MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 46/1992

Geometrie

18.10. bis 24.10.1992

Die Tagung fand unter Leitung von V. Bangert (Freiburg) und U. Pinkall (Berlin) statt. In 31 Vorträgen berichteten Teilnehmer über ihre neuesten Forschungsresultate. In den nachstehenden Vortragsauszügen spiegeln sich die Vielfalt des Tagungsthemas "Geometrie" und dessen enge Beziehungen zu anderen Teilen der Mathematik. Schwerpunkte des Vortragsprogramms lagen in der Geometrie der symmetrischen Räume, der Riemannschen Geometrie, der Variationsrechnung und der Theorie der Untermannigfaltigkeiten. Die Zeit zwischen den Vorträgen und die Abende boten lebhaft genutzte Möglichkeiten zu mathematischen Gesprächen und Diskussionen.

Vortragsauszüge

U. Abresch (joint work with V. Schroeder):
Examples of compact, real analytic manifolds of rank one

Let (M^n, g) be a compact manifold with $K_M \leq 0$. For this class there is a substantial difference between the C^∞- and and the C^ω-category. In the real analytic case one has Gromov's finiteness theorem, whereas in the C^∞-case there exist graph manifolds of infinite topological tye. A main tool in such investigations are flats $F^k\colon \mathbf{R}^k \hookrightarrow M^n$ and parallel sets of such flats. In the real analytic case, these parallel sets P_{F^k} extend to complete, convex submanifolds. If the geometric rank of M is ≥ 2 the configuration of such flats and their parallel sets is quite rigid, whereas there is still a lot of flexibility in the rank 1 case as the following result shows:

Theorem: Let $\overline{V}_i^{n-2} \subset H^n/\Gamma'$, $1 \leq i \leq N$, be compact totally geodesic submanifolds in a compact hyperbolic space. Any nonempty pairwise intersection is required to be orthogonal and of codimension 4. Moreover, there shall exist rotations

$\overline{\rho}_i: H^n/\Gamma' \to H^n/\Gamma'$ such that $\overline{V}_i^{n-2} = \mathrm{Fix}(\overline{\rho}_i)$ and such that each $\overline{\rho}_i$ permutes the submanifolds $\overline{V}_{i'}^{n-2}$, $1 \leq i' \leq N$. Then the blow-up $\pi: M \to H^n/\Gamma'$ along $\bigcup_i^N \overline{V}_i^{n-2}$ carries a real analytic metric g with the following properties:

i) $K_M < 0$ and even $\hat{R} < 0$ on the open dense set $M \setminus \pi^{-1}(\bigcup_i^N \overline{V}_i^{n-2})$,

ii) each $\hat{V}_i := \pi^{-1}(\overline{V}_i)$ is totally geodesic, and so are the $\hat{V}_I := \bigcap_{i \in I} \hat{V}_i$,

iii) the restriction $\pi|\hat{V}_I$ factors over a Riemannian submersion $\hat{\pi}_{(I)}: \hat{V}_I \to \overline{V}_I^*$ such that $K_{\overline{V}_I^*} \leq 0$, that the fibers are totally geodesically embedded product tori $\underbrace{\mathbf{R}P^1 \times \ldots \times \mathbf{R}P^1}_{\#I \text{ factors}}$, and that $\hat{\pi}_{(I)}$ itself is a flat bundle.

These manifolds (M^n, g) have as little zero curvature as permitted by their fundamental group. The $\hat{V}_I$ are the parallel sets of $\#I$-flats, and when starting with suitable data, they have maximal possible dimension. The metric g itself is not rigid, but the configuration of the $\hat{V}_I$ is determined by $\pi_1(H^n)$, and this in turn fixes the planes with zero curvature.

C. Bär:

Positively curved 4-manifolds

The only known compact connected 4-manifolds with positive sectional curvature are S^4, $\mathbf{R}P^4$, and $\mathbf{C}P^4$. On the other hand, only very few topological restriction of such manifolds are known. The most import one is the Synge-Lemma which says that the first fundamental group must be trivial if the manifold is oriented. Even the following "elementary" problem is still open.

Conjecture (H. Hopf): $S^2 \times S^2$ does not admit a metric of positive sectional curvature.

Using the Bochner method I can show:

Theorem: Let M^4 be compact, connected, and oriented. We assume (1) Sectional curvature $K \geq 1$, (2) $|\nabla R| \leq 2/\pi$. Then the intersection form of M is definite.

Corollary. $S^2 \times S^2$ cannot carry a metric satisfying (1) and (2).

J. Berndt (joint work with L. Vanhecke):

On generalizations of symmetric spaces

The analytic formulation of the expression of curvature of Riemannian manifolds in terms of variations of geodesics involves all the Jacobi operators R_γ along geodesics γ. The starting point for our studies is the following

Theorem: A Riemannian manifold M is locally symmetric if and only if for every geodesic γ in M (1) the eigenvalues of R_γ are *constant* along γ, and (2) R_γ is diagonalizable by a *parallel* orthonormal frame field along γ.

