MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 29/1994

Analysis und Geometrie singulärer Räume

03.07. bis 09.07.94

Die Tagung fand statt unter Leitung der Herren J.-M. Bismut (Paris), J. Brüning (Augsburg) und R. Melrose (Cambridge).

Diese Tagung mit 35 Teilnehmern gestaltete sich aufgrund hervorragender Beiträge als sehr fruchtbar. Thematisch schloß diese Veranstaltung an die beiden Tagungen über "Elliptische Operatoren auf singulären und nichtkompakten Mannigfaltigkeiten" aus den Jahren 1987 und 1991 an.

Schwerpunkte der Vorträge und Diskussionen lagen bei Indextheorie und Spektraltheorie sowie verschiedenen Invarianten von Mannigfaltigkeiten mit Singularitäten.

Bei den Invarianten findet die analytische und topologische Torsion zunehmendes Interesse (Lück, Köhler, Varghese). Eng verwandt hiermit sind Determinanten und η–Invarianten gewisser elliptischer Operatoren, deren verschiedene Aspekte teilweise sehr ausführlich diskutiert wurden (Bunke, Dai, Hassel, Vishik). In der Indextheorie wurden einerseits neue Beweise vorgestellt (Getzler, Zhang), andererseits wurden adiabatische Methoden zum Beweis neuer Resultate verwendet (Piazza, Lott). Des weiteren wurden verschiedene Aspekte der charakteristischen Klassen behandelt (Duflo, Jeffrey, Pardon, Rumin, Stanton, Stern), wobei auch Methoden der Nichtkommutativen Differentialgeometrie verwendet wurden (Moscovici). In der Spektraltheorie lag der Schwerpunkt auf asymptotischen Resultaten (Christiansen, Guillemin, Lesch, Zworski).

Weitere Beiträge behandelten mathematische Aspekte der Quantentheorie (Sjöstrand, Sunada, Vergne, Wolpert), Eichfeldtheorie (Eichhorn) sowie die nichtlineare Skalarkrümmungsgleichung (Mazzeo).

Die Teilnehmer waren sich darin einig, daß dies eine sehr erfolgreiche Tagung war und daß eine Tagung dieses Themenkreises nach Möglichkeit regelmäßig wiederholt werden sollte.

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Vortragsauszüge

Theta and zeta functions for compact locally symmetric spaces of rank one

Ulrich Bunke and Martin Olbrich

Let $\mathbf{G}$ be a semisimple Lie group of real rank one, $\mathbf{K}$ be its maximal compact subgroup, $X := \mathbf{G}/\mathbf{K}$ be the associated symmetric space and $M := \Gamma\backslash X$ be a compact locally symmetric space. Let $\mathbf{G} = \mathbf{KAN}$ be an Iwasawa decomposition and $\mathbf{M}$ be the centralizer of $\mathbf{A}$ in $\mathbf{K}$. To any pair of finite dimensional unitary representations χ of Γ and σ of $\mathbf{M}$ there is a zeta function of Selberg type $Z_\chi(s,\sigma)$. It is defined using the geometric data provided by the set of closed geodesics of M and their lifts to bundles associated with χ and σ. The zeta function has a meromorphic continuation to the whole complex plane. Its singularities are related to the spectral theory of elliptic differential operators on M. We distinguish spectral singularities mostly on the imaginary axis occuring symmetrically w.r.t zero and topological singularities located essentially on the negative real axis.

Let X^d be the compact dual space to X. Let $dim(M)$ be even. We construct an element γ in the integer representation ring of $\mathbf{K}$ restricting to σ under $\mathbf{M} \subset \mathbf{K}$. The associated bundles over M, X^d carry elliptic differential operators A_M^2, A_d^2 such that the eigenvalues of A_M correspond to the spectral singularities and the eigenvalues of A_d to the topological ones. In fact we represent the zeta function in terms of zeta regularized determinants of A_M, A_d. If $dim(M)$ is odd and σ is not invariant under the Weyl group of $(\mathbf{g}, \mathbf{a})$ the construction of γ has to be modified. This case involves a Dirac operator. Its eta invariant occurs naturally in the functional equation of the zeta function.

We employ the Selberg trace formula to study A_M. Using "analytic continuation of wave traces" we relate the identity contribution with the spectrum of A_d which is described by a generalization of the Cartan/Helgason theorem. Using this technique we avoid the trouble with the normalization constants of the Plancherel measure and obtain very explicit formulas allowing to show the meromorphicity of the zeta function by harmonic analysis methods.

We apply our results to the Ruelle zeta function relating its singularities with the spectrum of the Laplacian on differential forms and proving simple functional equations. In the odd dimensional case we identify its value at zero with the analytic torsion.

Our work sheads new light to the previous work of many people (D.Fried, R.Gangolli, A.Juhl, H.Moscovici, L.B.Parnovskij, R.Schuster, A.Selberg, R.Stanton, M. Wakayama, F.Williams). Nevertheless we were able to improve a couple of their results. We also think that our approach via the spectral theory of operators on M, X_d is very short, much less involved than more harmonic analysis oriented approaches and gives a derivation of *all* previous results in a unified framework.

The theta function associated to M, χ and σ is defined by Tre^{-tA_M}. It has a meromorphic continuation to the whole complex plane. Its singularities are located on the imaginary axis and the negative real axis and related to the closed geodesics of M and X^d. The functional equation relates the theta function of M with the theta function of X^d defined by Tre^{-tA_d}.

In order to see what happens in the finite volume case we study the theta function in the sperical case for Riemann surfaces with cusps. The theta function here is the sum of the spectral theta function defined by the discrete spectrum of A_M as above and a scattering theta function defined by the singularities of the scattering matrix. It has a meromorphic continuation to a Riemann surface of the logarithm. Its singularities on the imaginary axis can be explained in terms of the closed geodesics and the geodesics connecting the cusps. It also has the topological singularities on the negative real axis and singularities due to the cusps on the whole real axis. We believe that this is the general picture in the finite volume case of rank one.

