MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Nonlinear Evolution Equations, Solitons and the Inverse Scattering Transform

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This conference was the third in a series being held in Oberwolfach and was organized by Professors M. Ablowitz (Boulder), B. Fuchssteiner (Paderborn), M. Kruskal (Princeton) and V. Matveev (St. Petersburg).

The participants presented their most recent work in the meeting. This and the marvelous surrounding again created a lively scientific atmosphere with very many stimulating discussions which certainly will influence future directions and will contribute to further progress in the field.

The lecture program covered a broad range of topics such as integrability in multidimensions, inverse problems, continuous and discrete systems, particle and quantum systems, algebraic and geometrical aspects of nonlinear evolution equations as well as computational and algorithmic aspects, Painlevé analysis and Darboux transformations, Solitons and Positons, Surreal Numbers, soliton equations in relativity and differential geometry and various fields of applications from the study of water waves over symplectic integration to numerical chaos. Furthermore, in two sessions the computer algebra package 'MuPAD' was expertly demonstrated by Dr. Waldemar Wiwianka.

To our deepest regret we have to inform the conference participants that Dr. Wiwianka tragically died in a traffic accident shortly after the meeting.

VORTRAGSAUSZÜGE

M. Ablowitz:

Numerical Chaos, Roundoff Errors and Homoclinic Manifolds

The Nonlinear Schrödinger equation (NLS) is computed using various numerical algorithms. Depending on a parameter in the initial conditions we find two aspects of chaos. The chaos in the first case, using a "standard" numerical algorithm is due to inadequate grid refinement, and in this case good spatial temporal chaos is observed. The chaos disapppears as the mesh is refined. In this parameter regime, if we use a numerical scheme suggested by the inverse transform ("integrable discrete NLS") numerical chaos is not seen. In the other parameter regime, another – and perhaps more troubling – situation occurs. Here temporal chaos is generated by miniscule errors, such as those due to roundoff. The development of this temporal chaos occurs for both "integrable discrete" as well as higher order Fourier split-step methods. The fundamental analytical features inherent in this problem will be discussed. Other equations possess similar characteristics. These observations apply equally well to them as well.

R. Beutler

Positon Solutions of the Continuous and the Discrete Sinh-Gordon Equation

It has been shown for several integrable equations that they have solutions with very interesting properties, the so-called positon solutions, which are slowly decaying and oscillating at infinity in contrast to solitons with an exponential decay. After their mutual interaction positons recover their original shape without suffering any phase shifts. Solitons pass through positons without being shifted too, while the positons get changed by the solitons in a predictable way. The derivation of positon solutions is sketched for the case of the Sinh-Gordon equation and a discrete version of it. Their properties are discussed in detail. For the discrete case the positons are non-singular, which is in contrast to the positon solutions discovered so far for other nonlinear equations. It is shown that in the continuum limit the results of the discrete Sinh-Gordon equation reduce to those of the continuous case.

A. Bobenko:

Surfaces in Terms of 2 by 2 Matrices: Old and New Integrable Cases

Surfaces in a 3-dimensional Euclidean space are considered. The moving frame for a general surface is described in terms of quaternions $\psi \in H_*$. This description characterizes the spin structure of the immersion. The spin structure of the minimal surfaces is given in terms of the Weierstraß representation. Integrable cases and their deformation families are presented. Some of these cases are well known, some are not well known and some are possibly new. It is shown, in particular, that the surfaces with harmonic inverse mean curvature

$$\partial_Z \partial_{\bar{Z}}\left(\frac{1}{H}\right) = 0$$

are integrable. Here Z is a conformal variable of the first fundamental form. We show also that for all integrable cases considered the immersion function is given by $\psi^{-1}\frac{\partial}{\partial\lambda}\psi$, where λ is the corresponding spectral parameter.

M. Boiti

A new approach to the initial value problem of the Kadomtsev-Petviashvili I equation

It is shown that the theory of the Inverse Spectral Transform can be founded on a new mathematical object, called resolvent. In the specific example of the Kadomtsev-Petviashvili I equation it is shown that all the relevant quantities in the theory, the Jost and the advanced/retarded solutions together with the Spectral Data, can be obtained as a particular reduction of the resolvent. The resolvent satisfies a Hilbert-like identity, that can be used to derive the orthogonality and the analytic properties of the Jost solutions, the characterization equations for the Spectral Data and can also be used to solve the inverse problem.
(Joint work with F. Pempinelli; A. Pogrebkov, Steklov Mathematical Institute, Moscow, R.F.)

L. Bordag:

Qualitative Investigations of the Three Phase Solutions of the Sine-Laplace Equation

We give a complete description of all real three phase solutions of the sine-Laplace (SL) equation. We get smooth real solutions and singular solutions expressed through 3 dimensional θ-functions. All singularities of the solutions are vortices (with topological charge 8π) and anti-vortices (with topological charge -8π). The points, where the solutions are singular, form chains and these chains build up an almost periodic structure. We found one- and two-periodic solutions.
We developed a computer program for direct computation of the parameters of the Riemann surfaces and give graphical representations for all types of solutions. This program can be used not only for hyperelliptic surfaces, but for surfaces with genus > 3 too.
The real solutions of the SL equation are of interest in connection with superconductivity and superfluidity, as well as in geometrical applications.
(Joint work with M.V. Babich, Steklov Mathematical Institute, St. Petersburg, R.F.)

M. Bordemann:

2-dimensional nonlinear sigma models: Zero curvature and Poisson structures

Nonlinear sigma models are solutions of the action functional

$$S[\phi] = \frac{1}{2} \int dt\, dx\, \eta^{\mu\nu}\, h_{ij}(\phi)\, \partial_\mu \phi^i\, \partial_\nu \phi^j$$

i.e. pseudo harmonic maps $\phi : \mathbb{R}^{1,1} \to (M, h)$ where M is a homogeneous Riemann manifold $(M = G/H)$.
1. Define the current w.r.t. the isometry group G by

$$j_\mu[\phi](\xi) = -h_{ij}(\phi)\partial_\mu \phi^i (\xi_M)^j(\phi)$$

then

$$L_\mu = \frac{2}{1 - \lambda^2} \left(j_\mu + \lambda(*j)_\mu\right)$$

gives a zero curvature representation i.e. $(dL)_{\mu\nu} + [L_\mu, L_\nu] = 0$ if and only if M is a Riemannian symmetric space (idea goes back to Pohlmeyer (1976), proof by Giden-Forger (1979) and extended version by MB, Forger, Laartz, Schiper).
2. Using the symplectic structure of the initial conditions, there is a Lax representation:

$$D(x, \lambda) := \partial_x + L_?(x, \lambda),$$

i.e.

$$\dot{D} = [D, L_0]$$

and a classical r-matrix:

$$d_{12}(x,y;\lambda,\mu) = \frac{2\mu}{1-\mu^2}\left(\frac{\mu C}{\lambda-\mu} + \frac{\sigma(\phi(x))}{1-\lambda\mu}\right)\delta(x-y)$$

