MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 46/1993

Adaptive Methoden für partielle Differentialgleichungen

10. - 16. Oktober 1993

The conference was organized by R.E. Bank (San Diego), G. Wittum (Heidelberg) and H. Yserentant (Tübingen).

The topic of the conference was the construction and analysis of numerical methods for partial differential equations with singularities or boundary- and interior layers in the solution. Such differential equations cannot be successfully treated with the usual uniform or quasiuniform meshes. Nonuniform grids adapted to the local behaviour of the particular solution are essential to resolve the singularities. A lot of mathematical questions arise in this context. Refinement schemes for finite element subdivisions have to be found, which is still a difficult problem for three space dimensions. Reliable error estimators to control the refinement process are needed. Fast iterative methods for the solution of the resulting equations which do not degenerate for highly nonuniform meshes have to be constructed. Also, the implementation is a much more complicated issue than for uniform grids, especially on advanced parallel computers. Numerical methods have to be adapted to fill these needs.

Despite the fact that only slightly more than 20 participants could be invited, all relevant topics were discussed by the speakers who were, by a large part, leading experts in the field. The talks demonstrated that the theory for the case of selfadjoint coercive elliptic boundary value problems has reached a fairly mature state, and is now flexible enough to treat more difficult problems and to adapt the algorithms to specific computer architectures. On the other hand, it became clear that the really reliable treatment of problems in computational fluid dynamics is still in its infancy, whereas in continuum mechanics remarkable progress has been made during the last few years. Overall, the conference covered most of the current development and gave a good overview of the present state of the art.

Abstracts

Randolph E. Bank:

Some Nuts and Bolts of Adaptive Methods

In this lecture, we discuss some practical aspects of the implementation of adaptive finite element methods. First, we consider a strategy for local refinement and unrefinement of the mesh. The central issue is to develop a procedure which can create a sequence of good, but not necessarily optimal, meshes in which the number of degrees of freedom is growing geometrically. Second, we consider an algorithm for adaptively adjusting the placement of nodes in the mesh. We assume the mesh retains a fixed topology, and that second (in general order $p+1$) derivatives of the solution are locally constant. The algorithm itself is based on a nonlinear Gauss-Seidel iteration.

Peter Bastian:

Parallel adaptive multigrid methods

The parallel implementation of a fully unstructured locally refining multigrid code for MIMD computers with message passing and distributed memory is discussed. Multiplicative and additive MG methods as stand-alone iterations and preconditioners are considered in combination with several different smoothers. The parallelization uses the data partitioning approach and the problem of dynamic load balancing is discussed in detail. Two heuristic procedures have been developed – one for additive MG and one for multiplicative MG – and are compared for several examples.

Dietrich Braess:

Towards algebraic multigrid

There are finite element problems which are too large to be efficiently solved by pcg methods and which have so unstructured grids such that classical multigrid methods cannot be easily applied. Here, efficient algorithms are desired which require only the matrix of the linear system. We construct coarse grids by using the graph which is generated from the nonzero entries of the matrix. Variables, which are strongly connected, are put into groups (which contain 2 to 4 elements). The coarse grid functions are those which are constant on the elements of each group. The corresponding multigrid cycle is used as a preconditioner for a cg-method. We report on numerical results for the Poisson equation, for a membrane problem and for a problem of groundwater research.

Peter Deuflhard:

Cascadic conjugate gradient methods for elliptic partial differential equations

Cascadic conjugate gradient methods for the numerical solution of elliptic partial differential equations consist of Galerkin finite element methods as outer iteration and (possibly preconditioned) conjugate gradient methods as inner iteration. Both iterations are known to minimize the energy norm of the arising iteration errors. A simple but efficient strategy to control the discretization errors versus the PCG iteration errors in terms of energy error norms is derived and worked out in algorithmic detail. In a unified setting, the relative merits of different preconditioners versus the case of no preconditioning are compared. Surprisingly, the cascadic conjugate gradient method without any preconditioning (to be called CCG method) appears to be not only simplest but also fastest. The computational results clearly indicate that the cascade principle in itself already realizes some kind of preconditioning. Upon careful examination of the numerical observations, the usual Chebyshev convergence estimates appear to be unsatisfactory. A new theory in terms of an "effective condition number" is presented, which reflects the structure of the iterative errors. On this basis the observed CCG iteration patterns can be interpreted in detail.

