MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 26/1994

Mathematische Aspekte der Quantentheorie großer Systeme

19.06. bis 25.06.1994

Die Tagung fand unter Leitung von *Giovanni Felder* (Chapel Hill), *Jürg Fröhlich* (ETH Zürich) und *Horst Knörrer* (ETH Zürich) statt. Schwerpunkte waren einerseits die Verwendung von Methoden der konstruktiven Feldtheorie bei der Untersuchung von Vielteilchensystemen und andererseits die Theorie des Quantenhalleffekts. Aktuelle Arbeiten zu diesem Gebiet verwenden im beträchtlichen Umfang Chern-Simons-Theorie und konforme Feldtheorie. Dadurch ergab sich eine enge Verbindung zu der gleichzeitig stattfindenden Konferenz über die Quantentheorie integrabler Systeme. Dementsprechend wurden einige Vorträge gemeinsam mit dieser Tagung organisiert.

Dank der niedrig gehaltenen Teilnehmerzahl war es möglich, die Vorträge auf fünf "Halbtage" zu konzentrieren und somit viel Zeit für Diskussionen freizuhalten. Von dieser Möglichkeit wurde auch intensiv Gebrauch gemacht.

Vortragsauszüge

1. Hartree-Fock Theory for the Hubbard Model [V. Bach, E. Lieb, J.-P. Solovej]
We establish the generalized Hartree-Fock (HF) theory as a variational principle for the ground state (free) energy of quantum mechanical systems where the energy expectation is varied over quasifree states and we reformulate it in terms of one-particle density matrices (=2-pt-fns). In this formulation we apply it to the Hubbard Model with arbitrary positive couplings U_x and bipartite hopping matrix t_{xy}. We characterize the minimizers in terms of their behaviour under symmetry transformations of the Hamiltonian, provided the number of electrons equals the number of lattice sites (half-filling). It turns out that the global $SU(2)$ gauge symmetry is broken and the minimizers exhibit antiferromagnetic long range order. If the lattice is part of Z^2 with nearest-neighbour hopping, for example, this means that the spins of the minimizers are antiparallely oriented in a checkerboard fashion. We also consider negative couplings in which case the particle number is broken and one obtains a BCS-type minimizer. Finally, we prove for infinite positive coupling a Nagaoka-type theorem on the existence of a ferromagnetic minimizer for all electron numbers smaller than the number of lattice sites minus one.

Volker Bach, TU Berlin

2. Conformal Field Theory and Integrable Models on the Torus [G. Felder, Ch. Wieczerkowski]
The spaces of conformal blocks on the sphere or on the torus are introduced as spaces of invariant linear forms on tensor products of Kac-Moody algebra $\mathcal{G}$-modules under the Lie algebra of meromorphic functions with values in $\mathcal{G}$. These spaces are identified with certain spaces of functions with values on finite dimensional vector spaces. The corresponding K-Z equations are derived. The consistency condition on the torus leads to generalizations of the classical Yang-Baxter equation. Its quantization is discussed.

Giovanni Felder, FIM Zürich and UNC Chapel Hill

3. Two Dimensional Fermi Liquids [J. Feldman, H. Knörrer, D. Lehmann, E. Trubowitz]
Let $\varepsilon(\vec{k}) \in C^\infty$ be the (renormalized) dispersion relation, $\mu > 0$ the chemical potential and $\lambda\langle k_1, k_2|V|k_3, k_4\rangle \in C^\infty$ the interaction of a many-fermion model in two space dimensions. For simplicity suppose that the model has a fixed ultraviolett cutoff and that $F = \{\vec{k}|\varepsilon(\vec{k}) = \mu\}$ is compact. The main hypotheses are that $\nabla\varepsilon(\vec{k}) \neq 0$ for all $\vec{k} \in F$ and that, for all $\vec{q}$, $F \neq -F + \vec{q} = \{\vec{k}|\varepsilon(-\vec{k} + \vec{q}) = \mu\}$. Then there is an $\eta > 0$ such that for all $|\lambda| < \eta$ the thermodynamic limit of the Euclidean Green's functions of the model exist in the sense of distributions and are analytic in λ. The particle number density $n_{\vec{k}}$ is C^∞ in $\vec{k}$ except that it has a jump discontinuity at every $\vec{k} \in F$.