(C denoting the Casimir in g $\otimes$ g for semisimple g, $\sigma(\phi(x))$ denoting $Ad(g(x))\sigma Ad(g(x)^{-1})$ where σ is the involutive automorphism of g fixing the subalgebra and $\phi(x)$ is represented by $g(x) \in G$) such that

$$\{D_1(x,\lambda), D_2(y,\mu)\} = [d_{12}(x,y;\lambda,\mu), D_1(x,\lambda)] - [d_{21}(y,x;\mu,\lambda), D_2(y,\mu)].$$

The computation of higher Poisson brackets of the d's and D's yields a closed algebra in spite of the field dependence of d. The r-matrix d does **not** satisfy the classical Yang-Baxter equation, but

$$\begin{aligned}
&[d_{12}(x,y;\lambda,\mu), d_{13}(x,z;\lambda,\nu)] + [d_{12}(x,y;\lambda,\mu), d_{23}(y,z;\mu,\nu)] - [d_{13}(x,z;\lambda,\nu), d_{32}(z,y;\nu,\mu)] \\
&= [\frac{2\mu}{1-\mu^2} C_{12}\delta(x-y) + \frac{2\mu}{1-\mu^2} C_{13}\delta(x-z),\ d_{13}(x,z;\lambda,\nu)] \\
&-[\frac{2\nu}{1-\nu^2} C_{13}\delta(x-z) + \frac{2\nu}{1-\nu^2} C_{23}\delta(y-z),\ d_{12}(x,y;\lambda,\mu)].
\end{aligned}$$

R. K. Bullough

Quantum Groups, q-Bosons and Quantum and Classical Integrable Lattice Models

The "quantum groups" arise as natural algebraic structures underlying both the quantum and classical integrable dynamical systems. So far their role is understood in this context only in one space and one time (1 + 1) dimensions. In simplest form they arise as deformations ("q-deformations") of simple Lie algebras. Thus q-deformed $su(2)$ (or $sl(2)$) is, for generators S^+, S^-, S^z (corresponding to Cartan-Weyl basis e, f, h), the universal enveloping algebra deformation $U_q(su(2))$ with generators satisfying $su_q(2)$

$$[S^\pm, S^z] = \mp S^z\ ; \quad [S^+, S^-] = [2S^z] \tag{1}$$

where 'box' x, namely $[x]$, is

$$[x] \equiv (q^x - q^{-x})/(q-q^{-1})\ ; \quad q \in \mathbb{C}. \tag{2}$$

Evidently when $q \to 1$, $[x] \to x$, and the $su_q(2)$ algebra (1) is $su(2)$. "Dual" to this is the q-deformed monodromy matrix T which satisfies $RT_1 \otimes T_2 = T_2 \otimes T_1 R$ and R is the quantum R-matrix (and labels 1, 2 label spaces). The elements of T form a quantum group and there is a 'co-multiplication' $\Delta T \equiv T\dot{\otimes}T$ or $(\Delta T)_{ij} = \sum_k T_{ik}\dot{\otimes}T_{kj}$ (where $\dot{\otimes}$ is now 'co-multiplication') which is isomorphic to the quantum group formed by the T_{ij}. Likewise $su_q(2)$ has a co-multiplication isomorphic to the deformed algebra $su_q(2)$ and both the T_{ij} and the algebra $su_q(2)$ are Hopf algebras, co-algebras with antipodes acting like, but different from, inverse elements. The T_{ij} remain co-algebras (with co-multiplication) when spectral parameters $\lambda, \mu \in \mathbb{C}$ are introduced: $T \to T(\lambda)$ and $RT_1 \otimes T_2 = T_2 \otimes T_1 R \to R(\lambda,\mu)T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda)R(\lambda,\mu)$. This can be seen as a quantum integrability condition since the matrix trace is $[\hat{\Delta}(\lambda), \hat{\Delta}(\mu)] = 0$ where $\hat{\Delta}(\lambda)$ (and $ln\hat{\Delta}(\lambda)$) are generators of quantum integrable Hamiltonians $\hat{H}$ commuting with the $\hat{\Delta}(\lambda)$.

The q-bosons arise through 'group contraction' of the large dimensional representations of $su_q(2)$: three independent elements a, a^+, $N = \mathbb{N}^+$ satisfy

$$[a, N] = a \ , \ [a^+, N] = -a^+ \ , \ aa^+ - qa^+a = q^{-N}, \tag{3}$$

and this is evidently the appropriate q-deformation of the Heisenberg-Weyl algebra. To avoid complication we choose $q = e^\gamma$, $\gamma \in \mathbb{R}$. On the Hilbert space $\mathcal{H}$ spanned by the $|n\rangle$ $n = 0, 1, 2, \ldots$, where $N|n\rangle = n|n\rangle$, one proves for algebra (3) that $a^+a = [N]$, $aa^+ = [N+1]$ using the 'box' notation (so $[N]$ depends on q!). Then on $\mathcal{H}$: $aa^+ - q^{-1}a^+a = q^N$ also, i. e. for fixed q (3) becomes invariant under $q \to q^{-1}$. This algebra extends to a lattice of M points under periodic boundary conditions $M + n = n$, e. g. $a_n, a_n^+, N_n = N_n^+$ satisfy (3) for each n and elements **commute** for different m, n. By canonical transformation $B_n = q^{-\frac{1}{2}N_n}a_n$, $B_n^+ = (B_n)^+$

$$[B_n, N_n] = B_n \ , \ [B_n^+, N_n] = -B_n^+ \ , \ [B_n, B_n^+] = q^{-2N_n} \tag{4}$$

and on $\mathcal{H}^M$: $[B_n, B_n^+] = 1 - QB_n^+B_n$, $Q = 1 - q^{-2}$ with $N_n = -\frac{1}{2}\gamma^{-1}ln(1 - QB_nB_n^+)$. All elements commute on different sites.
We have constructed a number of quantum lattices which are **quantum** integrable. A fundamental one, which uses q-bosons eqn.(3) as dynamical variables, has a local Hamiltonian which however involves interactions on 4 sites n, $n+1$, $n+2$, $n+3$. But a simpler one derivable from this is the q-boson 'hopping model'

$$\widehat{H}_0 = -\frac{1}{2}\sum_{n=1}^{M}\{(B_{n-1}B_n^+ + h.c.) - 2N_n\}. \tag{5}$$