A natural question is: What happens if these two conditions split up? We call a connected Riemannian manifold M a C-space resp. P-space if for every geodesic γ in M condition (1) resp. (2) is satisfied. In the talk we discuss various analytic and geometric characterizations of such spaces, provide examples, give classifications in low dimensions and discuss a few applications. Further we state some open problems. The details can be found in our joint work "Two natural generalizations of locally symmetric spaces", Diff. Geom. Appl. 2 (1992), 57 - 80, and in forthcoming papers.

A. Bobenko (joint work with M. Babich):

Willmore tori with umbilic lines and minimal surfaces in hyperbolic space

The main result is a construction of the Willmore tori with umbilic lines (these tori are defined as extremals of the functional

$$W = \int_M H^2\, dS,$$

where H is the mean curvature and dS is an area element). The tori constructed possess the following properties:

(1) There is a plane A (infinity plane) intersecting M orthogonally and decomposing it into 3 parts $M = M_+ \cup M_0 \cup M_-$ (lying respectively above A, on A and below A),

(2) $M_\pm$ are minimal surfaces in hyperbolic spaces, realized as upper and lower (with respect to A) half spaces with the Poincaré metric,

(3) M_0 is an umbilic set.

The construction is based on the theory of integrable equations. Using the finite-gap solutions of the equation $\Delta u = \cosh u$ (Gauß equation) the explicit formula for the immersion is obtained. The simple tori constructed are of rectangular conformal type with closed mean curvature lines. The lowest possible genus of the spectral curve determining these tori is 3.

U. Brehm (joint work with K. Sarkaria):

Linear versus PL embeddability of simplicial complexes

Let K be a simplicial complex. A linear embedding of K in $\mathbf{R}^n$ is an injective mapping $f: K \to \mathbf{R}^n$, which maps each simplex of K linearly on a simplex in $\mathbf{R}^n$. A *PL (= piecewise linear) embedding* of K is a linear embedding of some subdivision of K. Let $K^{(r)}$ denote the r-th derived complex. The question arises, whether PL embeddability implies linear embeddability in the case $n = 2d$, where $d = \dim K$. Note that K can be always embedded in $\mathbf{R}^{2d+1}$.

Theorem: For each $d \geq 2$, $r \geq 0$, $n \geq 3$ with $d \leq n \leq 2d$ there exists a simplicial d-complex K, which is PL embeddable in $\mathbf{R}^n$, but $K^{(r)}$ is not linearly embeddable in $\mathbf{R}^n$.

Theorem: For each $d = 2^k$ ($k \geq 1$) there exists a triangulation K of $\mathbf{R}P^d \setminus \int B^d$ such that K is PL embeddable but not linearly embeddable in $\mathbf{R}^{2d-1}$. (B^d denotes a d-ball).

Yu. Burago:
New results on Alexandrov spaces

The talk may be divided in two parts.

1) A very short review of the Alexandrov spaces theory. The goal of this part is to show that this theory seems now to be developed almost as far as Riemannian geometry.

2) New results on quasigeodesics which mainly belong to my graduate student Anton Petrunin (and is not published yet). He gave a very convenient definition of quasigeodesic (shortly qug) in multidimensional case (the 2-dimensional case was studied by A.D. Alexandrov). Base results:

1. Converging (compactness) theorem.
2. Existence theorem (G. Perelman). Applications of qug.
3. Theorem on nonexistence of a collaps spaces with curvatures ≥ 1 to a space with $R > \pi/2$.
4. Generalized Libermann Lemma: A shortest of the boundary of an Alexandrov space is qug for this space.
5. Gluing theorem: If the boundaries of two Alexandrov spaces (with the same curvature restrictions) are isometric, then the result of gluing of those spaces along an isometry is an Alexandrov space.

F. Burstall:
Dressing orbits of harmonic maps

The fundamental results of Zakharov-Shabat and Uhlenbeck show that harmonic maps $\varphi: \mathbf{R}^2 \to G$ – a compact Lie group are in essentially bijective correspondence with certain holomorphic maps, the *extended solutions*, $\Phi: \mathbf{R}^2 \to \Omega G$ – the group of based maps $S^1 \to G$. This correspondence is realized in the diagram

$$\begin{tikzcd} \mathbf{R}^2 \arrow[rr, "\Phi"] \arrow[dr, "\varphi"'] & & \Omega G \arrow[dl, "ev_{-1}"] \\ & G & \end{tikzcd}$$

There is a polar decomposition of the free loop group $\Lambda G^{\mathbf{C}}$ as a product $\Omega G \cdot \Lambda^+ G^{\mathbf{C}}$ where the latter group consists of those loops extending holomorphically to the disc. Thus $\Omega G \cong \Lambda G^{\mathbf{C}}/\Lambda^+ G^{\mathbf{C}}$ and we have an action of $\Lambda G^{\mathbf{C}}$ on ΩG.

Ohnita-Guest have shown that this action preserves extended solutions.