Michael Griebel:

Adaptive point-block multilevel methods

Instead of the usual basis, we use a generating system for the discretization of PDEs that contains not only the basis functions of the finest level of discretization but additionally the basis functions of all coarse levels of discretization. The Galerkin-approach now results in a semidefinite system of linear equations to be solved. Standard iterative GS-methods for the system turn out to be equivalent to elaborated multigrid methods for the fine grid system.
Beside GS-methods for the level-wise ordered semidefinite system, we also consider block GS-methods for the point-wise ordered semidefinite system. These new algorithms show the same properties as conventional multigrid methods with respect to convergence and efficiency. Additionally, they possess interesting properties like host-parallelization on workstation networks.
Furthermore, using a tensor-product-type hierarchical basis so called sparse grid methods can be gained. In contrast to regular shell grids, the number of grid points is substantially reduced from $0(h^{-d})$ to $0(h^{-1}(\log(h^{-1}))^{d-1})$ where d denotes the dimension of the problem. The accuracy, however, deteriorates only slightly from $0(h^2)$ to $0(h^2(\log h)^{d-1})$ in the L_2-norm and even remains $0(h)$ in the E-norm, provided that the solution is sufficiently smooth.
The two major drawbacks of the sparse grids, namely the smoothness requirement and the simple quadrilateral shape of the domain can be overcome by adaptive refinement of the grid. We demonstrate that adaptively refined grids possess sparse grid properties on regions of the domain where the solution is smooth and show how adaptive refinement can be used to reduce the shape of domains with more general boundaries.

Wolfgang Hackbusch:

On the cutting of cycles in a graph

A discretization of a convection dominated problem in 2D leads to a certain graph. If the graph could be cycle-free, a suitable numbering of the unknowns can be determined. For this purpose, a minimal number of knots has to be taken out to avoid cycles. A corresponding algorithm is described requiring only linear time.

Jürgen Jäger:

Parallelization between Data Decomposition and Overlapping Domain Decomposition for a tensor product grid with refinement

For a realistic model problem (heating of a concrete beam) some parallelization methods are investigated: Data Decomposition, ODD and Data Decomposition preconditioned by ODD. Every method has specific advantages. DaDe: no additional arithmetical costs, good load balancing; ODD: low granularity, specific treatment of subregions. Furthermore ODD is a parallelization approach independent of local solvers (e.g. multigrid). The efficiency of (high granular) DaDe depends on hardware and communication software. In ODD losses arise by additional gridpoints and, using iterative methods, gain by saving iterations in most of the subdomains, 30% - 50% of overlap can be saved by an extrapolation approach.

Ralf Kornhuber:

Monotone multigrid methods for variational inequalities

Extending well-known linear concepts of successive subspace correction, we arrive at extended relaxation methods for elliptic variational inequalities. Extended underrelaxations are called monotone multigrid methods, if they are quasioptimal in a certain sense. By construction, all monotone multigrid methods are globally convergent. We take a closer look at two natural variants, which are called symmetric and unsymmetric multigrid methods, respectively. While the asymptotic convergence rates of the symmetric method suffer from unsufficient coarse-grid transport, it turns out in our numerical experiments that suitable application of the unsymmetric monotone multigrid method may lead to the same efficiency as in the linear, unconstrained case.

Ulrich Langer:

Load-balanced parallel DD solvers for F.E. equations on graded meshes

Domain Decomposition (DD) techniques are not only the basic tools for data partitioning but also methods for constructing new parallel pde solvers. Domain Decomposition, mesh adaptation, and load balance seem to be in contradiction to the efficient use of massively parallel computers with several hundred, or even several thousand powerful, distributed memory processors. In the case of plane boundary value problems producing solutions with fixed singularities caused by corner points at the boundary, by changing boundary conditions, or nonsmooth interfaces, it is possible to construct highly efficient, load balanced solvers based on non-overlapping DD techniques. The finite element discretization uses Courant's element on graded triangular meshes adapted to the singularities arising. The finite element equations are solved by the conjugate gradient method parallelized and preconditioned by DD techniques. The use of modified BPS preconditioners on the coupling boundaries and local multigrid methods with zero and specially chosen non-zero initial guesses in various Dirichlet DD preconditioners is studied. The numerical experiments carried out on transputer systems confirm the theoretical results obtained.

Wim Lenferink, Jos Maubach:

Global state-space parameter-space finite element methods for strongly monotone continuation problems

A new finite element scheme for numerical continuation of parameter dependent boundary value problems is proposed. It is a global discretization scheme in that it is based on triangulating the product of the state-variable domain and the parameter domain. In contrast with standard schemes, the present scheme allows local refinement in the product domain. Moreover, a discretization of multiparameter problems is possible. The new method is proved to be convergent. It is shown to be more efficient in some examples where the solution has steep gradients only locally.