We have solved this quantum lattice model (N. B. it is bilinear in the q-bosons (4) and becomes nonlinear to all orders in terms of ordinary canonically quantised bosons) – solved it in detail for eigenstates and eigenenergies by the so-called 'algebraic Bethe ansatz' and we have calculated its free-energy in thermodynamic limit.
By canonical transformation to the algebra (4) taken on $\mathcal{H}^M$: $\widehat{H}_0 \to \widehat{H}_1 = -\frac{1}{2}\sum_{n=1}^{M}\{(B_{n-1}B_n^+ + h.c) - \frac{1}{\gamma}ln[1 - QB_n^+B_n]\}$ and the solution of $\widehat{H}_0$ solves also $\widehat{H}_1$. The **semi-classical limit** of the algebra (4) on $\mathcal{H}^M$ is

$$\{C_n, N_n\} = iC_n \ , \ \{C_n^*, N_n\} = -iC_n^* \ , \ \{C_n, C_n^*\} = i[1 - 2\gamma C_nC_n^*]$$

$$N_n = -\frac{1}{2\gamma}ln[1 - 2\gamma C_nC_n^*]; \tag{6}$$

$[\cdots]$ are parentheses and $\{\cdots\}$ is the Poisson bracket. The 'q-bosons' (6) are now **classical** q-bosons. The equations of motion derived from the semi-classical limit of $\widehat{H}_1$ is then

$$-2i\frac{\partial}{\partial t}C_n = (C_{n-1} + C_{n+1})(1 - 2\gamma C_n^*C_n) - 2C_n^*C_n \tag{7}$$

which is the classical integrable Ablowitz-Ladik lattice. All of the quantum models become the quantum Nonlinear Schrödinger model in appropriate continuum limit and $\gamma > 0$ is the repulsive case. By combining a classical non-integrable term $\alpha C_n^*C_n^2$ (or a quantum non-integrable term $\alpha B_n^+B_n^2$ in the quantum case) with the classical or quantum Ablowitz-Ladik Hamiltonians it becomes possible to investigate classical or quantum chaos and the breakdown of integrability in these terms – classical or quantum KAM theory for M degrees of freedom including $M \to \infty$ in thermodynamic limit. The quantum lattices in terms of the q-bosons are strongly coupled ordinary boson

systems for large enough $|\gamma|$ becoming weakly coupled for small $|\gamma| > 0$. Their correlation functions have been calculated by conformal field theoretical methods and (for the repulsive cases) show a zero temperature 'phase transition' of super-fluid type i.e. correlation functions asymptotically algebraic → asymptotic exponential decay for small T (temperature) > 0.
(Joint work with N.M. Bogolinkov, Steklov Mathematical Institute, St. Petersburg; G.D. Pang; J. Timonen, Dept. of Physics, University of Jyväskylä, Finland)

F. Calogero

Solvable Dynamical Systems (Classical, Nonrelativistic "Many-Body Problems") in Multidimensions

An extension to multidimensions of (a generalized notion of) Lagrangian interpolation is used to introduce finite-dimensional matrix representations of the (partial) differential operators. It is thus possible to extend to a multidimensional environment various results which were obtained in the past by exploiting such a representation in a one-dimensional context. These applications include the construction of remarkable matrices, convenient techniques to solve numerically eigenvalue problems and partial differential equations of evolution in multidimensions, and the manufacture of (completely or partially) solvable dynamical systems, including some that look like (classical, nonrelativistic) "many-body problems" in multidimensions:

$$m_j \ddot{\vec{r}}_j = \vec{F}_j(\{\vec{r}_k, \dot{\vec{r}}_k \ ; \ k = 1, 2, \ldots, n\}), \quad j = 1, 2, \ldots, n.$$

These "Newtonian" equations of motion are generally rotation-invariant and possibly also translation-invariant, in N-dimensional space. Cases with $N = 1$ and $N = 3$, and with $n = 3$ and $n = 4$ ("few-body problems") have been exhibited, together with their complete solutions. The "forces" $\vec{F}_j$ are appropriately-chosen, time-independent, functions of the "particle coordinates" $\vec{r}_j$ and of their velocities $\dot{\vec{r}}_j$.

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H. W. Capel

Integrable quantum mappings

A review is given of a (nonultralocal) Yang-Baxter structure associated with integrable quantum mappings. The quantum mappings are generated by unitary operators which preserve the fundamental commutation relations and which provide the canonical transformation associated with the discrete-time evolution. The integrability is established by finding a sufficient number of quantum operators which are invariant under the mapping. As examples of the Yang-Baxter structure we present the integrable mappings associated with the lattice analogues of the Korteweg-de Vries and modified Korteweg-de Vries equation together with their quantum invariants.

P. A. Clarkson

Nonclassical Symmetry Reductions and Exact Solutions of Nonlinear Partial Differential Equations

In this lecture I discuss nonclassical symmetry reductions and exact solutions of nonlinear partial differential equations. These are calculated using an adaption of the technique first introduced by Bluman and Cole [J. Math. Mech., **18** (1969) 1025] in their study of symmetry reductions of the linear heat equation. The associated determining equations for the infinitesimals are an overdetermined nonlinear system of partial differential equations which are solved using the method of differential Gröbner Bases. The examples considered are:
(i) the nonlinear heat equation

$$u_t = u_{xx} + f(u),$$

for which several new symmetry reductions are presented for $f(u)$ a cubic and classified in terms of the roots of the cubic; and
(ii) a Boussinesq-type equation

$$u_{tt} + u_{xx} + \alpha u_x u_{xt} + \beta u_t u_{xx} - u_{xxxx} = 0$$

where α and β are arbitrary constants, for which a "two-soliton" solution is generated from a nonclassical symmetry in the case when $\alpha = \beta$, even though the equation appears to be non-integrable for all choices of α and β except the linear case when $\alpha = \beta = 0$.
(Joint work with Liz Mansfield (Boulder & Exeter))

A. P. Fordy

Hamiltonian Flows on Stationary Manifolds

We consider the relationship between the Hamiltonian structures of a bi- (or multi-) Hamiltonian system of PDEs and those of their stationary flows. For simplicity, the lecture is presented in the context of the KdV hierarchy.
In the usual formulation (Bogoyavlenski and Novikov) the first Hamiltonian structure of the KdV hierarchy gives rise to a Lagrangian formulation of the stationary flow. This requires the writing of the kernel of the Hamiltonian operator as the variational derivative of some local functional, which is not possible for the second Hamiltonian structure. Given the Lagrangian formulation, a (generalized) Legendre transformation gives formulae for canonical variables and the Hamiltonian. The first integrals obtained from the 'fluxes' q are the usual KdV integrals, but written in terms of the canonical variables. The second Hamiltonian structure is constructed by means of the Miura map and its differential consequences, when restricted to the finite dimensional stationary manifold (Antonowicz, Fordy, Wojciechowski). With this construction there is not apparent relationship between the Hamiltonian structures of the KdV hierarchy and those of their stationary flows. An alternative construction involves reversing the role of x and t, so that we treat x as the evolution parameter (Fuchssteiner and Oevel; Antonowicz and Blaszak). In this case the Hamiltonian structure is a square matrix whose size equals the x-order of the particular flow (3×3 for the KdV equation). Whereas the usual phase space is just the jet space generated by u and its x-derivatives, now we use the jet space generated by any set of coordinates on the stationary manifold (e.g. the canonical coordinates) and their t-derivatives. The resulting Hamiltonian structures reduce to exactly these of the stationary equations, when we take t-derivatives to be zero.