On the other hand, a large class of extended solutions arise as the orbit of the identity in ΩG under certain 1-complex-parameter subgroups of $\Lambda G^{\mathbf{C}}$. These solutions include all the harmonic maps of finite-type discussed by Burstall-Ferus-Pedit-Pinkall (+ thus most harmonic maps of tori in spheres and complex projective spaces).

The simplest non-constant harmonic maps $\mathbf{R}^2 \to G$ are the homomorphisms. One can show that essentially all the extended solutions obtained in the last paragraph are in a ΛG^+-orbit of such a homomorphism.

W. Degen:

Geometric Chebyshev Approximation

In computer aided geometric design (CAGD) good approximations of given curve or surface segments are needed, the approximants being polynomial or rational. Using a transversal vector bundle with Euclidean metric and suitable conditions for admissible approximants, one can define a differentiable deviation function d for each.

Applying the non-linear approximation theory (Rice, Meinardus/Schwedt) we get an alternant theorem analog to Chebyshev's famous theorem characterizing best approximants in the class of polynomial curves of a certain degree.

Another question concerns good (but not best) approximants and the problem to determine the approximation order (with respect to the length of the given segment tending to zero). Using rational cubic approximants instead of polynomial ones we improve a result of de Boor, Höllig and Sabin. They obtained, with a contact of order two at both endpoints, the order six. We could, with a third order contact, attain the order eight. A thorough analysis of the asymptotic behaviour answers the existence problem.

J.-H. Eschenburg (joint work with R. Tribuzy):

$(1,1)$-geodesic immersions of Kähler manifolds

Let $(M^{2m}, \langle\, ,\, \rangle, J)$ be a Kähler manifold. An isometric immersion $f: M \to$ into a Riemannian manifold N is called $(1,1)$-geodesic if $\alpha(X,Y) + \alpha(JX,JY) = 0$ for any $X, Y \in TM$. Such an immersion is minimal. In fact, it is characterized by the property that $f \circ c$ is a minimal immersion for any holomorphic curve $c: U \to M$, $U \subset \mathbf{C}$ open. If N is also Kähler, (anti-)holomorphic immersions are $(1,1)$-geodesic. We show that the geometry of N restricts the dimension m of possible $(1,1)$-geodesic immersions, and we give compact examples of such immersions with $m \geq 2$ which are not (anti-)holomorphic, where $N = G_p(\mathbf{C}^n)$. Some of these examples in $G_{n-2}(\mathbf{C}^n)$ have the maximal possible dimension m. We find examples also in $G_p(\mathbf{R}^n)$, e.g. he Gauß map of a standard embedded Hermitean symmetric space.

E. Heil:
The "Theorema egregium" for hypersurfaces

For surfaces in 3-dimensional Euclidean space the Gauß curvature, but not the mean curvature, is determined by the metric. For hypersurfaces in $(n+1)$-space the situation is different because of the rigidity theorem of Beez and Killing: If the Weingarten map has rank at least 3, then the hypersurface is determined up to motions. How can the curvature functions $K_1 = H, K_2, \ldots, K_n = K$ be calculated from the metric? Answers are collected which are scattered in the literature of the last 100 years.

G. Huisken:
Singularities of geometric evolution equations

The mean curvature flow $\frac{d}{dt}F = \vec{H}$, where a hypersurface $F: M^n \hookrightarrow \mathbf{R}^{n+1}$ is moved in direction of its mean curvature vector $\vec{H}$, has a very similar structure to the Ricciflow $\frac{d}{dt}g_{ij} = -2R_{ij}$, where a Riemannian metric g is deformed in direction of its Ricci curvature. Both are parabolic systems which imply nonlinear heat equations for the relevant curvatures of the hypersurface or the Riemannian metric. Singularities can occur, and the natural blowuprate for the curvature is $(T-t)^{-1}$, when T is the maximal time of existence for a smooth solution. It is useful then to distinguish between singularities, where the curvature can be bounded by $c(T-t)^{-1}$, (type I singularities), and those of a higher blowuprate, (type II singularities). It is shown that type I singularities of the mean curvature flow are asymptotically selfsimilar and a partial classification is given. Some results on type II singularities for these evolution equations are obtained under additional curvature assumptions.

M. Kanai:
Geometric structures invariant under dynamical systems

I would like to talk about interplay between geometry and the theory of dynamical systems: More precisely,

Problem 1: Find a geometric structure (e.g. an affine connection, a conformal structure, etc.) that is invariant under a given dynamical system (i.e., a smooth action of a noncompact group like $\mathbf{Z}$, $\mathbf{R}$, lattices in semisimple Lie groups), and

Problem 2: Show that the invariant geometric structure is "symmetric" under an appropriate condition on the dynamical system.

If we can solve these two problems for a given dynamical system, it can be expected that the "symmetry" of the dynamical system itself is concluded from that of the invariant geometric structure (rigidity phenomena). I shall exhibit a few examples in which this program works.

H. Karcher:
The genus one helicoid

All previously known embeddings of compcat punctured Riemann surfaces as complete minimal surfaces in $\mathbf{R}^3$ had finite total curvature and meromorphic Gauß map - except for the helicoid if parametrized by C. David Hoffman, Fushang Wei and I were able to add one handle to the helicoid producing a complete minimal embedding of a torus punctured in one point - with the Gauß map having an essential singularity there.