Spectral asymptotics for manifolds with cylindrical ends

Tanya Christiansen (joint with Maciej Zworski)

We give spectral asymptotics for the Laplacian on a manifold with cylindrical ends. Since the Laplacian has continuous spectrum one expects Weyl asymptotics for the sum of a term measuring the behaviour of the continuous spectrum and the counting function for embedded eigenvalues. The first term, $\sigma_t(\lambda)$, is an analogue of the scattering phase, and is expressed using an appropriately defined scattering matrix for manifolds with cylindrical ends. We denote by $N(\lambda)$ the number of embedded eigenvalues less than or equal to λ^2. If the manifold has a cylindrical end identified with $(-\infty, a)_t \times \partial X$ (and t is extended so that $t \geq a$ on the rest of X), then the asymptotics are given by

$$N(\lambda) + \sigma_t(\lambda) = c_n \lim_{M\to\infty} [\int_{X, t>-M} 1 - M\mathrm{Vol}(\partial X)]\lambda^n + O(\lambda^{n-1})$$

where n is the dimension of the manifold and c_n is the usual Weyl constant. We give an example that shows that the bound $O(\lambda^n)$ obtained on $N(\lambda)$ is optimal.

Eta invariants and determinant lines

Xianzhe Dai (joint with D. Freed)

We show that the eta invariant for a manifold with boundary lives naturally in the *inverse* determinant line of the boundary and we prove some properties of this invariant. More precisely, for a compact odd dimensional spin Riemannian manifold with product structure near the boundary, we construct a canonical element of the inverse determinant line of the boundary. In the case when the boundary is empty, the inverse determinant line is canonically identified with the complex numbers, and the canonical element is then (essentially) the reduced eta invariant.

The most important property the (inverse) determinant line-valued eta invariant satisfies is the *gluing law.* The gluing law we prove is more general than that obtained by cutting a closed manifold into pieces as in the works of Bunke, Hassell-Mazzeo-Melrose and Wojciechowski. Thus we must consider gluing along manifolds where the index of the Dirac operator may be nonzero. For this reason we use *graded* determinant lines, as introduced in Knudsen-Mumford where the idea is credited to Grothendieck.

For a family of Dirac operators this invariant is a smooth section of the inverse determinant line bundle over the parameter space. We prove a geometric variation formula which generalizes the usual formula for the variation of the eta invariant to a formula for the covariant derivative of this section. Here we use the natural connection on the (inverse) determinant line bundle defined by Bismut-Freed. This formula is crucial for the rest of the results.

As an application we give a new proof of the holonomy theorem of Witten (proved by Bismut-Freed and Cheeger) for the determinant line bundle. Our results also lead to a conjecture about the geometric index of families of Dirac operators on odd dimensional manifolds with boundary, which we verify for the degree one component.

Stratifications in infinite dimensions

Jürgen Eichhorn

Let (M^n, g) be open, G a compact Lie group with Lie algebra $\mathfrak{g}$, $P(M,G) \to M$ a G–principal fibre bundle, $G_P = P \times_{\mathrm{Ad}} G, \mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$. Assume (M^n, g) satisfies the conditions (I) $r_{\mathrm{inj}} = \inf_{x \in M} r_{\mathrm{inj}}(x) > 0$ and $(B_h(M,g))$ $|(\nabla^g)^i \mathcal{R}^g| \le C_i, 0 \le i \le k$. Let $C(B_h, f) = \{\omega \,|\, \omega \quad G\text{–connection}, |(\nabla^\omega)^i \mathcal{R}^\omega| \le D_i, 0 \le i \le k, \mathcal{YM}(\omega) = \frac{1}{2} \int |\mathcal{R}^\omega|^2 d\mathrm{vol} < \infty\}$. Then there exists for $k \ge r > n/2 + 1$ a metrizable Sobolev uniform structure on $C(B_h, f)$. Let $C^r(B_h, f)$ be the completion.

Theorem *$C^r(B_h, f)$ has a representation as a topological sum*

$$C^r(B_h, f) = \sum_{i \in I} comp(\omega_i) = \sum_{i \in I} \omega_i + \Omega^{1,r}(\mathfrak{g}_P, \nabla^{\omega_i}).$$

Here $comp(\omega_i)$ means the component and $\Omega^{1,r}(\mathfrak{g}_P, \nabla^{\omega_i})$ is the corresponding Sobolev space of 1-forms with values in $\mathfrak{g}_P$ of r-th order.

It is possible to adapt to each component $C^r(\omega_0) = comp(\omega_0)$ a gauge group $\mathcal{G}^{r+1}(\omega_0)$ which is a Hilbert-Lie group. The symmetry group $S(\omega)$ of any $\omega \in C^r(\omega_0)$ is a discrete subgroup. For a subgroup $S \subset \mathcal{G}^{r+1}(\omega_0)$ denote by (S) its conjugation class. Let $C^r_{(S)}(\omega_0) := \{\omega \in C^r(\omega_0) \,|\, S(\omega) \in (S)\}$, $\mathcal{J} = \{(S) \,|\, \exists \omega$ such that $S(\omega) \in (S)\}$, $\pi : C^r(\omega_0) \to C^r(\omega_0)/\mathcal{G}^{r+1}(\omega_0) = \mathcal{R}^r(\omega_0) =$ configuration space, $\mathcal{R}^r_{(S)}(\omega_0) = \pi(C^r_{(S)}(\omega_0))$.

Main Theorem *$\{\mathcal{R}^r_{(S)}(\omega_0)\}_{(S)\in\mathcal{J}}$ is a stratification of the configuration space.*

A local index theorem for pseudodifferential operators on $\mathbb{R}^n$

E. Getzler

Bargmann has constructed an isometry $Q : L^2(\mathbb{R}^n) \to L^2(\mathbb{C}^n, d\mu)$, where $d\mu$ is a Gaussian measure on $\mathbb{C}^n$, whose image is the space $H^2(\mathbb{C}^n, d\mu)$ of L^2 holomorphic functions on $\mathbb{C}^n$. Then Q^*Q is the identity, while $K = QQ^*$ is projection onto $H^2(\mathbb{C}^n, d\mu)$.

By Bargmann quantization, we mean the assignment to a symbol a on $T^*\mathbb{R}^n \cong \mathbb{C}^n$ of the operator $a^b = Q^*aQ$. On the one hand, a^b is unitarily equivalent to the Töplitz operator KaK acting on $H^2(\mathbb{C}^n, d\mu)$, and on the other hand, if a is a classical symbol (in the isotropic sense), then a^b is a pseudodifferential operator with the same leading symbol as a.

