MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Konstruktive Approximationstheorie

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The conference brought together researchers from approximation theory and related areas, such as finite elements, functional analysis, harmonic analysis, and partial differential equations. This resulted in many stimulating lectures and informal discussions. Some of the themes addressed in the lectures were: the construction of Schauder bases consisting of trigonometric polynomials; frames and wavelet decompositions; solution of an infinite system of linear equations and approximation power of shift-invariant spaces; radial basis functions and scattered data interpolation; image processing and noise removal; interpolation of operators and K-functionals; approximation in the complex; neural networks and ridge functions; shape-preserving and Bernstein operators; spline functions and spline projectors; subdivision and computer-aided geometric design.

The lectures reflect the continuing use of constructive approximation in the development of wavelets and multi-resolution as well as in the theory of radial basis functions and scattered data interpolation. Significant advances on long-standing questions in spline theory and subdivision were also reported, as were results in the emerging areas of neural networks and hyperbolic-cross approximation.

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Vortragsauszüge

Rick Beatson:

FAST EVALUATION OF 3-DIMENSIONAL POLYHARMONIC SPLINES

Far-field expansions and error estimates are presented for the $(m+1)$-harmonic spline

$$s(x) = \sum_{j=1}^{N} d_j |x - x_j|^{2m-1}$$

in 3 dimensions. These are related to the multipole expansion of the potential, to which the equation above reduces in the case $m = 0$. In combination with suitable hierarchical divisions of space, these expansions allow fast evaluation of these radial basis functions, and of the associated matrix-vector products

$$\sum_{j=1}^{N} d_j \phi(x_i - x_j), \quad i = 1, \dots, N.$$

Similar expansions are also presented for the d-dimensional generalized multiquadric

$$s(x) = \sum_{j=1}^{N} d_j (|x - x_j|^2 + c^2)^{(2k-1)/2}$$

where $k \in \mathbb{N}$.

Hubert Berens:

ONE MORE TIME – BERNSTEIN–DURRMEYER

The following result is well known:
Let $f \in C[0,1]$ and let $B_n f$ be its Bernstein polynomial of degree n, $n \in \mathbb{N}_0$. Then,

$$\begin{aligned} f \text{ is convex} \quad &\Longleftrightarrow \quad B_n f(x) \geq B_{n+1} f(x) \geq \ldots \geq f(x) \quad \text{on} \quad [0,1] \\ &\Longleftrightarrow \quad \limsup_n \, n\{B_n(x) - f(x)\} \geq 0 \quad \text{on} \quad [0,1] \end{aligned}$$

There is no analogous result for Bernstein–Durrmeyer polynomials (even with Jacobi weights). In 1987, W. Z. Chen considered the following modification:

$$V_n f = \sum_{k=0}^{n} v_{k,n}(f)\, p_{k,n}, \qquad v_{k,n}(f) = \begin{cases} f(0), & k = 0, \\ \frac{\int f p_{k,n} w^{-1}}{\int p_{k,n} w^{-1}}, & 1 \leq k \leq n-1, \\ f(1), & k = n. \end{cases}$$

where $p_{k,n}$, $1 \leq k \leq n$, are the n-th Bernstein basis polynomials and $w(x) = x(1-x)$, $x \in [0,1]$. This polynomial, on the one hand, has all the basic properties attributed to Durrmeyer polynomials, on the other hand it is, in some way, closest to the Bernstein polynomials, in particular it satisfies the equivalences stated above for the Bernstein polynomials.

In his doctoral thesis, Th. Sauer extended Chen's modification to functions on the standard simplex $\mathbb{S}_m := \{u = (u_0, u_1, \ldots, u_m) \in \mathbb{R}^{m+1} \mid u_k \geq 0 \text{ and } \sum u_k = 1\}$, rediscovering a polynomial transformation, say, $V_n f$, $f \in C(\mathbb{S}_m)$ and $n \in \mathbb{N}_0$, which had been defined earlier by T. N. T. Goodman and A. Sharma. For this sequence, he proved:

The following statements are equivalent:

(i) *f is subharmonic w.r.t. U on $\mathbb{S}_m$; i.e., f is subharmonic in the interior of $\mathbb{S}_m$ as well as in the relative interior of its subfaces (w.r.t. the restriction of U to the faces), where U is the elliptic differential operator*

$$2U = \sum_i u_i \frac{\partial^2}{\partial u_i^2} - \sum_{i,j} u_i u_j \frac{\partial^2}{\partial u_i \partial u_j}.$$

(ii) $V_n f(u) \geq V_{n+1} f(u) \geq \ldots \geq f(u)$ on $\mathbb{S}_m$.

(iii) $\limsup_n n\{V_n(u) - f(u)\} \geq 0$ on $\mathbb{S}_m$.

For the Bernstein polynomials on $\mathbb{S}_m$, the implications $(ii) \Rightarrow (iii) \Rightarrow (i)$ had been proved by H. J. Schmid already in 1974; the implications $(i) \Rightarrow (ii)$ and/or $(i) \Rightarrow (iii)$ are still open.

