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Chairmen of this meeting were Victor Bangert (Freiburg i. Br.), Yurii D. Burago (St. Petersburg) and Ulrich Pinkall (Berlin). Participants came from Canada, Germany, Great Britain, France, Russia, Switzerland and USA.

The 26 talks covered a wide field of differential geometry including e.g. Alexandrov Spaces, dynamics of the geodesic flow, surfaces of constant mean curvature and their discretization, spin-geometry, and affine differential geometry.

An important aspect of the meeting were the fruitful mathematical discussions outside the official program.

Abstracts

Compact manifolds of positive curvature

Uwe Abresch, Bochum

No abstract available.

Laminations of 3-tori by least area surfaces

Franz Auer, Freiburg

Let g be a Riemannian metric on $\mathbb{R}^3$ which is the lift of a metric on the torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Let $\mathcal{F}$ be the set of all properly embedded planes $F \subset \mathbb{R}^3$ which are homotopically area minimizing with respect to g (i.e. $A_g(D) \leq A_g(h)$ for every disk $D \subset F$ and every Lipschitz map $h \colon D \to \mathbb{R}^3$ with $h|\partial D = \mathrm{id}$, here A_g denotes area with respect to g) and do not have sefintersections when projected to the torus. Bangert (1990) proved: Every $F \in \mathcal{F}$ lies in a strip between two parallel Euclidean planes. Let $P(F) \in G(3,2)$ be the "direction" associated to these planes and $\mathcal{F}_P = \{F \in \mathcal{F} \mid P(F) = P\}$. Then $\mathcal{F}_P \neq \emptyset$ for all $P \in G(3,2)$. Assume $P \in G(3,2)$ is irrational, i.e. $\mathrm{rk}(P \cap \mathbb{Z}^3) < 2$, and let $\mathcal{F}(F)$ denote the closure of the orbit of $F \in \mathcal{F}_P$ under the $\mathbb{Z}^3$-action. Then the action of $\mathbb{Z}^3$ on $\mathcal{F}(F)$ has a unique minimal set $\mathcal{F}^{\mathrm{rec}}(F)$. It either forms a foliation or a lamination with gaps of $\mathbb{R}^3$. We show: If $P \in G(3,2)$ is irrational then $\mathcal{F}^{\mathrm{rec}}(F_1) = \mathcal{F}^{\mathrm{rec}}(F_2)$ for all F_1, $F_2 \in \mathcal{F}_P$. Thus there is a unique minimal set of the $\mathbb{Z}^3$-action on $\mathcal{F}_P$.

Topological Entropy of Geodesic flows on simply connected manifolds

Ivan Babenko, Moscow

The recent progress in lower bound estimations of topological entropy was discussed. The main results which were presented in the talk are the following:

If (M, g) is a C^∞ Riemannian simply connected manifold and $h(g)$ is the topological entropy of the corresponding geodesic flow, then

Theorem.

$$h(g) \geq \frac{1}{D^h(M,g)} \varlimsup_{k\to\infty} \frac{\ln \mathrm{rk}\, \pi_k(M)}{k},$$

where $\pi_k(M)$ is the k-th homotopy group of M and $D^h(M, g)$ is the total homological diameter of (M, g).

Corollary. If $\mathrm{rk}\Big(\sum\limits_{k=2\dim M}^{3\dim M-2} \pi_k(M)\Big) \neq 0$, then $h(g) > 0$.

Theorem 2. If $\dim M = 4$ and $b_2(M) = b_2 \geq 3$, then

$$h(g) \geq \frac{\ln\left[\left(b_2 + \sqrt{b_2^2 - 4}\right)/2\right]}{D_2^h(M,g)},$$

where $D_2^h(M, g)$ is the homological diameter of (M, g) in dimension 2.

Theorem 3. If the Ricci curvature of (M, g) is positiv, $\mathrm{Ric}(g) \geq kg$, then

$$h(g) \geq \frac{\sqrt{k}}{\pi\sqrt{\dim M - 1}} \varlimsup_{l\to\infty} \frac{\ln \mathrm{rk}\, \pi_l(M)}{l}.$$

The definition and the properties of $D^h(M, g)$ were discussed.

On Nodal Sets for Dirac and Laplace Operators

Christian Bär, Freiburg

Let M be a connected Riemannian manifold of dimension n, let D be a generalized Dirac operator on M. Examples are the classical Dirac operator acting on spinors, the operator $D = d + \delta$ acting on mixed differential forms, and $D = \bar{\partial} + \bar{\partial}^*$ acting on mixed $(0,p)$-forms. We show

Theorem. Let $s \not\equiv 0$ be a solution of $(D+h)s = 0$ where h is a potential. Then $\dim\big(\{x \in M \mid s(x) = 0\}\big) \leq n-2$.