Some older minimal surfaces were used to explain the connections between the geometry of the surface and its Weierstraß data - the Gauß map g and the complex differential dh of the height function - which determine the surface via

$$F(z) = \mathrm{Re} \int_{*}^{z} (\frac{1}{2}(\frac{1}{g} - g), \frac{i}{2}(\frac{1}{g} + g), 1)\, dh\,.$$

The present surface is parametrized by a *rhombic* torus because 180° rotation around the vertical axis is an antiholomorphic involution with only *one* fixed point component. The differential forms dg/g and dh have double poles at the puncture to immitate the asymptotic behaviour of the helicoid. Then dh has no other poles and its zeros at the two vertical points of the Gauß map; these have to be on the horizontal line on the surface, i.e. on one diagonal of the rhombic fundamental domain. Rotation around the normal at the intersection of the two lines on the surface is an involution, the corresponding quotient map is Möbius-normalized (almost) to the Weierstraß p-function $\wp$. Its differential equation

$$(\wp')^2 = -\frac{2}{\cos\rho}(\wp^3 - \wp - 2i\sin\rho\,\wp^2)$$

also describes the torus. We have

$$dh = e^{-i\pi/4}(\wp - i\varepsilon)\frac{d\wp}{\wp'}\,.$$

Since the other fixed points of this normal rotation are the finite-valued branch points of $\wp$ we have ε determined via period integrals of $\wp$. With more complicated arguments of this type also dg/g is first determined up to two parameters (the positions of the branch points of g) and then the parameters are determined from a linear system whose coefficients are elliptic integrals. - Next one has to prove that the so determined Weierstraß data have all the symmetries and other qualitative properties which we want the surface to have. This succeeds except for one property: The fixed points of the normal rotation - which we are forced to be at the same level - also have to lie on that normal. This is achieved by an intermediate value argument, where, at present, the occurence of opposite signs relies on (real) numerical integrations.

P. Kohlmann:
Two constant mean curvatures

In 1971 Münzner proved that a strictly convex hypersurface piece in $\mathbf{R}^{n+1}$ with constant first mean curvature $E_1 = K_1 + \ldots + K_n$ and one more constant higher order mean curvature $E_r = \sum K_{i_1} \cdot \ldots \cdot K_{i_r}$ is necessarily an open part of a sphere.

His method was based on a Weitzenböck formula for ΔE_r. A more algebraical approach in the general case E_r, E_s const. > 0 $(1 \leq s < r \leq n)$ leads to an identity containing a polynomial in the principal curvatures and a quadratic form in the coefficients of ∇II, where II is the shape operator. The semi-definiteness of the quadratic form can be shown for many combinations s, r, n by application of the Bullen/Marcus inequality and explicit representations as sums of squares involving combinatorial identities for binomial coefficient expressions. Generalized to spaces of constant curvature C one extracts

$$\sum_{i<j}(K_iK_j + C)(K_i - K_j)^2(\delta_{ij}E_{s+1}\delta_{ij}E_r - \delta_{ij}E_s\delta_{ij}E_{r+1}) = 0$$

Let $R := \{(n,r,s) \in \mathbf{N}^3 \,|\, s < r \leq n \text{ and } (s = 1 \text{ or } r = n \text{ or } (r,s) = (3,2) \text{ or } n \leq 6)\}$.

Theorem: Let $(n,s,r) \in R$ and M be a convex hypersurface piece in a standard space form with curvature C with E_s, E_r const. > 0 and sectional curvature not smaller $-C$ if $C > 0$. Then

$C = 0$: M is an open part of a cylinder or a sphere,

$C > 0$: M is an open part of a metric sphere,

$C < 0$: M is an open part of a metric sphere or a horosphere or a hypercylinder (tube around a geodesic).

O. Kowalski:
Nonhomogeneous relatives of symmetric spaces and 3-dimensional Riemann spaces with constant Ricci eigenvalues

A) All locally nonhomogeneous and locally irreducible Riemann spaces are determined which have the same curvature tensor as a fixed symmetric space. Such nonhomogeneous examples exist only if the symmetric space is of the form $H^2(-\lambda^2) \times \mathbf{R}^d$, or $S^2(\lambda^2) \times \mathbf{R}^d$, $d \geq 1$. Then all the solutions are given (locally) by an explicit formula depending on $d+1$ (essential) arbitrary functions of one variable. (Done in collaboration with E. Boeckx and L. Vanhecke.)

B) A classification has been done of all 3-dimensional Riemann spaces with the prescribed constant Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3$. It was proved that the germs of the metrics always depend on two arbitrary functions of one variable. Explicit examples have been given of locally nonhomogeneous spaces with the constant Ricci eigenvalues $(\frac{1}{4}\lambda^2, \frac{1}{4}\lambda^2, -2\lambda^2)$, or $(\frac{2}{9}\lambda^2, \frac{2}{9}\lambda^2, -2\lambda^2)$ respectively. Such Ricci eigenvalues are never acquired by any 3-dimensional homogeneous Riemann space (according to J. Milnor).