Yuri Brudnyi:

REAL INTERPOLATION OF SOBOLEV COUPLES

Let $W_p^k(Q_n)$ be a "homogeneous" Sobolev space on $Q_n := [0,1]^n \subset \mathbb{R}^n$. The problem of describing all real interpolation spaces of the couple $(W_{p_0}^{k_0}, W_{p_1}^{k_1})$ goes back to the beginning of the sixties. It relates to some essential problems of analysis (embedding and extension properties of Sobolev type, quantitative approximation, PDE's, etc.). For the solution of this problem, we have to find the K-functional of the above couple. In the lecture, we present the solution in the embedding case $W_{p_1}^{k_1} \subset W_{p_0}^{k_0}$ and describe some ideas of the proof. In particular we describe a covering theorem of new type that is the key point of the proof.

This is joint work with Natan Kruglyak.

Martin Buhmann:

LEAST–SQUARES APPROXIMATION WITH RADIAL BASIS FUNCTIONS

We study least-squares approximation by radial basis functions, i.e. approximations from spaces spanned by radially symmetric functions $\phi(\|\cdot - x_j\|_2)$. The x_j are given centres.

where we consider both the case when they lie on a grid and when they are scattered in $\mathbb{R}^n$, but satisfy a weak regularity condition. The inner products with respect to which the least-squares problem is considered are discrete and Sobolev, i.e. may involve derivative information. For the non-regular data case, the existence of orthonormal bases of the radial function space is established. In the gridded centres case, orthonormal bases are constructed and favourable estimates for the least-squares errors are found: they decrease as powers of the grid-spacing.

The work on scattered centres is joint work with Nira Dyn and David Levin from Tel-Aviv.

Charles Chui:

WAVELET FRAMES

The recent development of frames of the affine group will be discussed. This will include: (1) the what and why of frames; (2) necessary frame conditions; (3) sufficient frame conditions; (4) generating frames from oversampling; (5) affine operators; (6) identification of Bessel sequences; (7) oversampling theorems; (8) dual frames; (9) Weyl-Heisenberg frames; (10) multivariate results.

Zbigniew Ciesielski:

FRACTAL FUNCTIONS AND BASES

Two bases on the cube I^d are described: Haar functions supported by dyadic cubes and the interpolating diamond Schauder basis. By means of the coefficients of these Schauder and Haar series, sufficient conditions are given for the existence and the calculation of the box dimension of the graphs of the sums of the corresponding series. Applications to Weierstraß, Tagaki, and to fractional Brownian motion are presented.

Wolfgang Dahmen:

REFINEMENT EQUATIONS

Refinement equations play a pivotal role in several areas such as computer aided geometric design and wavelet analysis. After briefly describing some important consequences of the validity of the most familiar version of refinement equations, based on scaling by two, similar questions are addressed with respect to several other related functional equations. Specifically, the regularity of solutions for more general scaling matrices as well as properties of solutions to certain continuous refinement equations (characterizations, stability, reproduction of polynomials, approximate evaluation) are discussed and some examples are presented. For instance, one can construct compactly supported C^∞-functions whose

Laplacian can be written as a finite linear combination of its shifted dilates. Subdivision techniques are important ingredients of the analysis of all these problems.

Zeev Ditzian:

K-FUNCTIONALS AND REALIZATIONS

For $0 < p < 1$ and $f \in L_p[a,b]$,

$$K_r(f, t^r)_p := \inf_{g \in C^r[a,b]} \left(\|f - g\|_{L_p[a,b]} + t^r \|g^{(r)}\|_{L_p[a,b]} \right) = 0.$$

Some expressions which are realizations of the Peetre K-functional $K_r(f,t^r)_p$ for $1 \leq p \leq \infty$ are still useful measures of smoothness for $0 < p < 1$. This is joint research with V.H. Hristov and K.G. Ivanov.

Nira Dyn:

RADIAL BASIS FUNCTION APPROXIMATION: FROM GRIDDED CENTERS TO SCATTERED CENTERS

Approximation in the $L_\infty(\mathbb{R}^d)$-norm from a space spanned by a discrete set of translates of a basis function ϕ are studied. Attention here is restricted to functions ϕ whose Fourier transform is smooth on $\mathbb{R}^d \setminus \{0\}$, and has a singularity at the origin. Examples of such basis functions are the thin-plate splines and the multiquadrics, as well as other types of radial basis functions that are employed in Approximation Theory. The above approximation problem is well-understood in case the set Ξ of points used for translating ϕ forms a lattice in $\mathbb{R}^d$, and many optimal and quasi-optimal approximation schemes can already be found in the literature. In contrast, only few, mostly specific, results are known for a set Ξ of scattered points.

The main objective of this paper is to provide a general tool for extending approximation schemes that use integer translates of a basis function to the non-uniform case. We introduce a single, relatively simple, conversion method that preserves the approximation orders provided by a large number of schemes presently in the literature (more precisely, by almost all "stationary schemes"). In anticipation of future introduction of new schemes for uniform grids, an effort is made to impose only a few mild conditions on the function ϕ, which still allow for a unified error analysis to hold. In the course of the discussion here, the recent results of Buhmann, Dyn, and Levin on scattered center approximation are reproduced and improved upon.

The talk presents joint work with Amos Ron.