This gives a quick proof of

Corollary1. Let $f \not\equiv 0$ be an eigenfunction of the Laplace operator. Then $\{x \in M \mid f(x) = 0\} = N_{\text{reg}} \cup N_{\text{sing}}$ where N_{reg} is a smooth hypersurface and $\dim(N_{\text{sing}}) \leq n-2$.

We also discuss consequences for differential forms, i.e.

Corollary 2. Let ω be a Δ-harmonic k-form on a compact manifold. Then $\dim\big(\{x \in M \mid \omega(x) = 0\}\big) \leq n-2$.

Note that Corollary 2 does not hold if M is not compact, nor does it hold if ω is a Δ-eigenfunction for a positive eigenvalue even if M is compact.

T or not T

Werner Ballmann, Bonn

(joint work with Jacek Świątkowski (Wroclaw))

Let Γ be a finitely generated group and $S \subset \Gamma$ a finite set of generators. We say that Γ satisfies Property T if there is an $\varepsilon > 0$ such that the following holds: if ρ is a unitary representation of Γ on a Hilbert space H and if H contains a nonzero vector v such that $\measuredangle(\rho(g)v, v) < \varepsilon$ for all $g \in S$, then ρ contains the trivial representation. This is independent of the choice of S, but ε depends on S of course. Property T was introduced by Kazhdan. If Γ is a lattice in the isometry group of an irreducible symmentric space X of noncompact type, then Γ satisfies Property T iff $X \neq \mathbb{R}H^n, \mathbb{C}H^n$. This indicates connections with geometry and rigidity theory. In the talk I discussed sufficient and necessary conditions for a properly discontinuous group of automorphisms of a polygonal complex to satisfy Property T.

Isotropic Pluriharmonic maps

Jost-Hinrich Eschenburg, Augsburg
(joint work with R. Tribuzy)

Minimal surfaces in 3-space allow an isometric deformation preserving the principal curvatures but rotating the corresponding directions. The following theorem generalizes this situation:

Theorem 1: Let M be a simply connected complex manifold, S a symmetric space of compact, euclidean, or noncompact type, and $f: M \to S$ a smooth map. Then f is pluriharmonic iff for all (or some) $\theta \in (0, \pi)$ there exits a map $f_\theta: M \to S$ with $\phi_\theta \circ df_\theta = df \circ e^{i\theta}$, where $\phi_\theta: f_\theta^* TS \to f^* TS$ is a parallel bundle isomorphism preserving the curvature tensor R^S of S.

Now let $f: M \to S$ be a full pluriharmonic map. f is called *isotropic* if $f_\theta = f$ for all θ.

Theorem 2:

a) $S = \mathbb{R}^n$: f is isotropic iff n is even and f is holomorphic ($\mathbb{R}^{2k} = \mathbb{C}^k$).

b) S of compact [noncompact] type: f is isotropic iff S is inner ($S = G/K$ with $\sigma \in K_0$) and there is a flag manifold [domain] Z over S and a holomorphic superhorizontal lift $\hat{f}: M \to Z$ of f.

Superminimal surfaces in H^4

Thomas Friedrich, HU Berlin

Superminimal surfaces in the Euclidean space E^4 as well as in the sphere S^4 have been studied from the local point of view since the beginning of this century (Kommerell – 1905, Eisenhard – 1912, Boruvka – 1928). Global results we find in the papers of Cabalei (1967) ... Bryant (1982). In this lecture we construct a family of complete superminimal surfaces in the hyperbolic space H^4. This family depends on a holomorphic function defined in some domain $\Omega \subset \mathbb{C}$. The method of the construction is an application of the Twistor Theory. In the "generic" case the completeness of the surfaces follows from the well-known asymptotic expansion near the boundary of the Bergman kernel of the domain Ω.

A characterization of all CMC triunduloids

Karsten Große-Brauckmann, Bonn
(joint work with Rob Kusner, Amherst and John Sullivan, Minneapolis)

Delaunay determined the mean curvature 1 (CMC) surfaces of revolution in 1841, there are embedded unduloids and immersed nodoids. We consider their generalization: triunduloids are embedded CMC surfaces with 3 ends and genus 0. They are prototypical for all CMC surfaces which are embedded and of finite topology, in the sense that these surfaces can be thought to decompose into triunduloids. This is similar to graphs that decompose into trivalent nodes.

We characterize the CMC triunduloids by a spherical triangle whose edge lengths are exactly the necksizes of the triunduloid, that is the necksizes of Delaunay unduloids each end is asymptotic to. This is done using Lawson's theorem and an observation by Karcher. As a corollary we obtain bounds on the asymptotic necksizes $0 < x, y, z \leq \pi$: they have sum at most 2π, and satisfy the triangle inequalities. It is interesting that surfaces with one necksize cylindrical are possible. By some reflection we obtain numerically the existance of finite total curvature CMC surfaces, namely we have a surface with 30 cylindrical ends and genus 1.