This whole procedure is clarified and systematized by considering the **spectral problem** zero-curvature representation of the given equation, when written in terms of these new coordinates and when the x or t roles are reversed (Fordy and Harris). The resulting spectral problem is polynomial in the spectral parameter, and a highly reduced form of the general case. It is then not so straightforward to use all the standard constructions, but they can be done.

B. Fuchssteiner

The Camassa-Holm and similar equations

Recently, for the description of bores as well as for solitons, Roberto Camassa and Daryl Holm derived from first principles of water wave theory the equation

$$v_t - v_{xxt} = -3vv_x + 2v_{xx}v + v_{xxx}v \ .$$

They gave the Lax pair for that equation, proved its integrability and bihamiltonian structure. In the present lecture it is shown that the Camassa-Holm equation is related to a factorization with respect to terms obtained from a rescaled KdV-recursion operator. To be precise: The Bäcklund-transformation

$$u := (v - v_{xx}) = (I - D^2)v$$

relates the CH-equation to

$$u_t = \Phi_1(u)\Phi_2^{-1}(u)u_x$$

where $\Phi_1(u)$ and $\Phi_2(u)$ are the following sums of the KdV-recursion operator

$$\Phi_1 = (DuD^{-1} + u), \quad \text{and } \Phi_2 = (D^2 - I)$$

The same method applied to Gardner's equation yields the modified CH-equation as well as other generalizations.
Further application of this method gives the bihamiltonian structure of the new equation, as well as its Lax pair, the action angle map and, by hodograph link, a transformation to the KdV-hierarchy. The hodograph link provides additional information about the master symmetries and the (2+1)-dimensional generalizations.

C. H. Gu

Integrable systems and solitons in space-time $\mathbb{R}^{n+1}$

Integrable systems in $\mathbb{R}^{n+1}$ of the form

$$\frac{\partial\phi}{\partial x_i} = (\lambda_i + [J_i, P])\phi, \quad \frac{\partial\phi}{\partial t} = V[P,\lambda]\phi \qquad (1)$$

are considered, where ϕ, J_i, P, V are $N \times N$ matrices. Moreover, J_i's are diagonal constant matrices, P is off-diagonal and $V[P,\lambda]$ is a differential polynomial of P and a polynomial of the spectral parameter λ of m-th degree.
From the integrability condition of the system (1), we construct the explicit expressions for $V[P,\lambda]$ and obtain nonlinear evolution equations for P.
It is proved that the Darboux transformation for obtaining explicit solutions to the AKNS system in $\mathbb{R}^{1+1}$ is valid for system (1). For the case of $u(N)$, single and multisolitons are constructed. It is

proved that the interaction of solitons is elastic if we consider the amplitude of the complex-valued solutions.

B. Herbst

Symplectic integration of finite dimensional Hamiltonian systems and numerically induced chaos

Since symplectic integrators are designed to preserve the character of Hamiltonian flows they tend to be very efficient numerically. However, they are in general nonintegrable and spurious numerical chaos may occur in the vicinity of homoclinic orbits. Analytical arguments, based on the Melinkov function, as well as direct numerical measurements show that the width of the chaotic region decreases exponentially fast in the discretization parameter. This implies, and is confirmed by numerical studies, that the qualitative behaviour (suitably defined) of symplectic integrators does not depend on their order of accuracy. This is not the case for general (nonsymplectic) integrators.

J. Hoppe

The Dynamics of Relativistic Membranes and Higher Dimensional Integrability

1. I would like to explain how to increase (by one or two) the continuous (space time) dimensions of almost any class of (integrable) systems whose dynamics can be written in terms of $N \times N$ matrices.

2. The light cone gauge description of a relativistic membrane moving in Minkowski space can be greatly simplified by performing a field dependent change of variables which allows the explicit solution of all constraints and a Hamiltonian reduction to an $SO(3,1)$ invariant $2+1$ dimensional theory of isentropic gas-dynamics.

(Joint work with M. Bordemann)

H. Hu

Darboux transformation in differential Geometry

The construction of harmonic maps $R^{1,1} \rightarrow S^3$ with single soliton or multi-soliton property is considered. The solution of Sinh-Gordon equation and the construction of time-like surfaces of constant mean curvature in $R^{2,1}$ are discussed. These results are obtained by using Darboux transformation methods together with some other techniques.

B. Konopelchenko

$\bar{\partial}$-dressing in multidimensions: algebraic curves and nontrivial background

Recent new results obtained by the $\bar{\partial}$-dressing method are reviewed. $2+1$-dimensional and three-dimensional nonlinear integrable systems are discussed. Generalization of the $\bar{\partial}$-dressing method associated with the nonlocal $\bar{\partial}$-problem on algebraic curves of nonzero genus is considered. An integrable nonlinear system on the torus and a $2+1$-dimensional integrable generalisation of the Landau-Lifshitz equation are presented. The embedding of the Harry Dym equation into the $\bar{\partial}$-dressing scheme which allows to bypass the problem with the essential singularity of the eigenfunction is discussed. The extension of the $\bar{\partial}$-dressing technique which covers the case of nontrivial backgrounds in presented. The KP, mKP, DS, Ishimori and the $2+1$-dimensional integrable generalisation of the Sine-Gordon equation are considered as examples.

The Darboux-Zakharov-Manakov three-dimensional system is discussed. This system has important applications in the differential geometry of surfaces, hydrodynamics and other fields. The $\bar{\partial}$-dressing method provides a wide class of exact solutions of the DZM system on the background of the well-known 1 + 1-dimensional principal chiral field model equations. In the scalar case such solutions of the DZM system provide infinite and rich classes of systems of three surfaces with conjugate coordinate lines (old important problem of the theory of surfaces - Darboux ...) and infinite class of hydrodynamical Hamiltonian one-dimensional systems introduced by Dubrovin and Novikov.