Hans G. Feichtinger:

EFFICIENT METHODS FOR GABOR EXPANSIONS (NEW METHODS)

Let us define the time-frequency shift operators for functions in $L^2(\mathbb{R})$ as follows: $T_x f(z) = f(x - z)$, i.e. translation by x, and the modulation operator or frequency shift $M_u f(z) = \exp(2\pi i u z) \cdot f(z)$.

Given some building block g, i.e. some "nice" (smooth and well decaying) function, we are asking for so-called Gabor expansions, i.e. any $f \in L^2(\mathbb{R})$ should be expanded in a series involving as building blocks only terms of the form $M_{u_i} T_{x_i} g$, with (x_i, u_i) points in the time-frequency plane (identical with $\mathbb{R}^2$). Since the resulting 'sequence' $(M_{u_i} T_{x_i} g)_i$ is typically non-orthogonal, there is the question of the choice of the coefficient 'sequence' (a_i) (which should be square integrable) such that

$$f = \sum_{i \in I} a_i M_{u_i} T_{x_i} g.$$

In the talk, it is pointed out that the coefficients can be taken to be of the form $a_i = \langle f, M_{u_i} T_{x_i} h \rangle$ for a suitably chosen "dual" window h, which can be determined iteratively from g. For the finite-dimensional case (signals are interpreted as functions on the cyclic group of order n), numerical experiments have shown that the conjugate gradient method appears to be the best method. The choice of other subgroups of the time-frequency plane (besides the usual lattices which are products of subgroups of the two variables, time and frequency) gives more freedom to design families of building blocks (Gabor atoms) of the above form, with better time/frequency localization of the dual window.

Tim Goodman:

A GENERALIZED VARIATION DIMINISHING PROPERTY

It is shown that if T is a totally positive $m \times n$ matrix of rank n and A is an $n \times r$ matrix, then under certain conditions, the number of sign changes in the consecutive $r \times r$ minors of TA is bounded by the corresponding number for A. A sufficient condition is that all minors of order $r-1$ from the first $r-1$ columns of A are positive, but this can be relaxed if T has an appropriate band structure. Applications include bounds on the number of inflections or on the number of changes of sign of torsion in spline curves or limiting curves of subdivision schemes.

This is joint work with Jesus Carnicer and Juan Pena.

Rong-Qing Jia:

THE TOEPLITZ OPERATOR AND ITS APPLICATIONS

Let $A = (a_{ij})_{i\in I, j\in J}$ be a (possibly infinite) matrix. It is assumed that each row of A is finitely supported. The Toeplitz theorem says that the system of linear equations

$$\sum_{j\in J} a_{ij}x_j = b_i \quad (i \in I)$$

is consistent if and only if all its finite subsystems are consistent. It is shown that the Toeplitz theorem can be applied to the study of solvability of linear partial differential and difference equations with constant coefficients. These results, in turn, are applied to L_p-approximation ($1 \le p \le \infty$) of shift-invariant spaces. In particular, this algebraic approach leads to the following characterization of approximation order: If S is the shift-invariant space generated by a compactly supported function ϕ in $L_p(\mathbb{R}^s)$ with $\widehat{\phi}(0) \neq 0$, then S provides approximation order r if and only if S contains all polynomials of (total) degree less than r.

B. Kashin:

ON RANDOM SETS OF UNIFORM CONVERGENCE

The talk is devoted mainly to P. Ulyanov's problem about the possible "density" of sets of uniform convergence. Here, the set σ of integers is called a set of uniform convergence (**U.C. set**) if, for each continuous function f with a Fourier series of the form

$$f(x) = \sum_{k\in\sigma} \widehat{f}(k)\, e^{2\pi i k x}, \qquad (*)$$

this series converges uniformly.

Until now, only U.C. sets of the density $\le (\log N)^k$ are known, meaning that, for any known U.C. set σ and some k,

$$\sum_{k\in\sigma\cap[-N,N]} 1 \le (\log N)^k, \quad N = 3, 4, \ldots .$$

In the talk, it is shown (it is the result of joint work with L. Tzafriri) that any random set of integers of density greater than $\log N$ is not a U.C. set.

This subject is treated as a particular example of a theorem about the possibility of decreasing the operator norm by restriction to a space spanned by part of a given basis.

Burkhard Lenze:

ON LOCAL AND GLOBAL SIGMA-PI NEURAL NETWORKS — A COMMON LINK

In general, there is quite a difference between the usual three-layer feedforward neural networks with local basis functions in the hidden processing elements and those with standard sigmoidal transfer functions (often called global basis functions). The reason for this difference can be seen in the ridge-type arguments which are commonly used. It is the aim of this contribution to show that the situation completely changes when, instead of ridge-type arguments, we use so-called hyperbolic-type arguments. In detail, we show that the usual sigmoidal transfer functions evaluated at hyperbolic-type arguments – usually called sigma-pi units – can be used to construct local basis functions which vanish at infinity and, moreover, are integrable and give rise to a partition of unity, both in Cauchy's principal value sense. At this point, standard strategies for approximation with local basis functions can be used without giving up the concept of non-local sigmoidal transfer functions.