Thus, the index of an elliptic pseudodifferential operator on $\mathbb{R}^n$ equals the index of a Toeplitz operator on $\mathbb{C}^n$. We show that this index equals the index of a Dirac operator on $\mathbb{C}^n$, by a simple McKean-Singer style argument. We extend the local index theorem for Dirac operators to this setting, and thereby obtain a proof of Fedosov's index theorem.

(This talk was based on a paper to appear in Contemp. Math.)

Analytic surgery and analytic torsion

Andrew Hassel

'Analytic surgery' refers to metric degeneration of a closed manifold M to a pair of manifolds with cylindrical ends $M\pm$. This is achieved by stretching M across a hypersurface H which separates M.

I will present joint work with Rafe Mazzeo and Richard Melrose on the analysis of the Laplacian under analytic surgery, and then discuss a surgery formula for analytic torsion. From this surgery–formula, both the Cheeger–Müller theorem for closed manifolds and a combinatorial expression for the 'b–analytic torsion' (a regularized analytic torsion for manifolds with cylindrical ends) can be easily derived.

De Rham representatives of generators for the cohomology rings of moduli spaces

Lisa C. Jeffrey

We use group cohomology and the de Rham complex on simplicial manifolds to give explicit differential forms representing generators of the cohomology rings of spaces of conjugacy classes of representations of the fundamental group of a 2-manifold of genus g into the compact Lie group $K = SU(n)$. We treat the moduli spaces $M(n,d)$, which are defined as

$$M(n,d) = \{\rho \in \mathrm{Hom}(F_{2g}, K) : \rho(R) = \omega^d\}/K.$$

Here, F_{2g} (the free group on $2g$ generators) is the fundamental group of a genus g 2-manifold with one boundary component, and R is the element of the fundamental group which represents a loop around the boundary. The element $\omega = \mathrm{diag} e^{2\pi i/n}(1,\ldots,1)$ is the generator of the center $Z(K)$. We treat the case when n and d are coprime, for which $M(n,d)$ is smooth.

By choosing complex structures on 2-manifolds, $M(n,d)$ may be given an alternative characterization: it is identified with the moduli space of holomorphic vector bundles of rank n, degree d and fixed determinant over a closed genus g Riemann surface Σ. The generators of the cohomology ring of $M(n,d)$ were found by Atiyah and Bott (1983) using this holomorphic description, by taking the slant product of characteristic classes of the universal bundle (a holomorphic bundle over $M(n,d)\times\Sigma$) with the generators of $H_*(\Sigma)$. However this description of the generators is unclear from the point of view of representations of fundamental groups of 2-manifolds.

In this paper, explicit differential forms on $M(n,d)$ which represent these generators are constructed, by giving a construction of the universal bundle (together with its classifying map) in the language of simplicial manifolds. When G is a compact Lie group, differential forms representing the cohomology of classifying spaces BG (universal characteristic classes) were found by Bott (1973) and Shulman (1972), who used the standard simplicial manifold structure on BG. We pull the Bott-Shulman forms back under the classifying map, and apply the slant product to recover De Rham representatives for the generators of $H^*(M(n,d))$. In the course of this construction, we make use of a generalization of the Bott-Shulman construction to equivariant cohomology.

Our representatives for the generators are given explicitly in terms of the Maurer-Cartan form. They generalize the treatment of the symplectic form on these moduli spaces which has been given by Weinstein (1991, following Karshon and Goldman). The symplectic form is a de Rham representative of one of the generators in question: our work confirms Weinstein's conjecture that all the generators should have a description analogous to his description of the symplectic form.

References: L.C. Jeffrey, Group cohomology of the cohomology of moduli spaces of flat connections on 2-manifolds, preprint alg-geom/9404012; L.C. Jeffrey, Symplectic forms on moduli spaces of flat connections on 2-manifolds, preprint alg-geom/9404013, to appear in Proc. of the Georgia International Topology Conference 1993 (ed. W. Kazez).

Analytic torsion on symmetric spaces

Kai Köhler

We show how to calculate explicitly equivariant versions of the two Ray–Singer torsions for all equivariant vector bundles over symmetric spaces G/K of the compact type with respect to any isometry $g \in G$. In particular, we obtain the value of the usual non–equivariant torsions which are defined as

$$\tau := \exp \zeta'(0)$$

for the analytic continuation of the zeta function

$$\zeta(s) := \sum_{q\geq 1, \lambda\in \operatorname{spec} \Delta_q \setminus\{0\}} \frac{(-1)^q q}{\lambda^s} \quad (s > \dim G/K)$$

where Δ_q is the Kodaira– or Hodge–Laplacian acting on q–forms. First, we get for the equivariant holomorphic torsion for bundles over Hermitian symmetric spaces a closed expression in terms of some characters of G and another expression in characteristic classes evaluated on G/K. The result is shown to provide very strong support for Bismut's conjecture of an equivariant arithmetic Grothendieck–Riemann–Roch theorem in the case where g extends to a quasi–projective flat model $\mathcal{E} \to \mathcal{M}$ over spec $\mathbb{Z}$ of $E \to G/K$.

Also, we calculate the equivariant real torsion for all symmetric spaces G/K of the compact type with respect to the action of G. We show that it equals zero except for the odd-dimensional Graßmannians and for the space $\mathbf{SU}(3)/\mathbf{SO}(3)$. As a corollary, we classify up to diffeomorphism the actions of the elements of G of these spaces: in particular, we show that any two associated locally symmetric spaces (except S^1) are diffeomorphic iff they are isometric.

