MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Partielle Differentialgleichungen

06.07.—12.07.1997

This workshop was organised by L. Craig Evans (Berkeley), Gerhard Huisken(Tübingen), and Leon Simon (Stanford). 45 participants have been invited to discuss new developments in the broad field of partial differential equations.
The program consisted of four 45-minutes lectures in the morning and was supplemented by some 20- to 30-minutes talks before dinner. The afternoons and the evenings offered the possibility for informal discussions and joint work on projects.
The organisers and the participants are grateful to the Oberwolfach Institute for presenting the opportunity and the resources to arrange this interesting meeting.

Vortragsauszüge

BEN ANDREWS

Eigenvalue estimates with applications to geometric evolution equations

We consider a collection of evolution equations including
(1) Parabolic "Prescribed scalar curvature" equation:
If $\frac{\partial}{\partial t}g_t = -\frac{R[g_t]}{K}g_t$, then the metrics g_t converge (after rescaling to fixed volume) to a metric with $R = K$, if they remain comparable to g_0 up to scaling. The Kazdan-Warner obstructions to existence of such metrics imply that often the rescaled metrics *must* "blow up" for arbitrary initial data. The theorem of Hersch also shows that for *any* non-constant K the same result holds for generic initial data.
(2) Gauss curvature flows:
Consider hypersurfaces moving by $\frac{\partial x}{\partial t} = -\Psi(n)K^\alpha n$, $\Psi : S^n \to \mathbf{R}^+$ smooth. We prove an analogue of Hersch's eigenvalue estimate which shows any possibly limiting solution is unstable if $\alpha < \frac{1}{n+2}$ or if $\alpha = \frac{1}{n+2}$ and Ψ is non-constant. Hence the isoperimetric ratio must approach infinity for solutions starting from generic initial data. We also prove analogues of the Kazdan-Warner obstructions, which imply the same result for *arbitrary* initial data for many of these equations.

CLAIRE C. CHAN

Complete area-minimizing hypersurfaces with prescribed asymptotics at infinity

Given a minimal cone $C = \{r\omega : \omega \in \Sigma, r > 0\}$ with isolated singularity, if C is strictly minimizing, it is shown that there exists a large family of complete area-minimizing hypersurfaces asymptotic at infinity to C, that is, surfaces of the form $\mathrm{graph}_C\, u \equiv \{x \in \mathbf{R}^{n+1} : x = r\omega + u(r\omega)\vec{\nu}(\omega), r\omega \in \Omega \subset C\}$ outside a compact set, where $\vec{\nu}$ is a choice of unit normal on C, and $r^{-1}u(r\omega) = o(1)$ as $r \to \infty$. The result holds also in the strictly minimizing, non-strictly stable case, although for simplicity, only the strictly stable case is described here.
More specifically, the linearisation of the mean curvature operator, i.e. Jacobi field operator $\mathcal{L}_C$, admits separation of variables. Separated solutions of $\mathcal{L}_C\psi = 0$ are of the form $r^{\gamma_j^\pm+1}\phi_j(\omega)$. In case C is also strictly stable, the indices $\gamma_j^\pm$ are ordered as follows $\cdots \le \gamma_{j+1}^- \le \gamma_j^- \cdots < \gamma_1^- < \gamma_1^+ < \cdots \le \gamma_J^+ < 0 = \gamma_{J+1}^+ \le \cdots \gamma_j^+ \le \gamma_{j+1}^+ \cdots$. We denote by $\mathcal{J}$, the J-dimensional space of "slow-decaying" Jacobi fields $\left\{\sum_{j=1}^J c_j r^{\gamma_j^+ +1}\phi_j\right\}$. Corresponding to each minimizing hypercone with isolated singularity C are complete hypersurfaces $S^\pm \supset \mathrm{graph}_C u^\pm$ which are up to homotheties unique minimizers lying on either side of C. A *strictly* minimizing cone is characterised by the slow decay $r^{-1}u^\pm = \pm c^\pm r^{\gamma_1^+}\phi_1 + O(r^{\gamma_1^+ -\epsilon})$. We show that $\mathcal{J}$ parametrizes a J-parameter family $\mathcal{M}$ of distinct complete minimizing hypersurfaces. In particular, if $J > n+2$, there exists a continuum of non-congruent complete minimizing hypersurfaces asymptotic to C at infinity.
A sketch of the proof goes as follows: By the contraction mapping principle, the set $C \setminus B_1(0)$ is perturbed to a minimal surface (with boundary) having the desired "slow-asymptotics" at infinity. Specifically, given $\psi_1, \psi_2 \in \mathcal{J}$, with $\|\psi_1\|_\Sigma, \|\psi_2\|_\Sigma \le \epsilon$, there correspond minimal graphs u_{ψ_1}, u_{ψ_2} satisfying

$$u_{\psi_1} - u_{\psi_2} = \psi_1 - \psi_2 + o(\|\psi_1(r\cdot) - \psi_2(r\cdot)\|_{L^2(\Sigma)}). \qquad (*)$$