Dressing action on discrete and continuous CMC-surfaces

Guido Haak, SFB 288, TU Berlin

This is a report on work done in part with J. Dorfmeister (Univ. of Kansas). Starting from the integrable system approach to conformal CMC-immersions in $\mathbb{R}^3$, A. Bobenko and U. Pinkall introduced a definition for discrete CMC-surfaces. An infinite dimensional family of such discrete CMC-surfaces was produced by F. Pedit and H. Wu using a dressing action on a discrete cylinder. In the continuous case, by results of J. Dorfmeister, H. Wu, F. Burstall and F. Pedit, all finite type CMC-immersions, e.g. all CMC-tori, are contained in the dressing orbit of the cylinder. Thus, this class of discrete CMC-surfaces seems a natural candidate to search for discrete periodic CMC-surfaces, i.e. maps $\phi\colon \mathbb{Z}^2 \to \mathbb{R}^2$ satisfying the discrete CMC-condition such that for some $(k,l) \in \mathbb{Z}^2$ we have $\phi_{m+k,n+l} = T\phi_{mn}$, where T is a proper Euclidean motion in $\mathbb{R}^3$. Using the dressing method we rederive Pinkall and Sterling's classification of CMC-tori in terms of algebro-geometric data, and give an entirely parallel classification of discrete periodic CMC-surfaces in the dressing orbit of the discrete cylinder. In both cases the hyperelliptic curve C associated to a finite type surface is defined directly from the dressing matrix. The other main ingredient of the classification, an Abelian differential ω on C, is then obtained from the monodromy matrix in $\mathbb{R}^3$, i.e. the spinor representation of the Euclidean motion T. As opposed to the continuous case, where ω is an Abelian differential of the second kind, in the discrete case it is of the third kind, with poles which are determined by the lattice constants used in Pedit ans Wu's discretization of the cylinder.

Isoparametric submanifolds in Hilbert space

Ernst Heintze, Augsburg

It had been shown by Thorbergsson that (complete, connected, irreducible, full) isoparametric submanifolds of euclidean space of codimension at least 3 are homogeneous. In fact, they are orbits of the isotropy representation of some symmetric space. In recent years Terng considered infinite dimensional isoparametric submanifolds of a Hilbert space and developed a beautiful structure theory. This theory is not only interesting for its own sake but it is also a means to study submanifolds in compact symmetric spaces through a certain process of linearization. The outstanding problem is again the question of homogeneity. The purpose of the talk is to discuss a recent result together with Xiaobo Liu which says that a very large subset of an infinite dimensional isoparametric submanifold is homogeneous if the codimension is bigger than 1. Actually we conjecture that this subset coincides with the submanifold.

Darboux transforms of isothermic surfaces

Udo Hertrich-Jeromin, Berlin

After a short introduction to the classical approach to the Darboux transform we use the quaternion calculus to derive a Riccati type partial differential equation for Darboux transforms of isothermic surfaces. The quaternion calculus turns out to be very effective, as applications to Darboux and Bäcklund transforms of constant mean curvature surfaces show. Discretizing the Riccati equation we just derived provides a Darboux transform for discrete isothermic nets — the natural discretization of an isothermic surface given in conformal curvature line parameters. Returning to the starting point we see that this discrete Darboux transform reflects the geometric constellation of the smooth Darboux transform very well.

Values of manifolds in Gromov-Hausdorff convergence

Sergei Ivanov, St. Petersburg

Let M, M_k ($k = 1, 2, \dots$) be compact Riemannian manifolds of dimension $n \geq 2$. We write $M_k \to M$ if the M_k converge to M with respect to the Gromov-Hausdorff distance between metric spaces. Our question is for what topological types of M_k and M the convergence $M_k \to M$ implies that

$$\mathrm{Vol}(M) \leq \liminf \mathrm{Vol}(M_k). \qquad (*)$$

We prove:

Theorem 1. If M and the M_k are two-dimensional manifolds, possibly with boundaries, and $\sup_k |\chi(M_k)| < \infty$ (χ denotes the Euler characteristic) then $M_k \to M$ implies $(*)$.

Theorem 2. If the M_k are homotopy equivalent to M, and M admits either a map of nonzero degree onto the torus T^n or a map of nonzero degree modulo 2 onto $\mathbb{R}P^n$, then $M_k \to M$ implies $(*)$.

Both theorems 1 and 2 remain true if the limit space M is a Finsler manifold.

We also construct examples showing that

Theorem 3. For any Rimannian metric d on the sphere S^n, $n \geq 3$, there exists a sequence $\{d_k\}_{k=1}^{\infty}$ of Riemannian metrics on S^n, such that $(S^n, d_k) \to (S_n, d)$ and $\mathrm{Vol}(S^n, d_k) \to 0$ as $k \to \infty$.