Dany Leviatan:

SHAPE PRESERVING APPROXIMATION IN L_p

This is joint work with V. Operstein from the Technion, Haifa. We prove a direct theorem for shape preserving L_p-approximation, $0 < p < \infty$, in terms of the classical modulus of smoothness $\omega_2(f,t)_p$. This theorem may be regarded as an extension to L_p of the well-known pointwise estimates of the Timan type and their shape preserving variants of R. DeVore, D. Leviatan and X. M. Yu. It leads to characterization of monotone and convex functions in Lipschitz classes (and more general Besov spaces) in terms of their approximation by algebraic polynomials. Specifically, we prove the following

Theorem. *For every function $f \in L_p[-1,1]$, $0 < p < \infty$, there exists a sequence of algebraic polynomials P_k of degree not exceeding 2^k, preserving monotonicity and convexity, such that*

$$\left\| \frac{f - P_k}{\omega(\rho_{2^k})} \right\|_{\ell_p(L_p)} \le C \left\| \frac{\omega_2(f, 2^{-k})_p}{\omega(2^{-k})} \right\|_{\ell_p},$$

for every majorant $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying

$$M^{-1}\omega(t_2) \le \omega(t_1) \le M\omega(t_2), \qquad 0 < t_1 \le t_2 \le 2t_1,$$

for some $M \ge 1$. The constant $C = C(M)$ depends on M, and if $p < 1$, also on p.

R. A. Lorentz:

TRIGONOMETRIC WAVELETS: A SHORT SURVEY

Three different types of wavelets which consist of trigonometric polynomials have been constructed recently. First, those of Lorentz and Sahakian are described. They form an orthogonal (Schauder) basis of $C(K)$, the space of continuous 2π-periodic functions equipped with the uniform norm. Their most important property is that their degrees have minimal growth: for each $\epsilon > 0$ there is such a wavelet basis T_n with

$$\deg(T_n) \leq (1/2)(1+\epsilon)n.$$

Then the wavelets of Chui, Mhaskar are described. They are obtained by taking a modified partial sum of the Fourier series expansion of the Haar wavelet. One obtains a non-orthogonal trigonometric wavelet for which the coefficients of the decomposition and reconstruction equations can be explicitly given.

Finally, Privaloff's wavelets, which were investigated by Prestin and Quak, are described. As translates of a de la Vallée-Poussin kernel, they form an interpolatory (Schauder) basis of $C(K)$.

Also, a new orthogonal wavelet basis for the trigonometric spaces of Chui, Mhaskar is presented.

Brad Lucier:

REMOVING GAUSSIAN NOISE FROM IMAGES USING NONLINEAR WAVELET FILTERS

Certain "night vision" cameras amplify low levels of light to allow objects that cannot be seen by the unaided human eye to be seen using the camera. These cameras amplify any noise caused by thermal oscillation; this noise adds to the pixels in the image independent, identically distributed Gaussian random variables with mean zero and fixed variance.

We propose to remove some of this noise by applying a nonlinear filter to the wavelet coefficients of the noisy, observed image. This filter is derived as the solution of a non-quadratic minimization problem related to K-functionals. The associated linear filter used to remove Gaussian noise is called nonparametric spline approximation. We prove that our nonlinear method achieves the same rate of noise removal to within a power of a logarithm of the number of pixels as the nonparametric spline method, but for a much larger class of nonsmooth functions. Thus, our method is particularly suited for noise removal from images which are intrinsically not smooth.

These methods are nearly the same as some methods derived by David Donoho and Iain Johnstone: they prove various optimality results in various statistical settings for these methods.

This is joint work with Ronald A. DeVore.

W. R. Madych:

THE RECOVERY OF IRREGULARLY SAMPLED BAND LIMITED FUNCTIONS VIA TEMPERED DISTRIBUTIONS

We show that band-limited functions can be recovered from their values on certain irregularly distributed discrete sampling sets as the limits of piecewise polynomial spline interpolants when the order of the spline goes to infinity. This is an extension of the classical case when the sampling sequence is a lattice which was considered by L. Collatz, W. Quade, I. J. Schoenberg, and others.

To be more specific:

- The Paley-Wiener class PW_π is the collection of those square integrable functions whose Fourier transform has support in the interval $-\pi \le \xi \le \pi$.
- The sampling sequence $(x_n : n \in \mathbb{Z})$ is such that the corresponding sequence $(e^{-ix_n\xi} : n \in \mathbb{Z})$ of exponential functions is a Riesz basis for $L^2([-\pi,\pi])$.
- For f in PW_π, the function $S_k(f,x)$, $-\infty < x < \infty$, is the unique spline of polynomial growth and order $2k$ which interpolates f on $\{x_n\}$.

Theorem *If f is in PW_π, then $\lim_{k\to\infty} S_k(f,x) = f(x)$ in $L^2(\mathbb{R})$ and uniformly in x.*

Remarks:

- The estimates used to prove this can be applied to obtain similar results for wider classes of f's.
- This work was done in collaboration with Yu. Lyubarskii (Kharkov).