Rolf Mahnken:

Parameteridentification with finite-element methods

The development material laws for modelling elasto/visco plastic deformations consist of both development of a mathematical model and the determination of material dependent constants. The identification of these parameters from experimental data requires the solution of inverse problems. So far only uniaxial experiments were considered for this task. For minimization of the corresponding objective function stochastic methods such as the evolution strategy are usual. The approach in my talk is twofold: Firstly complex structures such as a plate with a hole are taken into account for determination of the material parameters. Secondly, for minimization of the objective function of least-square type a method based on gradient evaluations is applied. The specific algorithms are an SQP-method or, alternatively, a projection algorithm due to Betsekas. In order to determine the gradient of the objective function a sensitivity analysis has to be carried out.

Jos Maubach:

Adaptive local bisection refinement for n-dimensional simplicial grids

Grids of n-simplices have frequently been used for the computation of approximate solutions to various mathematical problems. In two and three space-dimensions, such grids of triangles (respectively tetrahedrons) are used for the finite element approximation of solutions of partial differential equations. In more dimensions, n-simplices are used to approximate solution manifolds of parametrized equations, for the approximation of fixed points of functions $f : \mathbb{R}^n \to \mathbb{R}^n$, or to manage graphical data for post processing. This lecture presents a bisection type refinement method which can be applied to grids of n-simplices such that the number of similarity classes is bounded above, independent of the level of refinement.

Peter Oswald:

Stable subspace splittings of Sobolev spaces and adaptivity

An additive splitting

$$\{V; a\} = \sum_j \{V_j; b_j\}$$

of a Hilbert space V with scalar product $a(.,.)$ into subspaces V_j (with their own scalar products $b_j(.,.)$ is called stable if a two-sided inequality

$$a(u,u) \approx \inf_{u_j \in V_j : \sum_j u_j = u} \sum_j b_j(u_j, u_j)$$

holds for all $u \in V$. The condition κ of such a splitting is basic for the convergence theory of additive and multiplicative subspace correction schemes.
We discuss stable splittings of Sobolev spaces and techniques to modify them. Applications are given to adaptive multilevel finite element solvers for elliptic boundary value problems.

Joe Pasciak:

Theory of multilevel methods on problems with mesh refinement

In this talk, I described recent results on additive and multiplicative multilevel methods and their application to problems on refined meshes. The canonical example of an additive method is the so-called BPX (Bramble, Pasciak, Xu) method, while multiplicative methods encompass variational multigrid algorithms. By the construction of an appropriate approximation operator, the analysis in the case of the refinement application can be reduced to that of the unrefined case. Recent advances by many researchers both here and abroad were surveyed. A comprehensive understanding of the multilevel technique must now been achieved.

Heiko Schulz, Wolfgang Wendland:

The problem of adaptivity for the boundary element method

After a short introduction into the model problem and its discretization some approaches for a posteriori error estimations are presented. The starting point for all approaches is the residual, which gives naturally a lower and an upper bound for the error in the energy norm. But there are three problems: The first one is the practical computation of the residual and of the used norms, the second one is the localization of the residual and the third one is the computation of the constants in the error estimation.
The first idea was to use local projections of the error, see the works of Rank 1984, Wendland, Yu, Göhner 1988 and the very new work of Foremann 1993. This approach works for operators of positive order, but for operators with negative order difficulties arise. Another approach is to start with a global projection of the error and to localize the result. This can be done in an efficient way by using an hierarchical ansatz. In that way it is possible to get sharp lower and upper bounds for the error and error indications. The third approach is to use ideas from Johnson and Eriksson. This was

done by Carstensen and Stephan, they proved an a posteriori error estimation with a step size function and the residual on the right side. But no lower bound for the error is given and the constants in the estimations are unknown, in general. At the end some results for local error estimation using pseudolocal properties are presented. This results were obtained together with Mr. Saranen.

Erwin Stein:

Adaptive finite-element-methods in structural engineering, esp. geometrical non-linearities and model adaptivity

Nested mesh refinement is used throughout for 2D- and 3D-problems. Diagonal colouring preserves symmetry conditions in the refinement process. Elastic shells with finite rotations are treated in a new methodology using rotated base vectors and direct analysis of strains. A non-linear extension of the a posteriori Babuška-Miller error indicator, proposed by Rheinboldt, is analyzed, and restriction for its applicability - esp. in bifurcation points - are derived. Examples of computer postcritical loading paths of shells with automatic local mesh refinement into the buckles show the power of the method. Furthermore, new concepts for model adaptivity of this structures in disturbed layers are shown, both with reduction from complete 3D-elasticity theory and expansion from 2D-theory, using a priori error analysis. Problems of mesh generation and computed examples, comparing reduction- and expansion-method, close the lecture.