N. Kuiper (joint work with P. Waterman):

Triangle group actions on hyperbolic 4-space H^4 and complete hyperbolic metrics on plane bundles over surfaces

Discrete group actions Γ on hyperbolic 4-space H^4, of the triangle group $T(2,u,n) = \{\zeta_1, \zeta_2, \zeta_3 : \zeta_1^2 = \zeta_2^u = \zeta_3^n = \zeta_3, \zeta_2, \zeta_1 = 1\}$ have been used to obtain metries $H^4/\pi_1(M_g)$ on a plane-bundle space with selfintersection $e = \chi^\perp \neq 0$ over a closed surface M_g, $\chi = -2(g-1)$, with $\pi_1(M_g) \subset \Gamma$ a subgroup without torsion of finite index. Recall that all cases $|\chi^\perp/\chi| < 1/3$ can be so realized as metrics on plane bundles.

Theorem: With this method $|\chi^\perp/\chi| < 1$.

Proof: The deformation component of an *algebraic* (not necessarily discrete) action is a rectangle or a point (P. Waterman). In the interesting cases Γ is characterized by (fixed point $=$) $\mathrm{Fix}\,\zeta_2 = X_2$, $\zeta_2 \sim (z_1, z_2) \to (z_1 e^{2\pi i/u}, z_2 e^{2\pi i p/u})$ in $(z_1\bar{z}_1 + z_2\bar{z}_2 < 1) = H^4 \subset \mathbf{R}^4 = \mathbf{C}^2$, $\zeta_1 \sim (z_1, z_2 \to (-z_1, z_2)$, $\zeta_3 \sim (z_1 e^{2\pi i/n}, z_2 e^{2\pi i p_3/n})$. That is by numbers p and p_3, $1 < p < u/2$, $1 \leq p_3 < n/2$. The action Γ has an immersed core surface $\gamma\colon H^2 \overset{\mathrm{PL}}{\hookrightarrow} H^4$ with vertices in $\Gamma(X_2)$ and in $\Gamma(X_3)$ with normal micro bundle $\mu(\gamma)$. The quotient by $\pi_1(M_g)$ gives a quotient normal bundle over M_g. $|\chi^\perp/\chi|$ can be calculated and it is $|\chi^\perp/\chi| < 1$. Then this holds in particular for discrete actions. For $\chi = -10000$ one finds realizations for all $|\chi^\perp/\chi| < 0.3760$. For $\chi = -2$, $g = 2$, $\chi^\perp = 1$ and $|\chi^\perp/\chi| = 1/2$ can be realized as complete hyperbolic (sectional curvature $K = -1$) manifolds.

R. Kusner (joint work with L. Hsu and J. Sullivan):

Minimizing the squared mean curvature integral for surfaces in space forms

We minimize a discrete version of the squared mean curvature integral for polyhedral surfaces in three-space using Brakke's surface evolver. Our experimental results support the conjecture that the smooth minimizers exist for each genus and are stereographic projections of certain minimal surfaces in the three-sphere.

K. Leichtweiß:

On inner parallel bodies in the euclidean and the equiaffine differential geometry

In 1941 G. Bol investigated inner parallel bodies of convex bodies in the Euclidean plane in order to get a simple proof of the isoperimetric inequality. It is remarkable that to the definitions and properties of such inner parallel bodies there exist analogous definitions and properties with respect to the equiaffine differential geometry. This will be pointed out, mostly in the two-dimensional case, for smooth and for arbitrary convex bodies with the help of computer drawings. The equiaffine inner parallel bodies have some practical meaning as "floating bodies" in the construction of ships.

E. Leuzinger:
The geometry of ends in locally symmetric spaces

A remarkable theorem of G.A. Margulis asserts that lattices in irreducible symmetric spaces of non-compact type and rank ≥ 2 are *arithmetically defined*. The geometry of locally symmetric spaces is thus inextricably linked with the theory of linear algebraic groups (defined over $\mathbf{Q}$). The key notions are those of $\mathbf{Q}$-*flats* resp. $\mathbf{Q}$-*Weyl chambers*. These are contained in (maximal) $\mathbf{R}$-flats (resp. $\mathbf{R}$-Weyl chambers) but in general have lower dimensions, i.e. $\mathbf{Q}$-rank $\leq$ $\mathbf{R}$-rank.

A lattice Γ is co-compact if and only if its $\mathbf{Q}$-rank is zero (Borel-Harish-Chandra). This is actually a relatively rare case; in general $V = \Gamma\backslash X$ is *not* compact - though still of finite volume - and thus has interesting ends which are "almost isometric" to *Siegel sets* in X. Such a Siegel set $\mathcal{S} \subset X$ in turn consists of asymptotic $\mathbf{Q}$-Weyl chambers. We determine an open subset of $\mathcal{S}$ which is embedded into V under the canonical projection $\pi: X \to V = \Gamma\backslash X$. The result can then be used to describe (the first) examples of "Tits geometry" for non-simply connected spaces of non-positive curvature.