M. Kruskal

Surreal Numbers

The surreal numbers are a vast generalization of the real numbers and also of the ordinal numbers (with their commutative Hessenburg arithmetic), yet also markedly simpler to define and prove properties of. They were discovered by John H. Conway (as described in his book "On Numbers and Games") as certain special combinatorial games, but received their name from Donald Knuth (in his short mathematical novel "Surreal Numbers").
A surreal number can most simply be constructed as consisting of arrows, $\uparrow$ or $\downarrow$, arranged in a sequence with the order type of some ordinal number (i. e. a well ordered sequence, finite or infinite). They are ordered in size lexicographically (so $\uparrow\uparrow\downarrow<\uparrow\uparrow<\uparrow\uparrow\uparrow\uparrow$, for example, because $\downarrow$ is less than blank, which is less than $\uparrow$). They also have a quite distinct partial ordering we call "earliness": x is earlier than y (written $x < y$) if x is a proper initial segment of y (where "proper" permits x to be null but not equal to y).
Addition is defined by transfinite recursion to have, at every stage, the earliest value consistent with the sum being a monotone-increasing function of each argument, with respect to the sums of all earlier pairs of arguments. Multiplication and other arithmetic functions have similar definitions, and turn out to have all properties that could reasonably be hoped for, as well as some entirely new ones (involving earliness). It is striking that these properties, even three expressing equalities (such as the commutative, associative and distributive laws of addition and multiplication), follow from such a definition based on inequalities (such as monotonicity) when combined with the "earliest" requirement.
The surreal numbers include the usual real numbers (0 is the null sequence, 1 is $\uparrow$, -1 is $\downarrow$, 2 is $\uparrow\uparrow$, $\frac{1}{2}$ is $\uparrow\downarrow$, $\frac{2}{3}$ is $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots$), infinite numbers (the earliest positive one, called ω, is $\uparrow\uparrow\uparrow\cdots$), infinitesimal numbers (the earliest positive one is $\uparrow\downarrow\downarrow\downarrow\cdots$), and (literally) innumberable combinations and extensions of these. Earliness leads to a lot more structure than is familiar to us.
An elaborate theory of surreal functions can be developed; the one serious missing ingredient at present is integration and related concepts. If that can be remedied, there should be important applications to conventional mathematics, in particular to a rigorous treatment of "asymptotics beyond all orders", the attempt to interpret asymptotic expansions with terms beyond the first infinitely many (such as exponentially small terms after infinitely many powers); this is already a growing field with contributions by R. B. Dingle, M. Berry, J. Ecalle, H. Segur and many others.

J. Leon

Algebraic Properties of the $\bar{\partial}$-Operator and General Integrable Systems

It is shown that a quite general space and time dependence of the spectral transform generically leads to a true (that is polynomial in the spectral variable) nonlinear evolution equation. The method makes use of some simple algebraic properties of the $\bar{\partial}$ operator and its inverse. The result is applied to physically interesting applications including the solution of initial-boundary value problems for coupled wave equations.

D. Levi

Levi-Civita theory for irrotational water waves in a one-dimensional channel and the Korteweg-de Vries equation

We review Levi-Civita theory for water waves that reduces the theory of irrotational water waves in a one dimensional channel with flat bottom to the study of the solution of a nonlinear differential-functional partial differential equation for the complex velocity function.
We show how, by considering small perturbations in a shallow water channel, the differential-functional partial differential equation can be reduced to a system of coupled Korteweg-de Vries equations in $2+1$ dimensions for the horizontal and vertical components of the velocity vector. By requiring that the vertical component of the velocity vector is small compared to the horizontal one the obtained system reduces to the standard Korteweg-de Vries equation.

S. De Lillo

Forced C-integrable equations: The nonlinear diffusion-convection case

A nonlinear diffusion-convection equation is considered, where the forcing is introduced through a time dependent boundary condition at the origin. This model, also known as Rosen-Fokas-Yortsos equation describes a two phase flow in a porous, semi-infinite medium. We solve the initial/boundary value problem with a general initial datum and a boundary condition at the origin representing a time dependent flux. The problem is reduced to a linear integral equation of Volterra type in one independent variable; in some cases of applicative interest this equation can be solved by quadratures.
(Joint work with F. Calogero)

L. Mason

Global solutions of the self-duality equations in signature (2,2) and the inverse scattering transform

First, a programme is reviewed that (1) attempts to classify integrable systems in one and two dimensions as symmetry reductions of the self-dual Yang-Mills equation and (2) attempts to derive the theory of these equations from the Ward construction for solutions of the self-duality equations from holomorphic vector-bundles on twistor space.
In order to address the second part of the programme, solutions of the self-dual Yang-Mills equations on $S^2 \times S^2$ in signature $(2,2)$ are studied as the 4-dimensional analogue of the rapidly decreasing boundary conditions for nonlinear evolution equations in $2+1$ dimensions.
The general solution is shown to give rise to a holomorphic vector bundle on $\mathbb{CP}^3$ together with a twisted analogue of a map from $\mathbb{RP}^3 \to G\mathbb{C}/G$ (G is the gauge group). The first part of the data projects out an instanton and generalizes the 'discrete spectrum' of the IST and the second part generalizes the scattering data (or reflection coefficient). This data determines the original solution and provides a paradigm in $2+2$ dimensions of the inverse scattering transform.

V. B. Matveev

Supertransparency and related Nonlinear phenomenae

There are two spectral miracles which are not widely known to the people working on solitons and spectral theory. One is the possibility to have in a linear problem a trivial scattering operator, i.e. $S = I$ for a nontrivial potential. Everybody knows that this miracle never occurs for the class of smooth rapidly decreasing potentials. The other exotic miracle is connected with the occurence of discrete eigenvalues embedded in the continuous spectrum of linear operators. The last phenomenon also cannot be realised for the class of rapidly decreasing potentials. In the case of the difference Schrödinger operator and in many other cases it is possible to construct a multiparametric family of potentials leading to coexistence of the afore mentioned miracles. The associated initial data for the integrable nonlinear evolution equations generate the remarkable explicit solutions called positons for the reason of their connection with positive eigenvalues in the continuum in the Schrödinger case. In a collision with a soliton, the position aquires two phase shifts but the soliton remains unchanged.

A brief summary of the results concerning positons and their properties in discrete and continuous systems is presented. Further details are described in the talks of R. Beutler and A. Stahlhofen.

G. Neugebauer and R. Meinel

The Einsteinian Gravitational Field of the Rigidly Rotating Disk of Dust

The gravitational field of a uniformly rotating stationary and axisymmetric disk consisting of dust particles is presented as a rigorous global solution to the Einstein equations. The problem is formulated as a boundary value problem of the Ernst equation and solved by means of inverse methods. The solution is given in terms of linear integral equations and depends on two parameters: the angular velocity Ω and the relative redshift z from the center of the disk. The Newtonian limit $z \ll 1$ represents the Maclaurin solution of a rotating fluid in the disk limit. For $z \rightarrow \infty$ the 'exterior' solution is given by the extreme Kerr solution. This proves a conjecture of Bardeen and Wagoner (1969, 1971).

(To appear in 'The Astrophysical Journal Letters')

F. W. Nijhoff

Integrable Lattice Systems and Discrete Painlevé Equations

In this talk I give a status report on discrete Painlevé equatios. Two approaches are exploited: the method of similarity reduction on the lattice, and the de-autonomitations of integrable mappings. In the similarity reduction approach one considers an integrable lattice equation, for instance the lattice (potential) modified KdV equation

$$p(v_{n,m}\, v_{n,m+1} - v_{n+1,m}\, v_{n+1,m+1}) = q(v_{n,m}\, v_{n+1,m} - v_{n,m+1}\, v_{n+1,m+1}), \qquad (1)$$

and one imposes on this equation a **compatible, nonautonomous** similarity constraint

$$n\,\frac{v_{n+1,m} - v_{n-1,m}}{v_{n+1,m} + v_{n-1,m}} + m\,\frac{v_{n,m+1} - v_{n,m-1}}{v_{n,m+1} + v_{n,m-1}} = 0. \qquad (2)$$

The system consisting of lattice equation (1) and similarity constraint (2) carries an isomonodromic deformation problem. In the full continuum limit this system goes over into the potential modified KdV equation and a linear similarity constraint, leading to a reducion to the Painlevé II equation.