On the spectral geometry of algebraic curves

Matthias Lesch (joint with Jochen Brüning)

Let M be the regular part of an algebraic curve $C \subset \mathbb{CP}^n$. We equip M with a Riemannian metric induced from some hermitian metric on $\mathbb{CP}^n$; we denote by $\mathcal{M}$ the set of all such objects. We have $C = M \cup \Sigma$ where the singular set, Σ, is finite. Near a point $p \in \Sigma$, C decomposes in $L(p)$ irreducible components providing the multiplicities $N_k(p) \in \mathbb{N}, 1 \leq k \leq L(p)$. If all $N_k(p)$ are one then p is just a multiple point which we do not regard as a singularity for the purpose of this study. If Σ is nonempty, the metric on M may be incomplete. The first analytic difficulty caused by this fact concerns the definition of "spectral data": the Laplacians derived from the de Rham complex,

$$0 \longrightarrow \Omega_0^0(M) \xrightarrow{d_0} \Omega_0^1(M) \xrightarrow{d_1} \Omega_0^2(M) \longrightarrow 0, \tag{1}$$

(where $\Omega_0^i(M)$ denotes smooth i–forms with compact support) may not be essentially self–adjoint in the respective Hilbert spaces. In a recent paper we have proved, however, that we are in the *case of uniqueness* in the sense that

$$d_{i,\min} = d_{i,\max},\ i = 0, 1. \tag{2}$$

Here, $d_{i,\min}$ denotes the closure of d_i and $d_{i,\max}$ the adjoint of $d_i^t := - * d_i *$. Thus we obtain a Hilbert complex from the closed operators in (2) with self–adjoint Laplacians Δ_i.

Theorem *Each Δ_i is discrete, and Δ_0 equals the Friedrichs extension of its restriction to $\Omega_0^0(M)$.*

If we put $\beta_i := \dim\ker \Delta_i, 0 \leq i \leq 2$, then a full set of spectral data is provided by spec Δ_0 *and*

$$\chi_{(2)}(M) := 2\beta_0 - \beta_1.$$

In view of Theorem , it is enough to compute the L^2–Euler characteristic, $\chi_{(2)}(M)$, and the spectral asymptotics of Δ_0. $\chi_{(2)}(M)$ has been determined by Brüning/Peyerimhoff/Schröder and Brüning/Lesch. Moreover, we have

Theorem 1) *For $t > 0$, e^{-tT_0} is trace class and we have the asymptotic expansion*

$$\mathrm{Tr}(e^{-tT_0}) \sim_{t\to 0+} \sum_{j\geq 0} a_j t^{j-1} + \sum_{j\geq 1} b_j t^{j-1} \log t + \sum_{\substack{p\in\Sigma \\ 1\leq i\leq L(p)}} \sum_{j\geq 0} c_j(i,p)\, t^{j/2N_i(p)}. \quad (3)$$

2) *In* (3), *we have*

$$a_0 = \frac{\mathrm{vol}\ M}{4\pi}. \quad (4)$$

and

$$b_1 = 0. \quad (5)$$

3)

$$\lim_{t\to 0+}(\mathrm{tr}\, e^{-tT_0} - a_0 t^{-1}) - \chi_{(2)}(M)/6 = \frac{1}{12} \sum_{\substack{p\in\Sigma \\ 1\leq i\leq L(p)}} (N_i(p) + N_i(p)^{-1} - 2). \quad (6)$$

4) *There are $M \in \mathcal{M}$ with $b_2 \neq 0$. More precisely, among the generalized parabolas $C^{k,l}$, b_2 distinguishes the parabolas of type $C^{1,l}, l \in \mathbb{N}$.*
5) *There are $M \in \mathcal{M}$ with $c_2(i,p) \neq 0$, for some $p \in \Sigma$ and $1 \leq i \leq L(p)$.*

R/Z-Index theory

John Lott

This talk was about an index theory in which the indices take value in $\mathbf{R}/\mathbf{Z}$. We first discussed the case of a single odd-dimensional manifold M. By general topological arguments, there is a pairing $K_{-1}(M) \times K^{-1}(M; \mathbf{R}/\mathbf{Z}) \to \mathbf{R}/\mathbf{Z}$. Using reduced eta-invariants, we gave an analytic version of this pairing. This used the Baum-Douglas description of K-homology and Karoubi's description of $K^{-1}(M; \mathbf{R}/\mathbf{Z})$. We gave applications to questions of the homotopy invariance of rho-invariants for the tangential signature operator, and the vanishing of rho-invariants for the Dirac operator on a manifold of positive scalar curvature.

We then described a families index theorem which gives an analytic description of the pushforward of an element of $K^{-1}(M; \mathbf{R}/\mathbf{Z})$ under a fiber bundle projection $\pi : M \to B$. The analytic pushforward was defined using the eta-form of Bismut-Cheeger. We gave an application to the case of the direct-image of a flat Hermitian vector bundle on M.

Combinatorial and analytic L^2–torsion

Wolfgang Lück

Given a connected finite CW–complex X, one can associate to the L^2–Laplacian $\Delta_p^{(2)}$ acting on the p–th cellular L^2–Hilbert space its L^2–Betti number $b_p^{(2)}(X)$, its Novikov–Shubin invariant $\alpha_p(X)$ and its (generalized) Fuglede Kadison determinant $\det(\Delta_p)$. We call X admissible if $b_p^{(2)}(X) = 0$ and $\alpha_p(X) > 0$ for all p. In this case one defines the combinatorial L^2–torsion $\varrho^{(2)}(X) = \sum(-1)^p p \det(\Delta_p) \in \mathbb{R}$. This invariant has the following properties: it is a homotopy invariant, satisfies a sum formula and a fibration formula. There are analytic counterparts which agree with the combinatorial one for $b_p^{(2)}$ and α_p. For a compact Riemannian manifold M the euqality $\varrho_{\text{an}}^{(2)}(M) = \varrho^{(2)}(M) + \log 2/2\chi(\partial M)$ is conjectured. The combinatorial L^2–torsion of a compact 3–manifold with incompressible torus boundary or empty boundary can be read of algebraically from a presentation of the fundamental group without knowing M. On the other hand the analytic L^2–torsion of a closed hyperbolic 3–manifold M is $-\frac{1}{3\pi}\text{Vol}(M)$. The L^2–torsion of an aspherical closed manifold seems to be related to the simplicial volume of Gromov and may be an obstruction for the existence of an S–foliation on an aspherical closed manifold.

Singular solutions of the scalar curvature equation

Rafe Mazzeo

The equation $\Delta u - u + u^{\frac{n+2}{n-2}} = 0$ is considered on S^n, and solutions are sought with prescribed singular set Λ, and such that the corresponding metric $g = u^{\frac{n}{n-2}} g_0$ is complete on $S^n \setminus \Lambda$. Previous work by Schoen, and Mazzeo–Smale give existence for a limited class of Λ. In recent joint work with F. Pacard, a solution is given for this whenever Λ is a disjoint union of submanifolds with dimensions in the range $1 \leq k \leq (n-2)/2$. The upper bound $(n-2)/2$ is sharp.