On the Margulis constant of hyperbolic manifolds

Ruth Kellerhals, Göttingen

Let M be a complete oriented hyperbolic n-manifold of finite volume and sectional curvature -1. Denote by $\varepsilon = \varepsilon_n > 0$ the Margulis constant which is defined to be the supremum of all $\varepsilon > 0$ such that for each discrete subgroup $\Gamma < \mathrm{Iso}(H^n)$ and for each $x \in H^n$ the group $\Gamma_\varepsilon = \langle \gamma \in \Gamma \mid \mathrm{dist}(x, \gamma x) < \varepsilon \rangle$ is elementary (i.e. contains a finite orbit in $\overline{H^n}$). According to work of Thurston, M admits a decomposition into a thick and a thin part, $M =$

$M_{(0,\varepsilon)} \cup M_{[\varepsilon,\infty)}$, where $M_{[\varepsilon,\infty)} := \{p \in M \mid \mathrm{injrad}_p(M) \geq \frac{\varepsilon}{2}\}$ is the thick and compact part of M, while $M = M_{(0,\varepsilon)} := \{p \in M \mid \mathrm{injrad}_p(M) < \frac{\varepsilon}{2}\}$ denotes the thin part consisting of tubes around geodesics of length $< \varepsilon$ (bounded components) and — if M is non-compact — of cusp neighborhoods (unbounded components).

For $n = 3$, Robert Meyerhoff found the estimate $\varepsilon_3 > 0.104$ by estimating the size of cusp neighborhoods and by constructing tubes around geodesics whose (real) length l is bounded by $\frac{\sqrt{3}}{4\pi}\log^2(1+\sqrt{2}) \sim 0.107$. He showed that for $l \searrow 0$ one has that the tube volume $\nearrow$. We discuss generalizations, making use of $\mathrm{Iso}^+(H^n) = PSL(2, C_{n-2})$, where C_{n-2} is the Clifford algebra in $n-2$ generators; the part of cusp neighborhoods can be generalized yielding a universal lower bound for $\mathrm{vol}_3(M)$ if M has cusps. For the tube part we mention arising problems.

New local and global uniqueness results for hypersurfaces in space forms

Peter Kohlmann, Dortmund

We consider isometric hypersurface immersions of manifolds with nonnegative sectional curvature into a standard space form M_c^{n+1}. Let φ, ψ be symmetric functions of n variables fulfilling certain ellipticity conditions, φ being convex and ψ concave.

If φ and ψ, as functions of the principal curvatures, are constant on the hypersurface piece M then M is isoparametric of a certain list of types.

The proof is based on a very generalized version of the Weyl-identity and allows extensions to Codazzi tensors on manifolds with positive sectional curvature. Globally — for compact parameter manifolds — the isoparametric hypersurfaces are characterized by the constancy of one function, either φ or ψ.

Special cases covered by these theorems are results of H. Weyl (1916), Nomizu/Smyth (1969), Münzner (1971, unpublished), Cheng/Yau (1977), Walter (1986), A. Li (1987), Ecker/Huisken (1989) and Konfogiorgos (1990).

Apart the convex–concave principle we prove results in the same spirit for pairs of elementary symmetric functions $(\varphi, \psi) = (E_r, E_s)$ with $s \in \{1, 2\}$ *or* $r \in \{n-1, n\}$ and the same functions of the principal curvature radii. The definiteness problems involved are solved via representations by square sums found with computer support, based on the simplex algorithm.

A triangulated K3 surface with the minimum number of vertices

Wolfgang Kühnel, Stuttgart
(joint work with Mario Casella, Boston)

Any (simplicial) triangulation of a K3 surface requires at least 16 vertices, and it has to contain all posible $\binom{16}{3}$ triangles if there are exactly 16 vertices. In this talk we present such a 16-vertex triangulation of a K3 surface. It has an automorphism group of order 240 acting transitively on the set of oriented edges. This triangulation is unique if one assumes a primitive group action on the set of 16 vertices. As a corollary we obtain that a K3 surface contains the 2-dimensional skeleton of the 15-dimensional simplex, and that there exists a tight embedding of a K3 surface into 15-dimensional Euclidean space.

Natural Lagrangian and Symplectic Structures for Constant Mean Curvature Moduli Spaces and Related Geometric Variational Problems

Rob Kusner, Amherst and Princeton/IAS

We investigate a natural symplectic manifold — a configuration space of weighted, parametrized geodesics — into which the moduli space of certain properly embedded constant mean curvature (CMC) hypersurfaces in Euclidean or hyperbolic space (or, of complete, conformally flat, positive constant scalar curvature (CSC) metrics on the complement of certain sets in the n-sphere) maps with isotropic image. This map α, which assigns to each CMC hypersurface (or CSC metric) the geodesic axes, weights and phases of the (asymptotically Delaunay) ends, is a kind of 'Gauß map' for the moduli spaces. We also introduce a natural Lagrangian submanifold — the space of ends of balanced geodesic trees — that serves as model for the image of the CMC (or CSC) moduli spaces under certain nondegeneracy hypotheses. This approach allows us to separate the analytic issues (such as those concerning the nondegeneracy hypotheses) from questions about the geometry of the moduli spaces themselves. By focusing on the latter, particularly the geometry of the target and image of α, we provide a common framework for understanding both the CMC and CSC cases, and thus clarify further some of the curious analogies between these cases.