C. Olmos (joint work with J.-H. Eschenburg):
A rank rigidity theorem for compact manifolds

In a recent work Heintze, Palais, Terng and Thorbergsson define the concept of k-flat homogeneous compact Riemannian manifold: every geodesic is contained in a k-flat and given two pairs (P_1, Σ_1^k), (P_2, Σ_2^k) there is an isometry of the manifold sending one pair into another, where P_i is a point in the manifold and Σ_i^k is a k-flat through it ($i = 1, 2$). They prove that those manifolds are symmetric, but their proof is rather long and uses classification results of Wolf. (For $k = 1$ it was well known: 1-flat homogeneous is equivalent to two point homogeneous.) We give a conceptual proof (relaying on Berger/Simons holonomy theorem) of a more general result. Namely,

Theorem: Let M be a compact manifold of rank $\geq k$ and assume that $I(M)$ acts transitively on the family of k-flats of M. Then M is locally symmetric. Moreover, if the k-flats are tori M is globally symmetric.

G. Paternain:

Complete integrability of convex. Hamiltonians topological obstructions on manifolds with such flows. Relations between topology and topological entropy

Completely integrable geodesic flows (and more generally convex Hamiltonians) appear to exist an very special manifolds, namely those manifolds for which the loop space homology grows polynominally. We have proved that this is the case for very important cases, particularly under conditions on the first integrals that seem "generic". This is accomplished by looking at the topological entropy of such flows. The vanishing of the entropy implies via Morse theory the topological restrictions.

L. Polterovich (joint work with Y. Eliashberg):

Bi-invariant Finsler metrics on the group of Hamiltonian diffeomorphisms

Consider a bi-invariant metric on the group $\mathcal{D}$ of compactly supported Hamiltonian diffeomorphisms of an exact open symplectic manifold M. Given a subset, say $X \subset M$, define its *displacement energy* as the distance between the identity map and the set of all diffeomorphisms from $\mathcal{D}$ which push X away from itself. Our basic observation is that the displacement energy of every open subset does not vanish.

As a consequence we show that the pseudometric ρ_p on the group $\mathcal{D}$ generated by the L_p-norm on the Lie algebra *is not a metric* provided $p < \infty$. Moreover, it turns out that in this case ρ_p is proportional to the absolute value of the Calabi homomorphism. These results demonstrate a contrast between the cases $p < \infty$ and $p = \infty$. As it was shown earlier by H. Hofer, ρ_∞ is a genuine metric on $\mathcal{D}$.

K. Polthier (joint work with U. Pinkall):

Computing discrete minimal surfaces and their conjugates

We presented a new algorithm to compute discrete minimal surfaces bounded by a number of fixed or free boundary curves. The algorithm works on an arbitrary triangulated data set, it is therefore independent of the genus and can handle singular situations as e.g. triple lines.

Additionally we gave for discrete harmonic maps an algorithm computing the discrete conjugate harmonic map. In case of a planar minimal surface this leads exactly to the 90° rotation, that means the conjugate minimal surface in this case. It also respects symmetry properties during the conjugation, i.e. straight lines become planar symmetry lines and vice versa.

Therefore it can be used to compute minimal surfaces whose existence is proved via the conjugate surface construction.

H. Scherbel:

Hamburger's theorem on umbilical points

The theorem of Hamburger states that the index j of an isolated umbilical point N is not greater than 1. Bol tried to prove this, and T. Klotz filled a gap that occurred in his arguments. But as it turned out, she didn't succeed completely. We use the same rather geometrical methods to solve this problem and provide the missing arguments. The main steps are the following:

(1) $j(N) \leq 1 \Leftrightarrow w(K) \geq 0$, where w is the winding number of K around $(0,0)$, and $K := z^2 f_{zz} =: x + iy$.

(2) For the study of the zeros of x and y we construct for the corresponding coefficients a_λ the main reference curves H_λ.

(3) Applying 2 sets of rules (the "circumvention rule" and the "closing rule") to each H_λ, we obtain a closed curve $\overline{H}_\lambda$. The sum of all the winding numbers of them provides us with a lower limit for $w(K)$.

D. Schüth:

Isospectrally deformable Riemannian manifolds diffeomorphic to Heisenberg manifolds

In 1991, Hubert Pesce and He Ouyang showed that continuous isospectral deformations $(\Gamma\backslash G, g^t)$, where G is a 2-step nilpotent Lie group, Γ a discrete cocompact subgroup of G and g^t a continuous family of left invariant metrics, can be obtained *only* in the "Gordon-Wilson-way", i.e. there is necessarily a continuous family of "almost inner" automorphisms $\phi_t \in \mathrm{AIA}(G;\Gamma) := \{\phi \in \mathrm{Aut}(G) | \forall\gamma \in \Gamma\ \exists a \in G : \phi(\gamma) = I_a(\gamma)\}$ s.t. $g^t = \phi_t^* g^0$. In particular, there are *no* non-trivial isospectral deformations $\Gamma\backslash H_m, g^t)$ where H_m is the $(2m+1)$-dimensional Heisenberg group, Γ a discrete cocompact subgroup and g^t left invariant metrics, because $\mathrm{AIA}(H_m;\Gamma)$ is equal to $\mathrm{Inn}(H_m)$, the group of inner automorphisms, which of course give rise only to trivial deformations.