Local formulae for Pontryagin classes of singular manifolds

Henri Moscovici

In joint work with F. Wu, we gave a solution to the problem of representing the topological Pontryagin classes by locally constructible cycles, based on quantizing the tangent microbundle into a "signature" K–cycle. The formulae thus obtained, although reminiscent of the classical Chern–Weil expressions in terms of matrix-valued curvature forms, involve instead traces of (infinite–dimensional) operator-valued "quantized curvatures".

The talk discussed the existence of "classical limits" for these formulae on one hand, and the possibility of extending them to pseudomanifolds on the other hand.

The family index theorem on odd-dimensional manifolds with boundary

Paolo Piazza (joint work with Richard B. Melrose)

Let $D = (D_z)_{z\in B}$ be a family of generalized Dirac operators acting on Hermitian Clifford modules E_z over the odd-dimensional compact manifolds with boundary, M_z, which are the fibres of a fibration with compact base $B : X \longrightarrow M \longrightarrow B$; $\dim X = n = 2k+1$. Let x be the normal variable to the boundary of M. We assume that each fibre is endowed with a metric which is of product-type near the boundary: $g_z = dx^2 + g_{\partial M_z}$.

Using the fibre cobordism invariance of the index on closed manifolds we show the existence of a Cl(1) spectral section P for the family of Dirac operators D^0 induced on the boundary. This means that P is a self-adjoint family of pseudodifferential operators of order zero with the following properties : (1) $P_z^2 = P_z$. (2) There exists a constant $R \in [0,\infty)$ such that $D_z^0 u = \lambda u \Rightarrow P_z u = u$ for $\lambda > R$ and $P_z u = 0$ for $\lambda < -R$. (3) $\sigma P + P\sigma = \sigma$ with $\sigma = \mathrm{cl}(dx)$.

Acting on $H^1(M_z, E_z; P_z) = \{u \in H^1(M_z, E_z); u|_{\partial M_z} \in \mathrm{Ker} P_z\}$ the family D defines, by Green formula, a family of self-adjoint Fredholm operators D_P and thus an index class $\mathrm{Ind}(D_P) \in K^1(B)$. In case the boundary family D^0 is invertible, P can be taken to be the Atiyah-Patodi-Singer projection; a formula for the Chern character of $\mathrm{Ind}(D_{APS})$ was conjectured by Bismut and Cheeger.

We prove such a formula in the general case by combining the superconnection formalism of Quillen and Bismut, the calculus of b-pseudodifferential operators, a suspension argument and the family index theorem in the even-dimensional case proved in the paper "Families of Dirac operators, boundaries and the b-calculus" (by R.B. Melrose and P. Piazza). Thus

$$\mathrm{Ch}(\mathrm{Ind}(D_P)) = \frac{1}{(2\pi i)^{k+1}} \int_{M/B} \hat{A}(M/B)\mathrm{Ch}'(\mathrm{E}) - \frac{1}{2}\eta_{odd,P} \tag{7}$$

with $\eta_{odd,P}$ a generalized eta form which depends on the choice of P. A relative index theorem, describing the effect of changing the spectral section, is also described.

The use of Witten Laplacians in high dimension

J. Sjöstrand

This talk describes some improvements of earlier work with B. Helffer, the new inspiration coming mainly from a work of Zhizhina–Minlos. Let Γ be a finite set, $\varphi \in C^\infty(\mathbb{R}^\Gamma; \mathbb{R})$. Under some technical assumption on φ at infinity (roughly that $\varphi(x) \sim$

$|x|^{1+\delta}$ there). $\int e^{-2\varphi(x)/h}dx < \infty$ and can be assumed to be 1 after adding a constant to φ. Witten complex: $d_\varphi = e^{-\varphi/h}hde^{\varphi/h}$, d = exterior differentiation. Witten(-Hodge) Laplacian in degree l: $\Delta_\varphi^{(l)} = d_\varphi^* d_\varphi + d_\varphi d_\varphi^*$, discrete spectrum$\subset [0,\infty)$. The lowest eigenvalue of $\Delta_\varphi^{(0)}$ is 0 with $e^{-\varphi/h}$ as the corresponding eigenfunction, while $\inf \operatorname{spec}(\Delta_\varphi^{(1)}) > 0$.

Put Cor(u,v) =Correlation of u and v=$<(u- <u>)(v- <v>)>$, where $<u>= \int ue^{-2\varphi/h}dx = (e^{-\varphi/h}u|e^{-\varphi/h})$. Then

$$\mathrm{Cor}(u,v) = h^2((\Delta_\varphi^{(1)})^{-1}(e^{-\varphi/h}du)|e^{-\varphi/h}dv)$$

(implicit in work with B. Helffer.) **Consequence:** A simple proof of the FKG inequalities in the ferromagnetic case ($\partial_{x_j}\partial_{x_k}\varphi \le 0, j \ne k$). (Cf. work by Herbst-Pitt.)

Let $\Gamma = (\mathbb{Z}/N\mathbb{Z})^d, d$ fixed, $N \to \infty$. Under certain assumptions on $\varphi = \varphi_\Gamma$ (implying strict convexity), uniform w. r. t. Γ, we get an asymptotic formula for Cor(x_j, x_k) when h is small, N, dist(j,k) large, describing more precisely the exponential decay when dist$(j,k) \to \infty$. The result applies to the case $2\varphi(x) = \sum_{j\in\mathbb{Z}/N\mathbb{Z}} x_j^2/4 - \log\cosh\sqrt{\nu/2}(x_j + x_{j+1})$ ("Kac potential") when $0 < \nu < 1/4$ (the strictly convex case).

The proof of this result is based on the study of some lower part (and not just the lowest eigenvalue) of $\Delta_\varphi^{(1)}$, performed via a stabilizing problem. The techniques can probably be applied to more general Schrödinger operators in high dimensions.

Hodge structure and perverse sheaves for locally symmetric varieties

Mark Stern

Let X be a locally symmetric variety. The $L^2 - \bar\partial$ cohomology gives a Hodge structure for the intersection cohomology of X. We compute the local cohomology groups of this complex.