Avraham A. Melkman:

CONVERGENCE OF NON-NEGATIVE STATIONARY MULTIVARIATE SUBDIVISION

Consider the stationary subdivision scheme $v^{(n+1)} = Av^{(n)}$, $v^{(0)} \in \ell_\infty(Z^d)$, with $A_{i,j} = c_{i-2j}$, $c_i \ge 0$, $\sum_{j\in Z^d} c_{i-2j} = 1$ for all $i \in Z^d$, and $c_i = 0$, $i \notin \Omega$. Denote $I := \{i : c_i > 0\}$, and $I_N := \{\sum_{i=0}^{N-1} t_i 2^i : t_i \in I\}$. Let Γ have the property that for any k the set $\{(Av)_i : i \in k+\Gamma\}$ is determined only by the values $\{v_i : i \in p_k + \Gamma\}$ for some p_k. Then the subdivision scheme converges if and only if, for some finite N, to each i there corresponds a j such that $i + \ell - j2^N \in I_N$ for all $\ell \in \Gamma$.

This leads to an investigation of I_N for various choices of Ω. In particular, in the univariate case we conjecture that, unless Ω is obviously bad, $\lim_{n\to\infty} \frac{1}{2^N} I_N = conv(I)$. Multivariate analogues of this statement are discussed.

This is joint work with Alfred S. Cavaretta.

H.N. Mhaskar:

DEGREE OF APPROXIMATION BY NEURAL NETWORKS

Let $\sigma : \mathbb{R} \to \mathbb{R}$. A network consisting of n neurons in one (hidden) layer is simply a function of the form

$$\sum_{j=1}^{n} c_j \sigma(\underline{w}_j \cdot \underline{x} + b_j)$$

where $c_j, b_j \in \mathbb{R}$. $\underline{w}_j \in \mathbb{R}^s$, $s \geq 1$ being an integer. By repeating this process, we get neural networks with more than one layer. For example,

$$\sum_{i=1}^{m} \gamma_i \sigma(\sum_{j=1}^{n} c_{ij} \sigma(\underline{w}_j \cdot \underline{x} + b_j) + \theta_i)$$

is a neural network having $m + n$ neurons, n in the first layer and m in the second.

We show that if σ satisfies the conditions

$$\lim_{x \to \infty} \frac{\sigma(x)}{x^k} = 1, \quad \lim_{x \to -\infty} \frac{\sigma(x)}{x^k} = 0, \quad |\sigma(x)| \leq K(1 + |x|)^k, \ x \in \mathbb{R},$$

then networks with more than one layer can be constructed to yield the usual Jackson rate of approximation. If $k \geq 2$, networks that yield a nearly geometric rate for approximation of analytic functions can also be constructed.

We then show that a network with a single hidden layer has severe limitations in approximating the characteristic function of the unit cube on compact sets of $\mathbb{R}^s$. Restricting our attention to the approximation of 2π-periodic functions, we examine the approximation power of networks with one hidden layer in terms of the degree of trigonometric approximation of the function involved and that of σ. Some dimension-independent bounds are also given.

Ferencz Moricz:

SATURATION CLASSES FOR STRONG APPROXIMATION BY DIRICHLET INTEGRALS

Let $f \in L^p(\mathbb{R})$, for some $1 \leq p \leq \infty$, and

$$s_\nu(f, x) := \frac{1}{\pi} \int_{\mathbb{R}} f(x - t) \frac{\sin \nu t}{t} \, dt$$

be its Dirichlet integral, and for $1 \leq \lambda \leq \infty$ and $T > 0$, let

$$h_T^\lambda(f, p) := \left\| \left\{ \frac{1}{T} \int_0^T |s_\nu(f, x) - f(x)|^p \, d\nu \right\}^{1/p} \right\|_\lambda.$$

It is easy to see that the saturation order is $O(T^{-1/p})$. However, it is tough to determine the saturation class

$$S_p^\lambda := \{f \in L^p \cap L^\lambda : h_T^\lambda(f, p) = O(T^{-1/p}), \ T \to \infty\}.$$

Our main results are:

Theorem 1. *If $f \in L^2 \cap L^\infty$, then $f \in S_2^\infty$ if and only if*

$$\int_0^\infty \left|\frac{f(x+u)+f(x-u)-2f(x)}{u}\right|^2 du \in L^\infty.$$

Theorem 2. *If $f \in L^p \cap L^\lambda$ for some $2 \le p < \infty$, $1 < \lambda < \infty$, then $S_p^\lambda = \mathcal{F}_{\lambda,p}^{1/p}$, the Lizorkin-Triebel space.*

Bernd Mulansky:

ON THE CONSTRUCTION OF DATA DEPENDENT TRIANGULATIONS

In bivariate scattered data interpolation by piecewise polynomials, the quality of the interpolant depends on the specific triangulation of the data sites. We report on numerical experiments concerning the construction of data-dependent triangulations. To find a global minimum of the corresponding combinatorial optimization problem, the method of simulated annealing is used.

Peter Oswald:

STABLE SPLITTINGS OF FUNCTION SPACES AND THE FAST SOLUTION OF VARIATIONAL PROBLEMS

Many iterative solvers for variational problems are based on appropriate splittings of the underlying Hilbert space. For Sobolev spaces, the theory of such splittings is essentially a byproduct of known descriptions via decomposition norms. The talk gives an introduction to the abstract convergence theory of additive and multiplicative Schwarz methods and describes the typical applications to finite element discretizations of elliptic boundary value problems in polyhedral domains. We touch on some problems of actual interest: decompositions for Sobolev spaces with weights, splittings for divergence-free finite element discretizations and for nonconforming subspaces.