Rüdiger Verfürth:

A posteriori error estimates for non-linear problems

We consider non-linear problems of the form $F(u) = 0$ with $F \in C^1(X, Y^*)$ and approximations there of the form $F_h(u_h) = 0$ with $F_h \in C(X_h, Y_h^*)$, $X_h \subset X, Y_h \subset Y$. In a first step, we show that under mild assumptions on F in a neighbourhood of a solution u_0 the error $\|u - u_0\|_X$ is equivalent to the residual $\|F(u)\|_{Y^*}$. Next we construct finite dimensional spaces $\tilde{Y}_h$ such that $\|F(u_h)\|_{Y^*}$ can be estimated by the finite dimensional quantity $\|F(u_h)\|_{\tilde{Y}_h^*}$ which in turn can easily be evaluated since the functions in $\tilde{Y}_h$ have a local support. The derived results are applied to scalar quasilinear elliptic equations of second order and yield residual error estimators which are upper *and* lower bounds for the error $\|u - u_h\|_X$.

Roland Vilsmeier:

Adaptive solutions of fluid flows on unstructured grids

The Euler and Navier-Stokes equations are solved on unstructured, anisotropically adapted meshes. The finite volume method is used for the discretizations. Time stepping is carried out with an explicit Runge-Kutta scheme. On unsteady problems, remarkable CPU-time saving is achieved using multiple time levels according to local stability conditions. This modification of the basic Runge-Kutta scheme is based on a grouping concept. Nodes and edges are grouped according to local stability limits. The time integrations for different groups, although running a different integration cycle, are synchronized to allow a proper discretization with neighbouring nodes on time.

Further attention is paid to the generation of triangular (2D) or tetrahedral (3D) meshes. First step of the generation is to close the triangulation within the boundaries. The mesh is "finished" using several tools, all of those working on a closed, developing triangulation. Adaptivity is introduced via virtual stretching. This approach allows to create strongly anisotropic triangulations, essential for high Re-numbers. The adaptivity is to be seen as a continued mesh generation, where more information is available. In 3D however, this adaption concept is not yet implemented.

Bruno Welfert:

Towards adaptive (pseudo-)spectral methods?

The nonlocality of basis functions used in spectral and collocation methods seems at first sight to prevent any possibility for adaptivity. On the pseudospectral side, basic results from interpolation theory show that the intuitive idea of adding collocation points where gradients in the solution are large, as is usually done in the finite element context, doesn't work. Nevertheless, assuming a set of collocation points has been chosen, it is interesting to find out how the matrix representation of basic operators such as differentiation operators can be determined numerically. In this talk we present a method which does so for particular (yet still quite general) basis functions. The complexity of the resulting algorithm for the p-th differentiation operator, based on a vansive computation of the matrix coefficients and a method by Fornberg to derive finite difference formulas, is $O(pn^2)$. The update of the resulting matrices upon changes in the collocation points is investigated. Results show that the modifications become rapidly global as the number of modified points is increased, especially for higher order matrices. Thus there is little hope for a general adapted collocation method.
On the other hand, wavelet basis functions appear to be suitable candidates to be used in an adaptive *spectral* method, due to their regularity (as high as desired), localization in space and frequency, and relations to hierarchical decomposition of the solution space.

Harry Yserentant, Ralf Kornhuber:

Multigrid methods for elliptic problems on domains with complicated boundaries

By definition, a multilevel method for finite element equations is based on a sequence of refined triangulations. One starts with a coarse initial triangulation crudely reflecting the properties of the boundary value problem under consideration. For the usual mathematical test problems like the solution of the Laplace equation on the unit square or on an L-shaped domain, this initial triangulation consists of only very few elements. Real-life problems, on the other hand, are often posed on very complicated regions which can only be described by hundreds or thousands of finite elements. This is probably one of the main reasons why multilevel algorithms are not widely accepted in the engineering community.
In the talk, a technique has been described which partly resolves these problems. We consider complicated subdomains Ω' of simple domains Ω and finite element spaces con-

sisting of functions defined on Ω and vanishing outside the subdomain Ω'. Beginning with a multilevel splitting of the space associated with the whole domain Ω, we construct stable splittings of the space of functions vanishing outside Ω'. Then one can derive a large class of multilevel algorithms following the ideas of Bramble, Pasciak and Xu, for example.

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