Theorem *The $L^2 - \bar\partial$ cohomology complex satisfies the vanishing and covanishing conditions of a perverse sheaf (appropriately shifted).*

Using this theorem we show how to construct a quasi-isomorphic complex on a toroidal resolution of X. We conjecture that this procedure might give a general approach to finding the Hodge structure of $IH^\cdot$ without using $\mathcal{D}$-modules.

Quantum ergodicity

Toshikazu Sunada

I shall introduce the notion of ergodicity at infinite energy level in both quantum and classical mechanics. It allows us to give a necessary and sufficient condition in terms of microlocal properties of eigenfunctions of a quantized Hamiltonian in order that the corresponding classical dynamical system is ergodic. Especially, a new insight into quantum ergodicity due to Snirelman, Zelditch and Colin de Verdiere is given.

Determinants of elliptic pseudo-differential operators

S. Vishik (results joint with M. Kontsevich)

The determinant of a linear operator $A \in \mathrm{End}(L)$ acting in a finite-dimensional linear space L is the product of the eigenvalues of A including their algebraic multiplicities. There are two different generalizations of $\det(A)$. The first is the Fredholm determinant $\det_{Fr}(\mathrm{Id}_H + K) := 1 + \mathrm{Tr}(K) + \mathrm{Tr}(\wedge^2 K) + \ldots$, where K is a trace class operator acting in a separable Hilbert space H. The series on the right is absolutely convergent for such K. This series is a finite sum for a finite-dimensional A, $K := A - \mathrm{Id}$, and it gives the value of $\det(A)$. The Fredholm determinant is a multiplicative one, i.e., $\det_{Fr}(AB) = \det_{Fr}(A)\det_{Fr}(B)$. Another generalization is the zeta-regularized determinant $\det_\zeta(A)$ of an elliptic PDO A introduced by D.B. Ray and I.M. Singer. It is defined as $\exp(\partial_s \zeta_A(s)|_{s=0})$, where the zeta-function $\zeta_A(s)$, $\mathrm{ord}\, A > 0$, is defined as $\mathrm{Tr}(A^{-s})$ for A^{-s} of trace class, i.e., for $\mathrm{Re}\, s \cdot \mathrm{ord}\, A < -\dim M$. To produce $\det_\zeta(A)$, we have to define a holomorphic family A^{-s} and to prove that $\zeta_A(s)$ has a meromorphic continuation in s which is regular at $s = 0$. The existence of A^{-s} is equivalent to the existence of a $\log A$. But the problem of existence of $\log A$ for a given elliptic A is unsolvable. If for general invertible A its $\log A$ exists, we can say nothing about the existence of $\log A_1$ for another invertible A_1 with the same complete symbol $\sigma(A_1) = \sigma(A)$. However we know that $\det_\zeta(A)$ depends not only on A but on a family A^{-s} of its complex powers. (For example, it is so for Dirac operators.) Hence we have to use something like $\log A$ in the definition of $\det(A)$. We use $\log \sigma(A)$, i.e., an element $b \in S_{\log}(M, E)$ of the Lie algebra of logarithmic symbols such that $\exp(b) = \sigma(A)$. The canonical determinant $\det(A, b)$ for an invertible elliptic PDO A, $\mathrm{ord}\, A \in \mathbb{C}^\times$, is defined as follows. Let $G = F_0 \backslash \mathrm{Ell}^\times(M, E)$ be the quotient by the normal subgroup $F_0 := \{\mathrm{Id} + K : K \text{ are smoothing}, \det_{Fr}(\mathrm{Id} + K) = 1\}$, and let $d_1(A)$ be the class of A in G. Then $\det(A, b) := d_1(A) \circ d_0(b)^{-1}$, where $d_0(b) \in G$ belongs to the same (as $d_1(A)$) fiber of the natural projection $p: G \to \mathrm{SEll}_{\mathrm{ind}\,0} = F\backslash\mathrm{Ell}$, $p(d_1(A)) \equiv \sigma(A) = p(d_0)$, $F := \{Q = \mathrm{Id} + K : \det_{Fr}(Q) \neq 0, K \text{ are smoothing}\}$. The group G is a central $\mathbb{C}^\times$-extension of the group $\mathrm{SEll}_{\mathrm{ind}\,0}$ of elliptic symbols of index zero. So $\det(A, b) \in \mathbb{C}^\times = F_0\backslash F$ (identification is given by $\det_{Fr}$). The element $d_0(b) := d_1(\exp B)\cdot(\det_{\mathrm{TR}}(\exp(-B))) \in G$ is defined with the help of any B from the Lie algebra of logarithms of elliptic PDOs with $\sigma(B) = b$. It is independent of a choice of B and it generalizes the $\det_\zeta(A)$ to its natural domain of definition. Here, $\det_{\mathrm{TR}}(\exp(-B)) := \exp(-\partial_s \mathrm{TR}(\exp(sB))|_{s=0})$, where TR is a new trace class functional defined for all classical PDOs of complex noninteger orders, $\mathrm{TR}([A, B]) = 0$.

Determinants II. Geometry of determinants. Description of the determinant Lie algebra in terms of symbols

S. Vishik (results joint with M. Kontsevich)

Claim. The Lie algebra $\mathfrak{g} := \mathfrak{f}\backslash\mathfrak{ell}$ of the determinant Lie group $G := F_0\backslash\mathrm{Ell}^\times$ can be explicitly described in terms of the Lie algebra $S_{\log}$ of logarithmic symbols only.