As such, the system (1)+(2) can be considered to be a **lattice version** of the Painlevé II equation, even though one cannot solve explicitly for a similarity variable from the **nonlinear** constraint (2). However, a partial continuum limit gives a reduction to a discrete version of the PII equation (for special parameter value).
Other discrete Painlevé equations have been derived using various methods, but not in all cases an isomonodromic deformation problem was given. I present a method that has yielded in some cases such isomonodromic deformation problems starting from **integrable mappings.** They are called "'de-autonomized" versions of reductions of a lattice Gel'fand-Dikii hierarchy. A particularly interesting example is a new isomonodromic deformation problem of a q-difference equation, the compatibility of which yields the discrete Painlevé III equation.
Finally, a q-deformed version of the discrete Painlevé I equation was presented.

W. Oevel

Darboux Transformations as Gauge Transformation

A framework for a systematic interpretation of Darboux transformations as gauge (dressing) transformations is suggested. In terms of the pseudo-differential symbol ∂^{-1} linear problems of the form $\phi_t = M\phi$ are left invariant by the transformation $M \to \widetilde{M} = (TMT^{-1}) + T_tT^{-1} = (TMT^{-1})_+$, where T is a suitable pseudo-differential operator satisfying the dressing equation

$$T_tT^{-1} = -(TMT^{-1})_- . \qquad (1)$$

Here, the subscripts $\pm$ denote the projection to the positive/negative differential orders of the operator. The operator equation (1) admits the following solutions:

Darboux transformation: $T = \phi \cdot \partial \cdot \phi^{-1}$,
adjoint Darboux transformation: $T = \psi^{-1T} \cdot \partial^{-1} \cdot \psi^T$,
binary Darboux transformation: $T = 1 - \phi/\Omega(\psi,\phi) \cdot \partial^{-1} \cdot \psi^T$.

Here Ω, defined by $\Omega_x(\psi,\phi) = \psi^T\phi$ and a suitable compatible time-derivative, is a bi-linear potential integrating the "squared eigenfunction" $\psi^T\phi$ with an eigenfunction ϕ and an adjoint eigenfunction ψ (i.e. $\phi_t = M\phi$ and $\psi_t = -M^*\psi$). These simple transformations can be iterated with several (adjoint) eigenfunctions and give rise to more complicated gauge operators, which are typically parametrized in terms of Wronskian determinants. A "squared eigenfunction" symmetry of the scattering problem is shown to generate a flow, which provides the binary Darboux transformation. (Joint work with W. Schief)

G.D. Pang

On the quantum analogues of classical $(2+1)$-dimensional integrable systems

We briefly review some of our recent works on the quantum analogs of classical $(2+1)$-dimensional integrable systems. These works include: (1) Exact solutions to the eigenvalue problem of the quantized Davey-Stewartson I (DSI) system (2) Conservation laws of the quantized DSI system (3) Thermodynamics and correlation functions of the quantized DSI system.
(Joint work with F.C. Pu, Institute of Physics, Chinese Academy of Science, Beijing, P.R.China and B.H. Zhao, Graduate School, Chinese Academy of Science, Beijing, P.R.China)
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F. Pempinelli

Soliton solutions of the Davey-Stewartson equations

It is shown that the Davey-Stewartson III (DSIII) equation, in addition to the so called DSI and DSII equations, is S-integrable and admits localized soliton solutions with properties similar to those of the DSI equation. Moreover, by introducing generalized Bäcklund gauge transformations, soliton wave soutions are found for the Hamiltonian DSI equation.

O. Ragnisco

Recent results on discrete integrable systems

The lecture consists of three points.
In the first part, a few results are recalled concerning the relation between Painlevé equations (continuous and discrete) and non-isospectral flows of soliton equations. In particular, it is shown that the discrete versions of Painlevé I and Painlevé II are stationary versions of suitable non-isospectral flows of the Volterra and the discrete mKdV hierarchy respectively.
In the second part, a method for constructing integrable maps is briefly described: it amounts to pick-up stationary flows of integrable evolution equations on lattices with "self-consistent sources": as an example, a discrete version of the Neumann system is presented and its integrability is discussed: a Lax representation is provided, and a sufficient number of independent invariants in involution is exhibited.
Finally, a model of integrable evolution equations on a one-dimensional lattice, the so-called "Tu-lattice", is briefly described. The associated hierarchy is bi-hamiltonian, and the first hamiltonian structure is the canonical one; the usual rational R-matrix exists. Moreover, the model admits a vector generalisation and can be considered as an integrable discrete version of the vector NLS hierarchy, to which it reduces in a suitable continuum limit.

C. Rogers, W. Schief

On a Novel Class of 2 + 1-Dimensional Systems

This survey describes developments emanating from a reinterpretation and generalisation of a class of infinitesimal Bäcklund transformations introduced by Loewner in 1952 is a gasdynamics setting. A novel class of integrable 2 + 1-dimensional equations is constructed which encorporates, in particular, 2 + 1-dimensional versions of the principal chiral fields model, Toda Lattice and notably the sine Gordon equation. The general class is conveniently parametrised in terms of four matrices. An auto-Bäcklund transformation is constructed for the 2 + 1-dimensional sine Gordon equation, the so-called KR equation, via a version of Moutard's theorem. The auto-Bäcklund transformation

itself constitutes a new integrable $2+1$-dimensional system incorporating in a one-dimensional reduction an SIT system.
It is noted that the Zakharov-Manakov system and its scalar version, the Darboux system of classical differential geometry, sit naturally within the Loewner system. The observation that the scalar Zakharov-Manakov system represents a squared eigenfunction symmetry of the $2+1$-dimensional AKNS system leads to a more general result wherein two important subcases of the Loewner system may be regarded as squared eigenfunction symmetries of the broad multi-component KP and mKP hierarchies of Sato Theory.
Finally, the celebrated HKX transformation of general relativity is set in the context of the Loewner formalism. This illustrates the fact that the linear infinitesimal Bäcklund transformations originated by Loewner may be seen as infinitesimal Darboux transformations.
(Joint work with B. Konopelchenko and W. Oevel)

S. Ruijsenaars

Sine-Gordon solitons vs. relativistic Calogero-Moser particles

The behaviour of the N-soliton solution to the classical sine-Gordon (sG) equation $\varphi'' - \ddot{\varphi} = \sin\varphi$ is reminiscent of an interaction between N relativistic point particles such that the set of momenta is invariant under a collision and such that the scattering phase shift is factorized in terms of the 2-particle shift. At the classical level this intuition can be concretized in terms of (a specialization of) so-called relativistic Calogero-Moser (CM(rel)) systems, a new class of integrable N-particle systems, discovered in collaboration with H. Schneider (1985).
After recalling the main features of these systems and the classical sG soliton/CM(rel) particle correspondence, we described how the systems can be quantized such that integrability is preserved. The commuting quantum Hamiltonians are built up from products of multiplication operators and shifts along the imaginary axis. Thus, the problem of explicit joint diagonalization reduces to the search for appropriate solutions to systems of analytic difference equations.
We presented explicit solutions to this problem for the 2-particle case. All of these solutions and some further tests are in agreement with the mathematically heuristic, but widely accepted lore concerning the quantum sG field theory. In particular, the scattering and bound state spectrum associated with the quantized particle systems agrees with that of the sG theory, in agreement with the expected physical equivalence of the particle theory and the field theory at the quantum level. The existence of unitarizing joint eigenfunctions for general N and general coupling constants with the expected Harish-Candra type asymptotics has not yet been proved, though.