Pencho Petrushev:

A NEW MODULUS FOR HYPERBOLIC-CROSS APPROXIMATION

The $L_p(\mathbb{T}^d)$ approximation of functions f by trigonometric sums T_n of exponentials with frequencies from the hyperbolic cross $\{k = (k_1, \ldots, k_d) : |k_1 \cdots k_d| \le n\}$ has received a lot of recent attention because of its various optimality properties. However, there has as of yet been no convenient characterization of the approximation order by such trigonometric sums. In this talk, we introduce a new modulus of smoothness in L_p depending on mixed differences and use it to characterize the approximation classes for hyperbolic-cross approximation in $L_p(\mathbb{T}^d)$ for $1 < p < \infty$.

This is joint work with Ron DeVore, Sergei Konjagin, and Vladimir Temlyakov.

Allan Pinkus:

SOME DENSITY PROBLEMS IN MULTIVARIATE APPROXIMATION

We consider the following idea for approximating multivariate functions. Let Φ be a family of reasonable, "nice", smooth functions in $C(\mathbb{R}^n)$. Set

$$G_\Phi := \operatorname{span}\{g(\phi(\cdot)) : \phi \in \Phi,\, g \in C(\mathbb{R})\}.$$

This space G_Φ is our approximating set. For example, if $\Phi = \{\sum_{i=1}^n a_i x_i : \forall (a_1, \ldots, a_n) \in \mathbb{R}^n\}$, then G_Φ is the space of Ridge Functions (plane waves). The main example we consider here is the following. Let h be a polynomial in n variables. Set

$$\Phi := \{h(\cdot - \underline{a}) : \underline{a} \in \mathbb{R}^n\},$$

i.e., the set of translates of h. For $n = 2$, G_Φ contains all polynomials if and only if $D_{x_1}h$ and $D_{x_2}h$ are linearly independent. The analogous result seems to be true for $n = 3$, but is not valid for $n \geq 4$.

M. J. D. Powell:

AN ITERATIVE METHOD FOR THIN PLATE SPLINE INTERPOLATION

Thin plate spline interpolation to functions of two variables is useful in many applications, because there are few restrictions on the positions of the data points. Further, some smoothness properties are achieved naturally, because the interpolant minimizes a second derivative norm subject to the interpolation conditions. On the other hand, full matrices occur, and the number of data points, n say, may be very large. Therefore we approximate each Lagrange function by a Lagrange function of interpolation to a small subset of the data. Thus each approximation usually has far fewer than n thin plate spline terms, and the approximations provide an initial estimate of the required interpolant which can be improved by iterative refinement. This procedure has been applied to several test problems. Some of the numerical results are presented, in order to illustrate the numbers of iterations and the amount of computation of the method. They suggest that interpolation to tens of thousands of scattered data points in two dimensions may soon become a routine calculation.

Jürgen Prestin:

POLYNOMIAL APPROXIMATION AND WAVELETS

We consider a wavelet decomposition of $L^2_w(-1,1)$, where w is the Chebyshev weight. Therefore, we use scaling functions defined as fundamental polynomials of Lagrange interpolation at the zeros of $(1-x^2)U_n(x)$, where U_n is the Chebyshev polynomial of the second kind with degree $n = 2^j - 1$. Hence the scaling functions are polynomials of degree 2^j. We construct corresponding polynomial wavelets of degree 2^{j+1}, which span the orthogonal spaces W_j. Then we describe decomposition and reconstruction algorithms for the nested polynomial spaces in a suitable matrix/vector form. Furthermore, we discuss Riesz stability and dual scaling functions and wavelets. This approach is strongly related to the recent results of C.K. Chui/H.N. Mhaskar, A.A. Privalov, R.A. Lorentz/A.A. Sahakian and J. Prestin/E. Quak in the trigonometric case.

Ulrich Reif:

SINGULAR SPLINES – A NEW METHOD FOR GENERATING FREEFORM SURFACES OF ARBITRARY TOPOLOGY

When using the standard parametric smoothness conditions for n Bézier patches sharing a vertex, only the regular case $n = 4$ leads to nondegenerate nonperiodic solutions in the space of Bézier points and consequently to smooth surfaces. So, until now, these smoothness conditions were thought to be too restrictive for the case $n \neq 4$ and geometric smoothness conditions were used instead. Since the concept of geometric continuity is not completely satisfying, the class of singular splines is introduced which provides the trivial solutions in the space of Bézier points when the standard parametric smoothness conditions are used. The corresponding degenerate Bézier patches (so called *D-patches*) are shown to be smooth under certain constraints on the neighboring Bézier points of the singular point. In particular, these points must be coplanar. The spline space constructed is invariant under subdivision, and this opens up for the first time the possibility of constructing subdivision algorithms with a known limit surface for meshes with arbitrary topology. Further, a linear map for projecting a set of arbitrary control points to the space of control points satisfying the constraints (the so called *quasi control points*) is given and a family of real-valued B–spline functions is constructed.