Corollary. *An element $d_0(b)$ in the definition of the canonical determinant* $\det(A, b)$ *can be defined algebraically in terms of $S_{\log}$.*

To realize this program, we define an adjoint-invariant quadratic form $T_2(x) := \mathrm{ord}\, x \cdot \partial_s \zeta_x^{\mathrm{TR}}(s)|_{s=0}$ on $\mathfrak{ell} \ni x$. Here, $\zeta_x^{\mathrm{TR}}(s) := \mathrm{TR}(\exp(-sx))$. This function is defined for $\mathrm{ord}\, x := \mathrm{ord}(\exp x) \neq 0$, and it is regular at $s = 0$. The right side of the expression for $T_2(x)$ is defined if $\mathrm{ord}\, x \neq 0$, and it is the restriction of a quadratic form defined on $\mathfrak{ell}$ to this domain. Let $B(x, y)$ be the associated with T_2 symmetric bilinear form on $\mathfrak{ell}$. Then $B(x + z, y) = B(x, y)$ for $z \in \mathfrak{f}_0$, and B defines a bilinear form on $\mathfrak{g}$. We have $B(1, 1) = 0$ for $1 \in \mathbb{C} = \mathfrak{f}_0\backslash\mathfrak{f}$, and $B(1, x) \equiv -\mathrm{ord}\, x \not\equiv 0$. Hence for any $b \in S_{\log}(M, E) = \mathfrak{f}\backslash\mathfrak{ell}$ with $\mathrm{ord}\, b \neq 0$ there is one and only one isotropic vector $b_1 \in \mathfrak{g}$ for B such that b_1 is mapped to b under the natural projection $\mathfrak{g} \to S_{\log}$, $B(b_1, b_1) = 0$.

Proposition. *We have $d_0(b) = \exp(b_1)$ in G.*

Proposition. *1. The Lie algebra $\mathfrak{g}$ has an explicit expression in terms of generators $\Pi_x y$, where $x \in S_{\log}$, $\mathrm{ord}\, x = 1$, $y \in S_{\log}$ (see Preprints MPI/94-30, MPI/94-57). 2. The form B on $\mathfrak{g}$ is given by an explicit formula*

$$B(\Pi_x y_1 + c_1 \cdot 1, \Pi_x y_2 + c_2 \cdot 1) = \mathrm{res}\,(a_1 a_2) - c_1 t_2 - c_2 t_1$$

for $x \in S_{\log}$, $\mathrm{ord}\, x = 1$, $y_j = t_j x + a_j$, where $a_j \in CS^0$ are classical PDO-symbols of order zero and $c_j \in \mathbb{C}$. Here, res *is the noncommutative residue.*

Corollary. *Let $b = tx + a \in S_{\log}$, $t \in \mathbb{C}^\times$, $a \in CS^0$. Then $b_1 = \Pi_x(tx + a) + c \cdot 1 = \log d_0(b) \in \mathfrak{g}$ for $c = \operatorname{res}(a^2)/2t$.*

Conclusion. The canonical determinant $\det(A, b)$ for $A \in \mathrm{Ell}^\times(M, E)$, $\operatorname{ord} A \in \mathbb{C}^\times$, $b = \log \sigma(A) \in S_{\log}$, is the product $d_1(A) \circ d_0(b)^{-1}$ of two elements of the determinant Lie group G. The factor $d_1(A)$ is the class of A in G and it is multiplicative, $d_1(AB) = d_1(A)d_1(B)$. The factor $d_0(b)^{-1}$ is computed in terms of symbols.

The "Classical Limit" for $PSL(2;\mathbb{Z})$

Scott Wolpert

Consider $H/PSL(2;\mathbb{Z})$ with hyperbolic metric, and D the Laplace–Beltrami operator acting in $L^2(H/\Gamma)$. Let $\{\varphi_j, \lambda_j\}$ be the collection of L^2-eigenpairs. We are interested in high-energy limits $\varphi_{jk}^2 dA \xrightarrow{w^*} \mu$, μ a measure, and particularly the question of "unique quantum ergodicity". S. Zelditch has shown that there is a full density subsequence $\{j_k\}$ with $\varphi_{jk}^2 dA \xrightarrow{w^*} dA/\text{area}$.

Are there "thin subsequences" with other limits?

What is the mechanism of convergence?

$SL(2;\mathbb{Z})$ contains a unit-translation. An eigenfunction invariant by unit-translation and in $L^2(y \geq a)$ has an expansion

$$\varphi(z) = \sum_{n \neq 0} a_n y^{1/2} K_{ir}(2\pi|n|y)e^{2\pi i n x} \quad \text{for} \quad z = x + iy,\ \lambda = 1/4 + r^2.$$

WKB can be applied to study the asymptotics of the product terms

$$yK_{ir}(2\pi ny)K_{ir}(2\pi my),$$

occuring in φ^2. In particular for $T \geq T_0 > 0, n, m > 0$ fixed, as $\lambda \to \infty$

$$\int_{T/t}^{\infty} yK_{ir}(2\pi ny)K_{ir}(2\pi my)\frac{dy}{y^2} = \frac{\pi e^{-\pi\lambda^{1/2}}}{2\sqrt{mn}} \int_{T}^{1} \frac{cos(2\pi/t|m-n|\sqrt{1-Y^2})}{\sqrt{1-Y^2}} \frac{dY}{Y} + o(1)$$

for $Y = yt$, $t = 2\pi n/\lambda^{1/2}$.

Let $G = \partial H \times \partial H \setminus \{\text{diagonal}\}$ be the space of geodesics on the upper half plane.

We are interested in the "integral"

$$(\mu, \Omega \subset\subset H) \mapsto \int_{\partial G} l(\gamma \cap \Omega)\mu(\gamma)$$

for μ a measure on G and $l(p)$ the hyperbolic length of an arc p.

Proposition Consider the "integral" for unit-translation invariant measures. The Right Hand Side of the K-Bessel integral is the $|m-n|$ Fourier coefficient of the kernel of the "integral".

Now if $\varphi_k(z) = \sum_n {}_k a_n y^{1/2} K_{ir}(2\pi|n|y)e^{2\pi inx}$ and $\varphi_k^2 dA \xrightarrow{w^*} \mu$, then μ is the push-down of a measure $\hat{\mu}$ on the space of geodesics

$$\hat{\mu}(\hat{\gamma}, t)|_{(a,b)\times[0,1]} = \sum_{m=-\infty}^{\infty} {}_{\mathrm{dist}} e^{2\pi i m \hat{\gamma}} w^* - \lim_k \frac{e^{-\pi\lambda_k^{1/2}}}{\lambda_k^{1/2}} \sum_{n\in(a\sqrt{\lambda}, b\sqrt{\lambda})} {}_k\bar{a}_n \, {}_k a_{n+m}.$$

In particular $\hat{\mu}$ is given by the w^*-limit of means of the numerical Fourier coefficients. Zelditch's result is reinterpreted to

$$\lambda^{-1/2} e^{-\pi\lambda^{1/2}} \sum_{|m|\le t\sqrt{\lambda}} {}_\lambda\bar{a}_n {}_\lambda a_{n+m} \longrightarrow \delta_{0m}\frac{12}{\pi^2}t$$

pointwise in $t > 0$, for the Kronecker delta and $\{{}_\lambda a_n\}$ the numerical Fourier coefficients of eigenfunctions from the full-density subsequence of Zelditch. The question of existence of "thin subsequences" is a matter of means of Fourier coefficients.