P.C. Sabatier

Nonlinear equations satisfied by solutions of a linear equation

We start from a linear integral equation E and deduce by **elementary linear algebra** non-linear evolution equations that are satisfied by the solutions of E when a given linear evolution is imposed to its kernel. The solutions Φ which are considered are $N \times N$ matrix valued functions and depend on M variables, these figures being arbitrary integers. Specifically, E has the form:

$$\Phi = 1 + \int\int \frac{d\sigma(\lambda,\Lambda)}{-k+\Lambda}\, T(\lambda,\Lambda,y)\, \Phi(\lambda,y) \qquad (E)$$

where Φ, 1, T are $N \times N$ matrices, $k \in \mathbb{C}$, the integration is on $\mathbb{C}^2$, where $d\sigma$ is a measure of compact support. A first study shows what linear operators L and what linear evolution of T are compatible and lead to the equality

$$L\Phi = \int\int \frac{d\sigma}{-k+\Lambda} T(L\Phi) \qquad (E_0)$$

for which we assume that the only solution is $L\Phi = 0$ (i.e. we shall consider only the $d\sigma$ and T such that this is true). It is shown that both the order of L and the linear evolution of T can be chosen with much arbitrariness and L can then be constructed algorithmically. The coefficients of L can depend on x, t, and even on functionals of Φ, so that $L\Phi = 0$ is already a non-linear relation. From several relations of this type, it is possible to find non-linear equations independent of k for the asymptotic value of $k(\Phi - I)$ as $k \to \infty$. Almost all known integrable equations can be obtained in this way. Their matrix generalization (with constraints in most cases) are obtained on the same foot and without additional work. Algorithms could work on a computer.

P.M. Santini

An elementary geometric characterization of the integrable motions of a curve

We show that the following elementary geometric properties of the motion of a curve select hierarchies of integrable dynamics:
(i) The curve moves in an N-dimensional sphere of radius R;
(ii) The motion of the curve is nonstretching;
(iii) The dynamics is independent of the radius of the sphere.
For $N = 2$ we obtain the mKdV hierarchy, for $N = 3$ we obtain the NLS hierarchy and for $N > 3$ we obtain integrable multicomponent generalizations of the above hierarchies.
(Joint work with A. Doliwa, Institute of Theoretical Physics, Warsaw University, Poland)

P.M. Santini

Cellular automata in multidimensions

We develop a general method to construct nonlinear cellular automata in arbitrary space dimensions from the compatibility of pairs of linear operators, giving examples in 1 + 1, 2 + 1 and 3 + 1 dimensions. These cellular automata possess a lot of constants of motion and exhibit a vast array of coherent structures, both particle-like (i.e., localized in space) and wave-like (not localized).
(Joint work with M. Bruschi, Dipartimento di Fisica, Universitá "La Sapienza", Roma, Italy)

J. Satsuma

On discrete soliton systems

There are many types of discrete systems: semi-discrete, fully-discrete, quasi-discrete and ultimate-discrete. Some of our recent results on such discrete systems are given in my talk.
(1) Trilinear form
It is well knwon that the most important soliton systems are written in terms of bilinear form. We would claim that some soliton systems are naturally expressed by trilinear form. A typical example is the Broer-Kaup system, for which we can show the existence of solutions exhibiting fusion and fission of solitons by means of the trilinear form. Moreover, for some discrete soliton systems, in particular nonautonomous systems, trilinear is quite essential. An example is the nonautonomous discrete KdV equation which arises in a discussion of the complete integrability of fully-discrete systems by the singularity confinement method.

(2) q-difference Toda equation
We propose a q-difference version of the 2D Toda lattice equation. Through a suitable reduction, it reduces to the q-difference version of the cylindrical Toda lattice equation. We show that the reduced equation admits solutions expressed by the q-Bessel function.
(3) Soliton cellular automaton
A soliton cellular automaton, which Takahashi and myself proposed three years ago, is considered to be the simplest soliton system at present. The system consists only of solitons and possesses infinitely many time invariants. Recently we found that the existence of time invariants can be explained by using combinatorics. The procedure is discussed in the last of my talk.

W. Scherer

A Nonlinear Schrödinger Equation for Quantum Mechanics and Some of its Solutions

Recently a family of nonlinear Schrödinger equations describing the time evolution of a quantum mechanical system has been derived by Doebner and Goldin. Various nonstationary solutions of this family of equations are presented. These solutions include quasiclassical waves which display damping and energy dissipation (where the energy is defined as $\langle E\rangle := \langle\psi \mid H_0\psi\rangle$, where $H_0 = -\frac{\hbar^2}{2m}\Delta + V$ is the linear part of the nonlinear Schrödinger equation). The solutions also include solitary waves displaying the behaviour of free particles.
(Joint work with P. Nattermann and A.G. Ushreridze)

A. Seeger

Bäcklund Transformations are Alive and Kicking!

The main message of the contribution is that notwithstanding the recent emphasis on the inverse scattering technique, the Bäcklund transformation approach continues to be extremely useful for the treatment of "practical" problems in the theory of integrable systems. This is illustrated by a recent treatment of the perturbed breather solution of the Enneper (sine-Gordon) equation by E. Moser (?).
In a historical survey the appearance of the Enneper equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2_{tt}}{\partial t^2} = \sin u \tag{1}$$

in physics is traced. It goes back to an attempt by L. Prandtl (1913) to derive a model for imperfections in crystals. (The sine-term accounts for the spatial periodicity of crystals). It was, however, not published in a journal until 1928 and (independently) rediscovered by U. Dehlinger 1928/29. The model is essentially the same as that discussed later by J. Frenkel (?) and T. Kontereno (?) (1938/39) who included time t explicitly without, however, searching for dynamic solutions. An important contribution to the physical significance of the model was made in 1940 by U. Dehlinger and A. Kochendörfer (?).
In 1948 U. Dehlinger asked the present speaker to look into the possibility whether the model could be extended to include "outside perturbations". I realized that much could be gained by "continuization" of the spatial ??? variable. Using a Lagrangian approach, eq. (1) was derived. To the best of my knowledge this was the first time that eq. (1) was **written down** in a physical context.