But one can show that there are nontrivially isospectrally deformable Riemannian manifolds arbitrarily close to certain such $(\Gamma\backslash H_m, g)$:

Let $H := H_2$ be the 5-dimensional Heisenberg group with Lie algebra $\mathcal{H}$ spanned by $\{X, Y, V, W, Z\}$ with non-trivial Lie brackets $[X, W] = Z$, $[Y, V] = Z$, and endow it with the left invariant metric g_1 s.t. the left invariant vectorfields X, Y, V, W, Z are orthonormal. Then (H, g_1) is isometric to (G, g_1), where G is the simply connected (solvable) Lie group with Lie algebra $\mathcal{G}$ spanned by $\{X, Y, V, W, Z\}$ and non-trivial Lie brackets $[X, W] = Z$, $[Y, V] = Z$, $[X, Y] = V$, $[X, V] = -Y$, and where g_1 is now the G-left invariant metric which makes the G-left invariant vectorfields X, Y, V, W, Z orthonormal. If we let $\Gamma \subseteq G$ resp. $\tilde{\Gamma} \subseteq H$ be generated by $\exp\{2\pi X, Y, V, \frac{W}{2\pi}, \frac{Z}{2}\}$ (where $\exp: \mathcal{G} \to G$ resp. $\exp: \mathcal{H} \to H$ are the group exponential mappings), then Γ and $\tilde{\Gamma}$ are isomorphic, and there is also an isometry between the quotients $(\Gamma\backslash G, g_1)$

and $(\tilde{\Gamma}\backslash H, g_1)$. $\phi_t \in \mathrm{Aut}(G)$ with $\phi_{t*}: X \mapsto X, Y \mapsto Y, V \mapsto V, W \mapsto X + tZ, Z \mapsto Z$ can be shown to be $\in \mathrm{AIA}(G;\Gamma)$, but not inner. Furthermore it fulfills a certain technical condition that allows us to apply a theorem of Carolyn Gordon and Dennis DeTurck and to conclude that $(\Gamma\backslash G, g^t)$ with $g^t = \phi_t^* g$ is an isospectral deformation for any G-left invariant metric g. If we especially choose $g := g_1$, then this deformation is trivial, because g_1^t happens to be equal to $I^*_{\exp tX} g_1$. But if we start with g_α $(0 < \alpha < 1)$ that makes the G-left invariant vectorfields $X, Y, \alpha V, W, Z$ orthonormal, then there is no obvious isometry between $g_\alpha^t = \phi_t^* g_\alpha$ and g_α on $\Gamma\backslash G$, hence $(\Gamma\backslash G, g_\alpha^t)$ with α fix, t varying is a good candidate for a non-trivial deformation if $\alpha < 1$.

Indeed, one can show the nontriviality of the deformation at least for every $(0.157\ldots =)$ $(4\pi^2+1)^{-1/2} < \alpha < 1$ by the following argument: Consider the two free homotopy classes of $\Gamma\backslash G$ corresponding to the conjugacy classes $[\exp Y]_r$ and $[\exp \frac{W}{2\pi}]_r$. The g_α^t-shortest geodesic loops in these classes foliate each a certain closed submanifold of $\Gamma\backslash G$ which is g_α^t-perpendicular to X (always under the above condition on α). The g_α-distance of these two submanifolds ist $\mathrm{dist}(t, 2\pi Z)$, thus non-constant in t. Now the non-triviality of the deformation follows by the countability of the set of free homotopy classes.

U. Simon (joint work with U. Pinkall and A. Schwenk-Schellschmidt):

Conformal and projective structures: Codazzi and Monge-Ampère equations

Let M be a connected and oriented C^∞-manifold of $\dim M = n \geq 2$. Let $\mathcal{C}$ be a conformal class of semi-Riemannian metrics and $\mathcal{P}$ a class of affine connections ∇ which are torsionless and Ricci-symmetric (which means: any connection admits a parallel volume form) and mutually projectively equivalent. -

Definitions: (i) A Codazzi tensor h relative ∇ is a symmetric $(0,2)$-field satisfying $(\nabla_u h)(v,w) = (\nabla_v h)(u,w)$ for tangent fields u, v, w. (ii) h is called to be generated by a function if there exists $f \in C^\infty(M)$ s.t. $h = h(f)$, where $h(f)(v,w) := (\mathrm{Hess}(f))(v,w) + \frac{1}{n-1} f \cdot \mathrm{Ric}(v,w)$ and Hess denotes the ∇-covariant Hessian. (iii) $\mathcal{P}$ and $\mathcal{C}$ are called Codazzi related if there exists a bijective mapping $B: \mathcal{P} \to \mathcal{C}$ such that $h := B(\nabla)$ satisfies Codazzi equations relative to ∇.