Generic metrics for Dirac operators on 2-manifolds

Stephan Maier, Basel

The Atiyah-Singer Dirac operator $\mathcal{D}_g$ defined on a closed compact manifold M depends upon the metric g on M, and in general $h^0(g) := \dim\ker\mathcal{D}_g$ is not constant in g. However, one has the inequality $h^0(g) \geq |\operatorname{Index}\mathcal{D}_g|$. Is it true that for the generic metric g (i.e. g taken from a suitable C^∞-dense and C^1-open subset of the set of metrics) equality holds? The mod 2-index-theorem shows that if $\dim M = 1$ or $2 \mod 8$ this cannot be true, so that in these dimensions the problem should be rephrased as follows: Is it true that for the generic g we have $h^0(g) = \text{constant} \in \{0,1\}$ if $\dim M = 1 \mod 8$, and $h^+(g) = \text{constant} \in \{0,1\}$ if $\dim M = 2 \mod 8$?

The answer is affirmative if $\dim M \in \{1,2,3,4\}$, and we prove this in dimension 2. The method of proof is to study those metrics g for which we cannot find an infinitesimal deformation which decreases h^0, and to show that these metrics are of a very special kind. In fact, if $\dim M = 3$ or 4 a metric of this kind with an excessive number of harmonic spinors forces (M,g) to be the flat torus.

If $\dim M = 2$ the result can be cast into the language of Riemann surfaces and Teichmüller spaces. It then asserts that once a square root L of the canonical bundle of a Riemann surface is fixed, the dimension $h_0(L)$ of the space of holomorphic sections of L is constant $\in \{0,1\}$ for the generic complex structure on M, i.e. complex structure in the complement of an analytic subset of Teichmüller space.

Stable norms for surfaces

Daniel Massart, Warwick

The stable norm on the q-homology of a compact manifold M endowed with a (Riemannian or smooth Finsler) metric g, is an asymptotic invariant, the most general definition of which is given by Federer.

For $v \in H_q(M,\mathbb{R})$, $\|v\| = \inf\{\operatorname{vol}_q(\lambda) \mid \lambda \text{ Lipschitz } q\text{-cycle}, [\lambda] = v\}$, where vol_q is the q-dimensional volume induced by the metric g.

For $q = 1$ we have a more convenient definition:

For $v \in H_1(M,\mathbb{Z})/\text{torsion} \hookrightarrow H_1(M,\mathbb{R})$, $\|v\| = \min\{\sum r_i \lg(\gamma_i) \mid \sum r_i[\gamma_i] = v\}$

We are interested in strict convexity and regularity properties of $\| \, \|$. In the case of surfaces of higher genus, our results are:

Theorem 1. Every point of the unit sphere lying on a rational direction belongs to a (genus$(M)-1$)-dimensional face of the unit sphere.

Theorem 2. At such a point x, the stable norm is differentiable only in the directions tangent to the maximal face containing x in its interior.

These results mean that at a rational direction the unit sphere is flat or angulous. This contrasts with the case of flat tori, where the stable norm is euclidean.

Scalar Curvature Rigidity

Maung Min-Oo, McMaster Univ.

This lecture is a short selected survey of results on scalar curvature rigidity of certain symmetric spaces, in particular, for the Euclidean, hyperbolic and spherical metrics.

The newer results presented are:

Theorem 1. Let (M^n, g) be a compact Riemannian spin manifold with simply connected boundary ∂M such that:

(i) ∂M is totally geodesic

(ii) the metric induced on ∂M has constant sectional curvature $K \equiv 1$

(iii) the scalar curvature of g satisfies $R(g) \geq n(n-1)$.

Then (M, g) is isometric to the round hemisphere with the standard metric.

Theorem 2. Let $(M^{2n}, \bar{g}, \bar{\omega})$ be an irreducible Hermitian symmetric space with Kähler form $\bar{\omega}$. If g is any Riemannian metric on M satisfying $|\bar{\omega}|_g < |\bar{\omega}|_{\bar{g}}$ then there exists a point in M where the scalar curvatures satisfy $R(g) < R(\bar{g})$.

The proofs, using spinors, are sketched, with an attempt to show the basic underlying structure.