Joel Feldman, UBC Vancouver

4. Chern-Simons Theory, Integral Lattices, and Fractional Quantum Hall Effect [J. Fröhlich, E. Thiran]
We consider two-dimensional gases of electrons in a strong, uniform, external magnetic field transversal to the plane of the system. We are interested in the physics of such systems at large distance scales and low frequencies, i.e., in the scaling limit. It is shown that in the scaling limit these systems are described by pure, abelian Chern-Simons theories, *provided* there is no dissipation in the system ($R_L = 0$). We then review the connection between Chern-Simons theory and the Knizhnik-Zamolodchikov equations and current (Kac-Moody) algebra. In passing, implications for knot theory are mentioned, but the physically important fact is that solutions of the K-Z equations span the physical state space of C-S theory. Returning to the physics of 2D-electron gases, one notices that those solutions of K-Z equations derived from C-S theories that describe physical states of such systems have special monodromy properties described by integral Euclidean lattices. These lattices are then studied and classified partially, and this leads to the prediction of allowed, rational values of the Hall conductivity, functional charges and statistics of "Laughlin vortices", etc..

Jürg Fröhlich, ETH Zürich

5. Conformal Field Theory in Higher Genera [K. Gawedzki]
The solution of the WZW (and coset) conformal field theory model on a higher genus Riemann surface may be encoded in the scalar product on the space of non-abelian theta functions. The latter are holomorphic sections of powers of the determinant bundle on the moduli space of holomorphic bundles of rank> 1. The scalar product of non-abelian theta functions is given by a formal functional integral. This integral may be effectively calculated (at least for the rank 2 case) and reduces to a finite-dimensional integral expression. As a by-product, one obtains integral expressions for higher genus conformal blocks of the WZW conformal model.

Krisztof Gawedzki. IHES Bures-sur-Yvette

6. Riemann Surfaces of Infinite Genus [J. Feldman, H. Knörrer, E. Trubowitz]
We introduce a class of Riemann surfaces of infinite genus to which many of the classical results on compact Riemann surfaces extend. They are special cases of "parabolic" surfaces in the sense of Alfohrs-Nevanlinna. So they admit a canonical homology basis $A_1, B_1, A_2, B_2, \ldots$ and a basis $w_1, w_2, \ldots$ of the Hilbert space of square-integrable holomorphic 1-forms such that $\int_{A_i} w_j = \delta_{ij}$. In addition, the associated theta function converges on a suitable Banach space, and there is an analogue of Riemann's Vanishing Theorem and Torelli's Theorem.

The definition of the class of Riemann surfaces considered is in terms of glueing "standard pieces". It is explicit enough to allow for a verification that Fermi curves of two-dimensional periodic Schrödinger operators as well as the spectral curves for the periodic KP II equation belong to it. The latter fact is used to prove that the initial value problem for this equation has solutions that are almost periodic in time.

Horst Knörrer, ETH Zürich

7. A Microscopic Derivation of the Critical Magnetic Field in a Superconductor [D. Lehmann]
The propagator of a non-interacting many-electron system is computed. A BCS equation with magnetic field is derived which no longer has a solution Δ if the magnetic field is sufficiently high. Perturbation theory around the magnetic field propagator is discussed.