Amos Ron:

THE ADJOINT OF A WEYL-HEISENBERG FRAME

Let (K, L) be a pair of lattices in $\mathbb{R}^d$ and let $f \in L_2(\mathbb{R}^d)$. Consider the set

$$X := \{M_\ell T_k f : \ \ell \in L, \ k \in K\},$$

with $T_x : f \mapsto f(\cdot + x)$, $M_\ell : f \mapsto e_\ell f$, $e_\ell : x \mapsto e^{i\ell x}$. Associated with X is the operator R_X defined on a dense subspace of $\ell_2(X)$:

$$R_X : c \mapsto \sum_{x \in X} c(x)x.$$

If R_X is bounded, then X is called a **Bessel set**; if, in addition, the range of R_X is closed, then X is a **frame**. A frame whose corresponding R_X is injective is a **Riesz basis**.

We introduce in this talk the notion of the **adjoint** X^* of a given X. With

$$\widetilde{K} := \{k \in \mathbb{R}^d : \; k \cdot \ell \in 2\pi\mathbb{Z}, \; \forall \ell \in K\}$$

the lattice dual to K,

$$X^* := \{M_\ell T_k f : \; \ell \in \widetilde{K}, \; k \in \widetilde{L}\}.$$

We prove basic relations between X and X^*.

The talk represents joint work with Zuowei Shen.

E. B. Saff:

OPTIMAL RAY SEQUENCES OF RATIONAL FUNCTIONS

Given two compact disjoint subsets E_1, E_2 of the complex plane, the third problem of Zolotarev concerns estimates for the ratio $\sup_{z \in E_1} |r(z)| / \inf_{z \in E_2} |r(z)|$ where r is a rational function of degree n. We consider, more generally, the infimum Z_{mn} of such ratios taken over the class of all rational functions with numerator degree m and denominator degree n. For any such "ray sequence" of integers (m, n); that is, $m/n \to \lambda$, $m + n \to \infty$, we show that $Z_{mn}^{1/(m+n)}$ approaches a limit $L(\lambda)$ that can be described in terms of the solution to a certain minimum-energy problem with respect to the logarithmic potential. For example, we prove that $L(\lambda) = \exp(-F(\tau))$, where $\tau = \lambda/(\lambda + 1)$ and $F(\tau)$ is a concave function on $[0, 1]$ and we give a formula for $F(\tau)$ in terms of the equilibrium measures for $E_1^* \cup E_2^*$ and the condenser (E_1^*, E_2^*), where E_1^*, E_2^* are suitable subsets of E_1, E_2. Of particular interest is the choice for λ that yields the smallest value for $L(\lambda)$. In the case when E_1, E_2 are real intervals, we provide for this purpose a simple algorithm for directly computing $F(\tau)$ and for the determination of near optimal rational functions r_{mn}. Furthermore, we discuss applications of our results to the approximation of the signum function and to the generalized ADI iterative method for solving Sylvester's equations.

Robert Schaback:

LOCALIZED INTERPOLATION BY "RADIAL" BASIS FUNCTIONS

For "radial" basis function interpolation of scattered data in $\mathbb{R}^d$, the error bounds known so far can be retained even if the interpolant is calculated using only a few neighboring data points around each evaluation point. This approach has many computational advantages, but produces nonsmooth interpolants. Results include error bounds, convergence orders, and numerical methods. Numerical examples are provided.

Alexei Shadrin:

ON L^p BOUNDEDNESS OF THE L^2 SPLINE PROJECTOR

Let T be an arbitrary knot sequence on $\mathbb{R}$ and P_T be the orthogonal $L_2(\mathbb{R})$ projector onto the space of splines of order r with knots from T. Then clearly P_T is a norm-one operator on $L_2(\mathbb{R})$. A conjecture of de Boor is that P_T is also bounded on L_∞ with a bound depending only on r. We show that for each r there is an $\epsilon = \epsilon(r) > 0$ such that P_T is bounded as an operator on L_p for all p satisfying $|p - 2| < \epsilon$ and with norm bounds depending only on r.

Herbert Stahl:

NORMALITY IN NIKISHIN SYSTEMS

We discuss simultaneous rational approximants to a vector $(f_1, \ldots, f_m)$ of analytic functions, and explore the connection of their definition with the multiple orthogonality of the common denominators of the approximants. For a special system of Markov functions, namely for the Nikishin system, we prove the normality of all multi-indices $n = (n_0, \ldots, n_m)$ with $n_j \geq \max(n_1, \ldots, n_{j-1})$, $j = 2, \ldots, m$. Normality means that the Hermite-Padé polynomials of type II are unique up to a non-zero constant factor. The normality result is the basis for investigations of the convergence of simultaneous rational approximants.

Joachim Stöckler:

LOCALIZATION PROPERTIES OF NON-STATIONARY WAVELET BASES

Wavelets have proved to be a very effective mathematical tool for analyzing functions f with certain irregular local behaviour. For instance, a wavelet with $m+1$ vanishing moments can be used to detect the location of a point x_0 at which a function is only $m-2$-times continuously differentiable while it is in C^{m-1} in a deleted neighbourhood of x_0. The effectiveness of such methods depends very much on the localness of the wavelet ψ. When a single wavelet ψ is used to generate an orthonormal basis of $L^2(\mathbb{R})$, this localness is sacrificed, if m is required to be large.