Surfaces, Quaternions and Spinors

Franz Pedit, Amherst

(joint work with Ulrich Pinkall, Berlin and George Komlikov, Amherst)

The lecture dealt with proposing an extension of complex function theory to, what we call, a quaternionic (valued) function theory in the following sense: regarding $\mathbb{R}^4 = \mathbb{H}$ and $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ we observe that an immersion $f\colon M \to \mathbb{R}^3$, M a Riemann surface, is conformal if there exists a map $N\colon M \to \mathbb{H}$ so that

$$*df = N df \tag{1}$$

where $*\alpha = \alpha \circ J$ for an $\mathbb{H}$-valued 1-form $\alpha \in \Omega^1(M, \mathbb{H})$. Here $J\colon TM \to TM$ is the complex structure on M and thus $*$ is negative the usual Hodge-star operator. For $f = hj$, $N = i$, the above condition is the Cauchy-Riemann equation for $h\colon M \to \mathbb{C}$. From equation (1) we derive the fundamental equations of surface theory by successive differentiation:

$$\begin{aligned} d{*df} &= 2HN|df|^2 \\ dN &= H\,df + \omega, \qquad *\omega = -N\omega \\ d\omega &= (*dH + N\,dH)df \qquad \text{(Codazzi)} \\ |\omega|^2 &= (H^2 - K)|df|^2 \qquad \text{(Gauß)} \end{aligned}$$

In all computations we use wedge-product over $\mathbb{H}$, i.e. $\alpha \wedge \beta(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$, and identify 2-forms with quadratic functions on TM via

$$\omega(X) = \omega(X, JX), \quad \omega \in \Omega^2(M, \mathbb{H}).$$

The main idea is what we call a *spin-transform* on a conformal surface $f\colon M \to \mathbb{R}^3$: let $\lambda\colon M \to \mathbb{H}$ and consider the 1-form

$$\alpha = \bar{\lambda}\, df\, \lambda \tag{2}$$

Then α is closed, i.e. $d\alpha = 0$, if and only if

$$*d\lambda + N\,d\lambda = \rho\, df\, \lambda \tag{3}$$

for some real $\rho\colon M \to \mathbb{R}$. Thus, if $\pi_1(M) = 0$, we can integrate to a new conformal immersion $\tilde{f}\colon M \to \mathbb{R}^3$ with

$$d\tilde{f} = \bar{\lambda}\, df\, \lambda. \tag{4}$$

Equation (3), when applied to $f = zj$, the plane with $N = i$, gives the Dirac equation for $\lambda = \lambda_1 + j\lambda_2$ with potential ρ:

$$\begin{aligned} \lambda_{1,z} &= \frac{i\rho}{2}\lambda_2 \\ \lambda_{2,\bar{z}} &= \frac{i\rho}{2}\lambda_1. \end{aligned} \tag{5}$$

$d\tilde{f}$ is a quadratic expression in λ_1, λ_2 which reduces to the classical Weierstraß representation for minimal surfaces in case $\rho \equiv 0$. It should be pointed out that equation (5) has already been discussed by Uwe Abresch in the mid 80's for constant mean curvature surfaces, it can also be found in Sasha Bobenko's work and has been shown to us by Iskander Taimanov (who told us that it has been rediscovered by B. Konopelchenko in the 90's). Apparently, Eisenhard knew of it already in some form or the other.

Finally, we apply our viewpoint to solve a classical problem of surface theory: we classify all Bonnet pairs, i.e. pairs of surfaces with same mean curvature and induced metric, in terms of isothermic surfaces. The relationship is rather simple: if $f\colon M \to \mathbb{R}^3$ is an isothermic surface with dual surface $f^*\colon M \to \mathbb{R}^3$, i.e. $df \wedge df^* = 0$, then $f_\pm\colon M \to \mathbb{R}^3$, given by

$$df_\pm = \overline{(\pm\varepsilon + f^* + a)}\, df\, (\pm\varepsilon + f^* + a)$$

for $\varepsilon \in \mathbb{R}$, $a \in \mathbb{R}^3$, give a Bonnet pair. All Bonnet pairs arise this way. We obtain a 4-parameter family of Bonnet pairs to each isothermic surface.

Where do spinors come in? Well, the λ's appearing in the spin-transform can be viewed as quotients of sections of a quaternionic line bundle, a spin bundle over M. Then (5) becomes

$$D\varphi = U\psi$$

where D is the conformal Dirac operator and U is a half density on M.

Gromov-Levy Isoperimetric Inequalities and Subharmonic functions on Alexandrov space

Anton Petrunin, St. Petersburg

Theorem 1 (Generalized Gromov-Levy Inequalities). Let M^n be an Alexandrov space with $K \geq 1$ and let Σ divide $\operatorname{Vol} M^n$ in ratio α. Let S^n be the standard sphere and let Σ^* be a sphere in S^n which divides S^n in ratio α. Then

$$\frac{\operatorname{Vol}\Sigma}{\operatorname{Vol}\Sigma^*} \geq \frac{\operatorname{Vol}M^n}{\operatorname{Vol}S^n}.$$

Main technical tool is subharmonic functions on the Alexandrov space.