Detlef Lehmann, ETH Zürich

8. Two Dimensional Superconductivity [J. Feldman, J. Magnen, V. Rivasseau, E. Trubowitz]
We consider a gas of non-relativistic interacting electrons in $2+1$ dimensions. The propagator $(ip_0 + \vec{p}^2/2m - \mu)^{-1}$ (where m is the electron mass and μ is the chemical potential) is singular around the Fermi surface $\vec{p}^2 = 2m\mu$ For a two-body interaction with short range the model is just renormalizable. The interaction is taken of the form $-\lambda/2 \sum_{\sigma,\tau} \bar{\psi}_\sigma(p_1)\psi_\sigma(-p_3)v(p_1 - p_3)\bar{\psi}_\tau(p_2)\psi_\tau(-p_4)\delta(p_1 - p_3 + p_2 - p_4)$ and so the model is $U(1)$ invariant. The model is formally given in terms of an integral over Grassmann variables, i.e., in terms of a sum of determinants. We then decompose the Fermionic fields in scales according to the distance of its momentum from the Fermi surface, the angle of the momentum and its space localization. This enables us to obtain uniform bounds in each scale. We can then compute in each scale the effective action coming from the "integration" over the scales farther from the singularity. There is a renormalization of the Fermi surface, and also of the vertex, which eventually break the $U(1)$ invariance. It is of the form $\bar{\psi}_\sigma(p)\bar{\psi}_\tau(-p)G(p-q)\psi_\sigma(q)\psi_\tau(-q)$. So if we decompose $v(p-q)$ according to the angular momentum $v = \sum \lambda_i \Pi_i(p,q)$ and if $\lambda_0 < 0$ and $|\lambda_0| > |\lambda_i|$ for $i = 1, 2, \ldots$ then the model is attractive, and will remain attractive, so a symmetry breakdown takes place, that, however, is not discussed here.

Jacques Magnen, Ecole Polytechnique Palaiseau

9. Fluctuations of Hall Conductance [Y. Avron, R. Seiler, P. Zograf]
A model for fermions in a strong magnetic field in a topologically non-trivial configuration space is considered. The dynamics depend on the fluxes passing through the handles of a Riemann surface. The conductivity relating the electromotive force around loop α and current around loop β is split into a universal term constant in the flux ϕ and a fluctuating one as an application of Quillen's family index theorem.

Ruedi Seiler, TU Berlin

10. Some Mathematical Problems of Non-Linear Dynamics [I.M. Sigal]
Consider a non-linear Schrödinger or wave equation. Assume a solution exists for as long a time as required. The question we address is to describe properties of such a solution, especially its localization in space and time. We addressed the following three topics: I. Periodic Solutions; II. Resonances; III. Dynamics of vortices.

We describe one of the results. Consider NL Schrödinger or wave equations that are small perturbations of linear ones. Assume the corresponding linear equations have solutions periodic or quasiperiodic in time and L^2 in space. Then

WE: periodic or quasiperiodic solutions are unstable under generic NL perturbations.

SE: periodic solutions are stable under all reasonable perturbations while quasiperiodic solutions are unstable under generic perturbations.

Israel Michael Sigal, Univ. of Toronto

11. Schrödinger Operators and Classical Almost Integrability [L. Thomas, S. Wassell]
Let $H = -\Delta + V$ be a Schrödinger operator acting on $L^2(M)$, M either a d-dimensional torus or sphere, V analytic.

For the case of the torus, we construct asymptotic expansions for the eigenfunctions and eigenvalues of H at high energy E, via WKB methods, the expansions in inverse powers of E. The classical action appearing in the eigenfunctions is the solution to a classical Hamilton-Jacobi equation. KAM methods assure that this classical action can be constructed, at least for a Cantor set V^∞, whose intersection with the subset $V(E)$ of the phase space with energy $\leq E$ satisfies $|V^\infty \cap V(E)|/|V(E)| \to 1$ as $E \to \infty$. It follows that the dimension of the subspace of the approximate eigenfunctions corresponding to energy $\leq E$ is asymptotic to the actual dimension of the subspace corresponding to $H|_{\leq E}$, $E \to \infty$. Similar results are obtained for the case of the $2-sphere$, but just to $O(E^{-1})$ (which should be compared to the typical spacing between eigenvalue clusters which is $O(E^{-1/2})$). Turning point difficulties and problems associated with the high eigenvalue degeneracy of the unperturbed operator are addressed by first transforming the operator to a Bargmann space representation.

A theorem of Kac-Spencer, Weinstein, Wilson and others states that the l^{th} cluster of eigenvalues (for H on the sphere) clustered about $E = l(l+d-1)$ has a limiting (probability) distribution for $l \to \infty$ equal to the distribution of the Radon transform of V. We give an example of a (Hölder-) continuous potential V for which this limiting distribution of eigenvalues is singular continuous.

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