A.M. Shelekhov)

26. Classify G-webs given on a homogeneous space G/H, where G is a semisimple group. (V.B. Lazareva, A.M. Shelekhov)

27. Describe G-webs given on a Lie group G. (V.B. Lazareva, A.M. Shelekhov)

28. Study the automorphism group of a multidimensional three-web. (V.B. Lazareva, A.M. Shelekhov)

29. Is a C^{∞}-differentiable loop defined on an analytical manifold analytical if all left and right translations are analytical diffeomorphisms ? (K. Strambach, P. Nagy).

30. Let G be a Lie group and g be the Lie algebra of G. Investigate the singulaties of $\exp : g \to G$. When is exp surjective ? Sample question: Suppose that $\exp g$ is a neighborhood of $x \in G$. Does there exist an open ball B around 0 in g such that $\exp B$ is a neighborhood of x in G ? (K.H. Hofmann)

31. Given a three-web with a geodesic complete Chern connection defined on a simply connected global manifold, does such a web have global coordinates loops ? (P. Nagy)

Reporting : V.V. Goldberg

List of participants

Prof.Dr. Maksa A. Akivis
Belomorskaya ul., dom 4
kv. 37

Moscow 125 195
RUSSIA

Prof.Dr. Karl Heinrich Hofmann
Fachbereich Mathematik
TH Darmstadt
Schloßgartenstr. 7

W-6100 Darmstadt
GERMANY

(e-mail: hofmann@mathematik.th-darmstadt.de)

Prof.Dr. Adriano Barlotti
Dipt. di Matematica "U.Dini"
Universita di Firenze
Viale Morgagni 67/A

I-50134 Firenze

(e-mail: udini@ifidg.bitnet)

Prof.Dr. Michihiko Kikkawa
Department of Mathematics
Shimane University

Matsue 690
JAPAN

Dr. P. Dean Gerber
IBM Corporation
Thomas J. Watson Research Center
P. O. Box 218

Yorktown Heights , NY 10598
USA

Prof.Dr. Valentina B. Lazareva
Department of Functional Analysis
Tver State University

Tver 170000
RUSSIA

Prof.Dr. Vladislav V. Goldberg
Dept. of Mathematics
New Jersey Institute of Technology

Newark , NJ 07102
USA

(e-mail: vlgold@m.njit.edu)

Prof.Dr. Pavel O. Mikheev
Department of Mathematics
Friendship of Nations University
M.-Maklaya 6

117198 Moscow
RUSSIA

Prof. Dr. Alain Hénaut
Centre de Rech. en Mathématiques
Université Bordeaux 1, C.N.R.S.
351, cours de la Libération

F-33405 Talence-Cedex

(e-mail: henaut@frdbdx11.bitnet)

Prof.Dr. Peter T. Nagy
Bolyai Institute
Jozsef Attila University
Aradi Vértanúk Tere 1

H-6720 Szeged

Prof.Dr. Alexander M. Shelekhov
Department of Functional Analysis
Tver State University

Tver 170000
RUSSIA

Prof.Dr. Jonathan D.H. Smith
Dept. of Mathematics
Iowa State University
400 Carver Hall

Ames , IA 50011
USA
(e-mail: jdhsmith@pollux.iastate.edu)

Prof.Dr. Karl Strambach
Mathematisches Institut
Universität Erlangen
Bismarckstr. 1 1/2

W-8520 Erlangen
GERMANY
(e-mail: formath@cnve.rrze.uni-erlangen.dbp.de)

Dr. Markus Stroppel
Fachbereich Mathematik
T.H. Darmstadt
Schloßgartenstr. 7

W-6100
GERMANY
(e-mail: stroppel@mathematik.th-darmstadt.de)

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