Lemma: $\mathcal{P}$ and $\mathcal{C}$ are Codazzi related if there exists one pair (∇, h) satisfying Codazzi conditions.

We study (1) Codazzi tensors on projectively flat spaces; (2) Generating Codazzi tensors by functions within a class $\mathcal{P}$; (3) Monge-Ampère operators on projectively flat manifolds.

As application we study the problem to find, for a given projectively flat connection $\nabla \in \mathcal{P}$ on $S^n(1)$, the existence and uniqueness of hyperovaloids in real affine space with ∇ as equiaffine conormal connection. This problem is an affine version of the Minkowski problem for hyperovaloids in Euclidean space.

I. Sterling:
Constant mean curvature cylinders

Multisoliton solutions to the sinh-Gordon equation $\Delta w + \frac{1}{2}\sinh(2w)$ arise via a geometric construction of constant Gauß ($K > 0$) curvature surface due to Bianchi and Bäcklund.

The parallel surfaces have constant mean curvature and their isospectral deformations appear as colliding bubbles.

A classification theorem of Pinkall-Sterling on constant mean curvature tori partially extends to cylinders.

E. Teufel:
On integral geometry in Riemannian spaces

The classical integral geometry in the sense of W. Blaschke and L. A. Santaló happens in spaces of constant curvature. The integral geometry on surfaces was initiated by W. Blaschke and M. Haimovici and further developed by L. A. Santaló.

Here we add a few contributions to the integral geometry in Riemannian spaces: densities of geodesics and strips; integral formulas (intersection formulas); isoperimetric inequalities for closed curves in spaces of non-positive sectional curvature, generalizing an isoperimetric inequality of Th. F. Banchoff and W. F. Pohl; an extrinsic version of an isoperimetric inequality of Ch. B. Croke for hypersurfaces in spaces of non-positive sectional curvature; inequalities for Gleichdicke on surfaces, related to Barbiers theorem.

K. Voss:
Bonnet surfaces in spaces of constant curvature

An immersion $S: M^2 \to M^3(k)$ (space of constant curvature k) is called a *Bonnet surface*, if there is a second surface S^*, which has the same induced metric and mean curvature function, but is not congruent to S. For $k = 0$ the problem has first been posed and treated by O. Bonnet (1867). There are pairs S, S^* and 1-parameter families of Bonnet surfaces. E. Cartan (1942) determined all real Bonnet families with $H \neq c$. I give simple proofs of the following results:

1) Bonnet pairs correspond to the solutions of two PDEs, depending on 4 initial value functions of one variable.

2) Generalization of Cartan's results on Bonnet families with non constant H to all spaces $M^3(k)$.

3) An explicit example of a Bonnet family with $H \neq c$ (formulas and pictures).

4) In hyperbolic space ($k = -1$), there exist Bonnet families, which do not consist of Weingarten surfaces, but these surfaces are Willmore surfaces. This surface class is

more general than the Weingarten Bonnet surfaces.

C. Wang:
Mõbius geometry for hypersurfaces in S^4

Let $x: M \to S^4$ be an immersed hypersurface with different principal curvatures λ, μ and ν at each point. Let $\{t_1, t_2, t_3\}$ be the unit principal vector fields on M corresponding to λ, μ and ν. Let $\{\omega^1, \omega^2, \omega^3\}$ be the dual basis for $\{t_1, t_2, t_3\}$. Then we have the Mõbius invariant function W and 1-forms $\theta^1, \theta^2, \theta^3$ defined by

$$W = \frac{\nu - \mu}{\lambda - \mu}, \quad \theta^1 = (\mu - \nu)\omega^1, \quad \theta^2 = (\lambda - \nu)\omega^2, \quad \theta^3 = (\lambda - \mu)\omega^3.$$

Theorem: $\{\theta^1, \theta^2, \theta^3, W\}$ forms a complete invariant system for $x: M \to S^4$ which determines x up to Mõbius transformations.

Using this theorem we can determine all Mõbius homogeneous hypersurfaces in S^4.

G. Wiegmink:
Vector fields of minimal total bending on S^3

On compact orientable Riemannian manifolds (M, g) of Euler number zero we want to distinguish those globally defined unit vector fields that are most parallel (globally parallel vector fields need not exist) in the sense that the so-called total bending

$$B(X) := \frac{1}{\mathrm{vol}(T^1M)} \int_{T^1M} g(\nabla_v X, \nabla_v X)\, d\,\mathrm{vol}(v)$$

is minimal (∇ the Levi-Civita covariant derivative).

It is open wheter in general a minimum of B exists, or when $\inf B > 0$.

It turns out that at least on Einstein manifolds Killing fields of length 1, if existing, play a special role. On the standard sphere S^3 these are exactly the Hopf vector fields. The problem of minimizing B here is completely solved for a certain class of vector fields; together with some variational results one is led to the conjecture that for S^3 the functional B attains a minimum exactly at the Hopf vector fields.

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