By the notion of "non-stationary" wavelet bases which use different generators ψ_j on different scales, we gain freedom in designing orthonormal bases for $L^2(\mathbb{R})$ with good localization properties. The functions ψ_j are taken as (infinite) combinations of shifts of one function ϕ with respect to the lattice $2^{-j-1}\mathbb{Z}$. Their $2^{-j}\mathbb{Z}$-shifts define an orthonormal basis of a shift-invariant subspace W_j of $L^2(\mathbb{R})$ whose orthogonal sum gives all of $L^2(\mathbb{R})$. In particular, if ϕ is an even function whose Fourier transform is positive and decays exponentially, we obtain analytic wavelets ψ_j. With such a basis we can detect C^∞-singularities of a piecewise analytic function. As a measure of localness we give quantitative estimates for the standard deviation of ψ_j and $\hat{\psi}_j$. The results are given in terms of certain decay parameters of the function ϕ. Surprisingly different results are obtained for the case of the multiquadric and the Gaussian function.

V. Temlyakov:

APPROXIMATION OF MULTIVARIATE PERIODIC FUNCTIONS

The problem of finding an optimal and universal system of approximation is discussed. The problem of optimization is investigated for classes of functions with bounded mixed differences. This investigation is based on the concept of width: Kolmogorov width and orthowidth (or, Fourier width). The concept of universality is introduced and applied to the study of the anisotropic Nikol'skii classes. As a result, we get that the approximation by means of trigonometric polynomials with frequencies in hyperbolic crosses is natural.

Vilmos Totik:

APPROXIMATION ON THE BOUNDARY AND STRICTLY INSIDE

Let K be a compact subset of the complex plane with connected complement and f a continuous function on K which is analytic inside K. We are interested in the problem of approximating f by near-best polynomials for which the approximation is better inside K, i.e., we ask if there are polynomials P_n such that $\|f - P_n\| \le CE_n(f)$ and at the same time locally uniformly on compact subsets of the interior of K we have $|f(z) - P_n(z)| \le Ce^{-cn}E_n(f)$. The answer to this problem is YES if K is bounded by an analytic curve, and NO if its boundary has an external angle smaller than π at some point z_0. In fact, in the latter case the polynomials P_n can exist only when f is analytic at z_0. Possible rate of approximation in the form $|f(z) - P_n(z)| \le Ce^{-cn^\omega}E_n(f)$ is discussed along with related results concerning local improvements of the order of best polynomial approximation for arbitrary compact sets.

Joe Ward:

p-NORMS OF RADIAL BASIS INTERPOLATION MATRICES

We discuss results concerning l_p bounds on $\|A^{-1}\|_p$, $1 \le p \le \infty$, where

$$A = \left(\phi(x_j - x_k)\right)_{j,k=1}^{n}$$

is the interpolation matrix associated with a radial basis function (or, rbf) ϕ and scattered distinct data $\{x_j\}_{j=1}^n$. Here, we specify that ϕ is either an order-zero or order-one rbf. It is known that the bounds on $\|A^{-1}\|_2$ are independent of n and depend exactly on ϕ, the dimension d and the minimal separation q.

We show, in case ϕ decays to zero "sufficiently rapidly" as $\|x\| \to \infty$, that $\|A^{-1}\|_p$, $1 \leq p \leq \infty$, is independent of n and again depends exactly on ϕ, the dimension d and q.

Moreover, in special cases, we can apply preconditioning techniques to demonstrate that even if ϕ grows at infinity, $\|A^{-1}\|_p$ is independent of n, where n denotes the cardinality of the data set. As a sample result, we obtain another proof of a theorem of M. Powell. Namely, if $\phi(x) := \sqrt{c^2 + \|x\|^2}$ and $A_n := \left(\phi(i-j)\right)_{i,j=1}^{n}$, then $\|A_n^{-1}\|_\infty \leq K$, where K is independent of n.

Yuan Xu:

MULTIVARIATE ORTHOGONAL POLYNOMIALS AND APPROXIMATION

In this talk I report recent work of mine on the subject of multivariate orthogonal polynomials and its application to approximation. The main results concerning multivariate orthogonal polynomial include: three-term relation in a vector-matrix form which leads to a Farvard's theorem that characterizes the orthogonality; definition of block Jacobi matrices which, through an operator-theoretic approach, makes it possible to establish the existence of orthogonal measure; the common zeros of multivariate orthogonal polynomials which are identified as joint eigenvalues of truncated block Jacobi matrices; new formulation of the characterization of Gaussian cubature formulae which enables us to find two classes of weight functions that lead to Gaussian cubature formulae, the latter one provides a positive answer to the question proposed by Radon in 1948. Results concerning approximation include: asymptotics of Christoffel functions; uniform convergence of partial sums and de la Vallée Poussin means, where the measures of orthogonality are subject to certain conditions but kept general otherwise.

Kang Zhao:

APPROXIMATION ORDER ACHIEVED BY SEMI-DISCRETE CONVOLUTION

Without appealing to the argument of polynomial reproduction, we show how to construct a function from a given shift-invariant subspace in L_2 with which the corresponding semi-discrete convolution mapping achieves the approximation order provided by the shift-invariant subspace if it also provides some simultaneous approximation order.

Berichterstatter: R.A. DeVore

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