Theorem 2. Let $f\colon \Omega \to \mathbb{R}$, $\Omega \subset M$ open, be a harmonic function (i.e. minimizing energy $E = \int_\Omega |\operatorname{grad} f|^2\, dh_n$), and $C \subset \Omega$ a compact subset, then $f|C$ is Lipschitz.

Discrete Surfaces of higher Polynomial Order

Konrad Polthier, Berlin

In the last years the study of triangulated surfaces has led to successful algorithms, e.g. for the study of discrete minimal and discrete constant mean curvature surfaces. These results are based on the strategy of "geometric discretization" (rather than e.g. finite differences) where one defines on triangulated surfaces discrete equivalents of smooth differential geometric terms such as minimality, harmonicicty, conformality etc.

In the talk I presented a new approach of generalizing the above ideas to triangulations of piecewise higher polynomial order, and corresponding maps between them. This concept provides beside higher order algorithms a more flexible adaption of geometric questions and also error estimates.

We use Bezier elements which come together with a net of controlpoints containing geometric information. As an application we derive a discrete version of the Dirichlet energy and harmonic maps given in terms of the control net.

Some of the ideas came out in discussions with U. Pinkall and G. Greiner.

Centroaffine surfaces in $\mathbb{R}^4$ and feedback equivalence of control systems

Christine Scharlach, TU Berlin

We study the centroaffine geometry of surfaces in $\mathbb{R}^4$, i.e. we study immersions $x\colon U \to \mathbb{R}^4 \setminus \{0\}$, $U \overset{\circ}{\subset} M^2$, and their invariants under general linear transformations ($GL(4,\mathbb{R})$). As usual we restrict to nonconical surfaces, for which the position vector never becomes tangential to the surface and thus can be considered as a transversal vector. We classify surfaces with respect to their two wellknown invariant semiconformal structures:

– the affine semiconformal structure given by

$$\phi_F(X) = \frac{[dx(X_1), dx(X_2), D_X dx(X_1), D_X dx(X_2)]}{[dx(X_1), dx(X_2), \xi, \eta]},$$

for an arbitrary constant determinant form $[\,,\,,\,,\,]$ on $\mathbb{R}^4$ and $F = (dx(X_1), dx(X_2), \xi, \eta) \in GL(4, \mathbb{R})$, and

– the h^3-semiconformal structure given by

$$h^3_F(X) = \frac{[dx(X_1), dx(X_2), D_X dx(X), X]}{[dx(X_1), dx(X_2), \xi, X]}.$$

For h^3-non-degenerate surfaces we introduce a centroaffine invariant symmetric bilinearform, the centroaffine metric g, by

$$g(X,Y) = \varepsilon \frac{1}{\sqrt{|\det_{\{X_1,X_2\}} h^3_F|}} \phi_F(X,Y).$$

Then we investigate the class of (positive) definite oriented surfaces. We show that there exists a uniquely determined centroaffine frame field $F = (dx(X_1), dx(X_2), n, x)$ such that

$$\begin{aligned} D_{X_i} dx(X_j) &= dx(\nabla_{X_i} X_j) + (-1)^{i+1}\delta_{ij} n + (1-\delta_{ij})x, \\ D_{X_i} n &= -dx(S_n X_i) + \tau(X_i)x, \end{aligned} \tag{$*$}$$

i.e. the second fundamental forms are normalized w.r.t. $\{X_1, X_2\}$ and $[dx(X_1), dx(X_2), D_{X_i} n, x] = 0$. To formulate the fundamental theorem (uniqueness and existence) we need either the induced connection ∇ or the (totally-symmetric) cubic form $C^3 = \nabla h^3$ and a pair of the three invariant symmetric bilinearforms (the metric and the two second fundamental forms). Now we can classify surfaces with planar ∇-geodesics (joint with L. Vrancken) and surfaces with vanishing cubic forms. As an application of our theory we study complex curves $x\colon U \to \mathbb{C}^2$, $U \overset{o}{\subset} \mathbb{C}$, which we can characterize by centroaffine invariants. The centroaffine structure equations $(*)$ in complex notations are:

$$d\begin{pmatrix} x' \\ x \end{pmatrix} = \begin{pmatrix} \kappa\, ds & -i\, ds \\ ds & 0 \end{pmatrix} \begin{pmatrix} x' \\ x \end{pmatrix}.$$

The 0-curvature curves are given by $x(z) = (z, \sqrt{z^2+1})$ and $x(z) = (z, \frac{1}{z})$.

Finally we explain how centroaffine geometry arises in the problem of (static state) feedback equivalence of control systems (cp. R. B. Gardner and G. Wilkens, Classical Geometries arising in Feedback Equivalence, Proc. 32 conf. of IEEE-CDC, San Antonio, 1993, 3437–3440).