In a search of the first appearance of (1) anywhere I found that Enneper (1870) wrote the equation in the form

$$\frac{\partial^2 u}{\partial p \partial q} = \sin u \tag{2}$$

in his theory of surfaces of constant negative curvature. I suspect that U. Dini (?) was aware of (2) even before or least simultaneously with Enneper, but in his very extensive writings on differential geometry of the late 1860ies I could not find an explicit statement of (1) or (2). In 1881 L. Bianchi discovered, by entirely geometric reasoning, what was later called the "Bianchi transform", a special case of the transformation discovered by V. Bäcklund in 1883. The decisive discovery in the field, in my mind, is the formula by Bianchi giving explicitly solutions in terms of two different Bäcklund transforms of a common starting solution of (1). For Bianchi this was a "side product" of his proo (1892) of his "theoremo di permutabilité ", but actually it meant that (1) is completely integrable.
I discovered the significance of the results of Bäcklund, Bianchi and Darboux for physics during my Ph.D. work (1949/50), which was mainly concerned with working out a perturbation theory based on the kink solution of (1) and the extension of the results obtained on (1) to the so-called Peierls (?) model (in which a non-linear integro-differential equation takes the place of (1)).
In subsequent work together with H. Dorth (?), who did his Diplomarbeit on the subject, the consequences of the "superposition principle" for (1) were worked out fully. We called the persistent solutions of (1) **"Eigenbewegungen"** as an extension of "Eigenschwingungen" ("characteristic vibrations"). We distinguished between "translatorische Eigenbewegungen" ("solitons") and "oscillatorische Eigenbewegungen" ("finite-amplitude waves"). For the latter I found both the moving and standing varieties. The Bäcklund transform for the moving finite wave were published, those for the standing wave (work with Z. Wesolovski) are unpublished.
For the "translatorische Eigenbewegungen" we found the N-soliton solution (including the breather, which was studied in detail) and the collision laws. The analogy to elastic particle collisions was recognized, also the suitability of the theory for ??? problems in statistical mechanics (work which has continued up to the present).
The perturbation theory mentioned at the beginning makes intensive use of a linearization of the Bäcklund transformation which allows us to solve both the initial-value and the perturbation problem even if the homogeneous perturbation equation is not separable. By this approach E. Moser (?) has recently given a complete solution of the perturbed-breather problem. It revealed a number of surprising features as well as the inadequacy of the so-colled adiabatic approximation.
Literature:

- A.Seeger in "Continuum Models of Discrete Systems (ed. by E. Kröner (?) and K.-H Anthony), University of Waterloo Press.
- E. Moser, to be published.

(As the handwriting was difficult to read, some words might be wrongly guessed and therefore are indicated by a ?, G. Oevel)

H. Segur

The Kadomtsev-Petviashvili equation and water waves

The nonlinear partial differential equation due to Kadomtsev and Petviashvili (KP) generalizes the famous Korteweg-de Vries equation to two spatial dimensions plus time. It is completely integrable, and it also describes approximately the evolution of waves of moderate amplitude in shallow water.

This talk surveys work done in a joint theoretical-experimental program to use exact KP solutions to describe waves in shallow water. The survey includes detailed comparisons among exact solutions of the KP equation, laboratory data of waves in shallow water, and field observations. (Joint work with J. Hammack, D. McCallister and N. Scheffner)

W.M. Seiler

Formal Theory of Partial Differential Equations: Applications to Symmetry Theory

The problem of determining the number of arbitrary functions (and the number of their arguments) in the general solution of an involutive system of partial differential equations is treated. A necessary criterion for the existence of an algebraic representation of the general solution is given. As applications normal systems and gauge systems are considered. For normal systems the loss of generality in a symmetry reduction is determined; for gauge systems the number of physically distinguishable arbitrary functions in the general solution is computed.

A. Stahlhofen

Supertransparency and Related Nonlinear Phenomena III: The modified Korteweg-de Vries equation

The mKdV equation provides again a simple example that the miracles mentioned in the talk by V. B. Matveev hold also for continuous linear matrix operators.
The positon solutions of this equation exhibit in principle the same characteristic features as those mentioned in the two talks before leading now to positive eigenvalues embedded in the continuum of the Dirac operator. When extending the Darboux Transformations underlying this construction to higher orders, one obtains again positon solutions of a more complicated structure having the same properties as before or exotic solutions like Negatons representing – loosely speaking – soliton compounds of mKdV. The properties of these solutions as well as their interaction behaviour are discussed.

H. Steudel

Relations between integrable systems by Darboux pairing

It is shown that a "Darboux pair" with respect to the Sturm-Liouville problem – i.e., two solutions connected by a Darboux transformation – is equivalent to one solution to a Zakharov-Shabat problem (with $r = q$). In an analogous way we found that a Darboux pair with respect to the AKNS-problem is equivalent to one solution for some more general spectral problem which we call a "W-problem".
The extension of this concept to linear systems with two independent variables leads to a method to generate new integrable nonlinear equations (or systems of such equations) by "Bäcklund pairing". Surprisingly many examples were found, physically applicable integrable equations are connected in this way. Two new integrable equations of MKdV and nonlinear Schrödinger types are derived by this procedure.

Y.B. Suris

General quadratic Poisson brackets on Lie groups and bi-Hamiltonian structure of Toda and relativistic Toda lattices

We introduce a general quadratic Poisson bracket on the Lie group. With the help of this bracket we obtain the interpretation of the Toda, discrete time Toda, and relativistic Toda lattices as the restrictions of one and the same bi-Hamiltonian system to two different low-dimensional manifolds, which are Poisson submanifolds with respect to two brackets simultaneously.

M. Wadati

Integrable particle systems with long-range interactions

We consider quantum integrable particle systems with long-range interactions.
1. For the quantum Calogero-Moser model,

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{g}{2}\sum_{\substack{j,k \\ j\neq k}} \frac{1}{(x_j - x_k)^2}, \quad p_j = -i\frac{\partial}{\partial x_j},$$

we construct a set of conserved operators [1] and another set of operators, named boost operators, from its Lax operator. We prove that each conserved operator satisfies both the Lax equation and a remarkable relation named additional relation. By using them, we show that the conserved operators are involutive [2, 3]. Moreover, the conserved operators and the boost operators constitute the U(1) current algebra [2, 3].
2. Spin $-\frac{1}{2}$ particle systems with long-range interactions are considered in one dimensional space. Conditions for the integrability of the systems are shown through the quantum inverse scattering method [4]. Among the solutions, integrable spin particle systems, which we call the XXZ-type model and the Ising-type model, are newly found. A set of conserved operators is obtained from the Lax operator. Further, the ground state is shown to be the solution of a Knizhnik-Zamolodchikov-like equation [5]. For the $SU(\gamma)$ Calogero-Moser spin system, we obtain a series of new conserved operators from the Lax operator. These operators realize the $su(\gamma)$ Kac-Moody algebra. The coexistence of $U(1)$ and $SU(\gamma)$ currents corresponds to the charge-density separation in the model [6].

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