A proof of the Atiyah–Patodi–Singer index theorem

W. Zhang

In this talk, we report a joint work with X. Dai, in which we give a new proof of the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary.

Our proof is based on techniques developed by Bismut et. al. and the cone method of Cheeger. We embed the manifold with boundary to a higher dimensional ball, and reduce the problem to a ball, on which it is trivial.

This approach is different from the previous approaches, which are based on McKean–Singer type formulas. The Bott periodicity theorem is also avoided by the triviality of vector bundles over balls.

Further talks

M. Duflo: Equivariant differential forms and fixed points

V. Guillemin: Wave trace invariants for elliptic operators

W. Pardon: Hodge structure and boundary conditions

M. Rumin: Differential forms on contact manifolds

R. Sjamaar: Symplectic cross section and classification of Hamiltonian group actions

R. Stanton: The cohomology of flag manifolds revisited

M. Varghese: Amenability and L^2-invariants

M. Vergne: Geometric quantization and multiplicities

M. Zworski: Scattering matrix and geodesic flow at infinity

Berichterstatter: Matthias Lesch

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Tagungsteilnehmer

Prof.Dr. Jean-Michel Bismut
Mathématiques
Université de Paris Sud (Paris XI)
Centre d'Orsay, Bâtiment 425

F-91405 Orsay Cedex

Prof.Dr. Michel Duflo
U. F. R. de Mathématiques
Case 7012
Université de Paris VII
2, Place Jussieu

F-75251 Paris Cedex 05

Prof.Dr. Jochen Brüning
Institut für Mathematik
Universität Augsburg

D-86135 Augsburg

Prof.Dr. Jürgen Eichhorn
Fachrichtungen Mathem./Informatik
Universität Greifswald
Jahnstr. 15 A

D-17489 Greifswald

Ulrich Bunke
Institut für Reine Mathematik
Humboldt-Universität Berlin

D-10099 Berlin

Christian Gantz
Balliol College

GB-Oxford OX1 3BJ

Prof.Dr. Tanya Christiansen
Department of Mathematics
University of Pennsylvania
209 South 33rd Street

Philadelphia, PA 19104-6395
USA

Dr. Ezra Getzler
Dept. of Mathematics
RM 2-267
MIT

Cambridge, MA 02139
USA

Prof.Dr. Xianzhe Dai
924 West Campus Lane

Goleta, CA 93117
USA

Prof.Dr. Victor W. Guillemin
Dept. of Mathematics
RM 2-267
MIT

Cambridge, MA 02139
USA

Prof.Dr. Andrew Hassell
MIT
77 Massachusetts Avenue

Cambridge, MA 02139
USA

Prof.Dr. Wolfgang Luck
Fachbereich Mathematik
Universitat Mainz

D-55099 Mainz

Prof.Dr. Lisa C. Jeffrey
Mathematics Department
Princeton University
Fine Hall
Washington Road

Princeton NJ 08544 - 1000
USA

Prof.Dr. Rafe R. Mazzeo
Department of Mathematics
Stanford University
Building 380

Stanford, CA 94305
USA

Dr. Kai Köhler
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26

D-53225 Bonn

Prof.Dr. Richard B. Melrose
Department of Mathematics
Massachusetts Institute of Technology

Cambridge, MA 02139-4307
USA

Dr. Matthias Lesch
Institut für Mathematik
Universität Augsburg

D-86135 Augsburg

Prof.Dr. Henri Moscovici
Department of Mathematics
Ohio State University
231 West 18th Avenue

Columbus, OH 43210-1174
USA

Prof.Dr. John Lott
Department of Mathematics
University of Michigan

Ann Arbor 48109-1003
USA

Prof.Dr. Werner Müller
Mathematisches Institut
Universität Bonn
Wegelerstr. 10

D-53115 Bonn

Prof.Dr. William L. Pardon
Dept. of Mathematics
Duke University

Durham NC 27708
USA

Prof.Dr. Johannes Sjöstrand
Centre de Mathématiques
Ecole Polytechnique
Plateau de Palaiseau

F-91128 Palaiseau Cedex

Prof.Dr. Yiannis N. Petridis
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26

D-53225 Bonn

Prof.Dr. Robert J. Stanton
Department of Mathematics
Ohio State University
231 West 18th Avenue

Columbus, OH 43210-1174
USA

Prof.Dr. Paolo Piazza
Università "La Sapienza"
Istituto Matemàtico "G.Castelnuovo"
P.le Aldo Moro 2

I-00185 Roma

Prof.Dr. Mark A. Stern
Dept. of Mathematics
Duke University

Durham NC 27708
USA

Michel Rumin
Département de Mathématiques
Université de Paris Sud

F-91403 Orsay Cedex

Prof.Dr. Toshikazu Sunada
Department of Mathematics
Tohoku University

Sendai 980
JAPAN

Prof.Dr. Reyer Sjamaar
Dept. of Mathematics
RM 2-163
MIT

Cambridge, MA 02139
USA

Prof.Dr. Mathai Varghese
Department of Mathematics
University of Adelaide

Adelaide 5005
AUSTRALIA

Prof.Dr. Michele Vergne
Département de Mathématiques et
Informatiques
Ecole Normale Supérieure
45, rue d'Ulm

F-75230 Paris Cedex 05

Prof.Dr. Simeon Vishik
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26

D-53225 Bonn

Prof.Dr. Scott A. Wolpert
Department of Mathematics
University of Maryland

College Park , MD 20742
USA

Prof.Dr. Wei-Ping Zhang
Mathematical Science Research
Institute
1000 Centennial Drive

Berkeley , CA 94720
USA

Prof.Dr. Maciej Zworski
Department of Mathematics
The John Hopkins University

Baltimore , MD 21218
USA