Tits boundary of analytic Hadamard spaces

Viktor Schroeder, Zürich
(joint work with Christoph Hummel)

Let $M = \Sigma\backslash X$ be a compact, real analytic Riemanian manifold with non-positive sectional curvature. On the ideal boundary $X(\infty)$ of the universal covering X of M one can define the Tits metric Td. The metric space $(X(\infty), \mathrm{Td})$ reflects much of the asymptotic geometry of X. We investigate the relation between the fundamental group Σ and the space $(X(\infty), \mathrm{Td})$. It is shown that up to dimension 3 the space is completely determined by Σ while in dim 4 new and interesting phenomena occur. It turns out that $(X(\infty), \mathrm{Td})$ has "non-standard" components, which are not determined by the structure of the flat subspaces. However, these components have diameter $< \pi$ and are contractible. Thus the homology of $(X(\infty), \mathrm{Td})$ is completely determined by Σ. Applying results of B. Kleiner we then obtain a complete description of all quasiflats in X.

Higher order Codazzi tensors and topology

Udo Simon, Changping Wang, TU Berlin
(joint work with H. L. Liu)

Definition. (i) Let M be a connected, oriented C^∞-manifold, $\dim M \geq 2$, ∇ an affine connection, Φ a $(0, m)$-tensor field, totally symmetric, $m \geq 1$. Φ is called ∇-Codazzi of order m if $\nabla\Phi$ is totally symmetric. If $\nabla = \nabla(g)$ is the Levi-Civita connection of a metric g we call Φ Codazzi relative to g.
(ii) Φ is called traceless relative to a metric g if $\mathrm{trace}_g\,\Phi = 0$.

Theorem. Let M be a compact, oriented surface of genus γ and g a Riemannian metric on M. For $m \geq 2$ define the $\mathbb{R}$-vector space $\mathcal{R}_m := \{\Phi \mid \mathrm{trace}_g\,\Phi = 0,\ \Phi \text{ Codazzi of order } m \text{ relative to } g\}$. Then

$$\dim \mathcal{R}_m = \begin{cases} 0 & \text{for } \gamma = 0 \\ 2 & \text{for } \gamma = 1 \\ 2(2m-1)(\gamma - 1) & \text{for } \gamma > 1 \end{cases} .$$

Theorem. Let (M, g) be a Riemannian surface with curvature K and Φ a $(0, m)$-Codazzi-tensor relative to g, $\mathrm{trace}_g\,\Phi = 0$. Then $\frac{1}{2}\Delta\|\Phi\|^2 = \|\nabla\Phi\|^2 + mK\|\Phi\|^2$.

Theorem. Let (M, g) be Riemannian, $n = \dim M \geq 3$. Let Φ be a traceless $(0, m)$-Codazzi tensor relative to g and assume $0 \equiv W$ (Weyl conformal curvature tensor). Then

$$\frac{1}{2}\Delta\|\Phi\|^2 = \|\nabla\Phi\|^2 + \frac{n+2m-4}{n-2}\alpha^{ij}\Big(R_{ij} - \frac{(m-1)n}{n+2m-4}K g_{ij}\Big),$$

where R_{ij} = components of the Ricci-tensor, K = normal scalar curvature, $\alpha_{ij} := \Phi_{ii_2\ldots i_m}\Phi_j^{i_2\ldots i_m}$.

We list many examples of Codazzi tensors for order $m = 2, 3, 4$ and give applications of the above results to (i) Weyl geometries, (ii) c-mc h-surfaces in space forms, and related results, (iii) hypersurfaces in affine differential geometry, (iv) isoparametric hypersurfaces.

The Minimax Sphere Eversion

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A sphere eversion is a regular homotopy which turns a sphere inside out (without tearing or pinching). Smale proved this was possible in the 50's, but it took a long time before they (notably Morin) found explicit ways to do it. We look for an optimal eversion, in the sense of requiring the least bending at any stage. We measure bending energy by the Willmore integral $W(S) = \int_S H^2\,dA$. The round sphere is the minimum for this energy, with $W = 4\pi$. W is invariant under Möbius transformations, except that if we send a k-fold point on the sphere to ∞ we lose $4\pi k$ (as k spheres are converted to planes). A theorem of Banchoff-Max says any sphere eversion must have a quadruple point at some stage, so we look for one in which the maximum of W is 16π, at a saddle point for W. Bryant classified all spheres critical for W, which arise as Möbius transforms of minimal surfaces in $\mathbb{R}^3$ with k flat ends (and these have $W = 4\pi k$). Kusner found one such surface with $k = 4$ and the right symmetry to be a "Morin halfway model": a 90° rotation turns the surface inside out, since it has 4 lobes of alternating orientation. We use Brakke's evolver to numerically produce our minimax eversion, starting with the halfway model and flowing downhill by conjugate gradient methods to minimize W. The eversion we get is topologically equivalent to one by Morin, and has the simplest possible sequence of events.

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