Now fixing some $\psi \in \mathcal{J}$ the perturbed surface $T_1 \supset \mathrm{graph}_C u_1$ is then arbitrarily completed. Using T_1 as an obstacle, a complete "one-side area-minimizing" surface T_2 is found amongst surfaces lying on one side of T_1. Then $T_2 \supset \mathrm{graph}_C u_2$ is used as an obstacle to find a complete "one-sided minimizer" $T_3 \supset \mathrm{graph}_C u_3$ (on the T_1 side). The resulting T_3 must then be actually minimizing. In the process of obtaining the complete minimizer, a sequence of obstacle problems are solved while moving the prescribed boundaries to infinity. The strictly minimizing property is used to guarantee that the sequence of solutions does not diverge to infinity. In addition, by means of a "fast-decaying" barrier previously employed by Hardt and Simon, each one-sided minimizer is shown to be "rapidly" asymptotic to the obstacle, in the sense that $r^{-1}(u_1 - u_2), r^{-1}(u_2 - u_3) = O(r^{\gamma_1^+ -\epsilon})$. Thus the minimizers $\{T_3(\psi)\}_{\psi\in\mathcal{J}}$ finally obtained still have distinct slow-asymptotics in the sense of $(*)$.

Gerhard Dziuk

Anisotropic mean curvature flow of graphs

Anisotropy of space is introduced via an anisotropy function $\gamma : \mathbf{R}^{n+1} \to \mathbf{R}^+$ which is positively homogeneous of degree one. The problem of mean curvature flow for the graph $\Gamma(t) = \{(x, u(x,t)) | \; x \in \Omega\}$, $\Omega \subset \mathbf{R}^n$ bounded, is to find a function $u = u(x,t)$ which solves the equation

$$u_t - \sqrt{1 + |\nabla u|^2} \sum_{j,k=1}^{n} \gamma_{p_j p_k}(\nabla u, -1) u_{x_j x_k} = 0$$

in $\Omega \times (0,T)$. It is proved that under natural assumptions, including adequate convexity of γ and γ–mean convexity of $\partial\Omega$, a global classical solution exists. A Finite Element Method for the numerical solution of the problem is derived from a variational form of the equation and asymptotic convergence is proved. The algorithm can be applied experimentally to crystalline anisotropy. This is joint work with Klaus Deckelnick, Freiburg.

Klaus Ecker

Interior estimates for mean curvature flow of spacelike hypersurfaces

We present new interior gradient and curvature estimates for spacelike hypersurfaces in Lorentzian manifolds evolving by mean curvature. The technical problems involved are very different from those arising in mean curvature flow in Riemannian manifolds.
These estimates are then used to establish the existence of a global smooth solution of the flow for arbitrary non–compact spacelike initial data in Minkowski space. Selfsimilar solutions of the flow will also be discussed.

L. Craig Evans

Monge-Kantorovich mass transfer and stochastic sandpile models

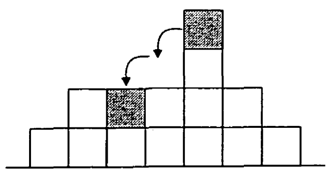
A problem posed by Monge in the early 1780's is, in modern terms, this: Given 2 measures $d\mu^+ = f^+ dx$ and $d\mu^- = f^- dy$, with $\int f^+ \, dx = \int f^- dy$, find a mapping $\underset{\sim}{s}^*$ which minimizes the work

$$w[s] := \int c(x, \underset{\sim}{s}(x)) f^+(x) \, dx \qquad (*)$$

among all maps satisfying the constraint $\underset{\sim}{s}_\#(\mu^+) = \mu^-$, i.e.

$$f^-(\underset{\sim}{s}) \, det D\underset{\sim}{s} = f^+ \qquad (**)$$

If $\underset{\sim}{s}^*$ is a minimizer, a first-variation calculation shows $D_x C(x, \underset{\sim}{s}^*(x)) = Du^*$ for some potential v^a (= Lagrange multiplier for the constraint (**)). The cases $C(x,y) = \frac{1}{2}|x-y|^2$, $|x-y|$ are particularly interesting.
Some (crude) models of "sandpile growth" can be interpreted as a Monge-Kantorovich mass transfer on a "fast" time scale, e.g. $f - u_t \in \partial I[u]$, where u = height, $I[u] = 0$ if $|Du| \leq 1$ a.e., $= +\infty$ otherwise . Recently Rezakhanlou and I have introduced a related stochastic model:



MARTIN FLUCHER

Construction of comparison functions by p-harmonic transplantation

We define the p-harmonic transplantation of a radial function $U = \Phi \circ G_0$ to be $u := \Phi \circ G_x$. Here G_x denotes the p-Green's function

$$\begin{aligned} -\operatorname{div}\left(|\nabla G_x|^{p-2}\nabla G_x\right) &= 0 \qquad \text{in } \Omega \\ G_x &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

The properties

$$\int_\Omega |\nabla u|^p = \int_{B_0^{\rho(x)}} |\nabla U|^p$$
$$\int_\Omega F(u) \geq \int_{B_0^{\rho(x)}} F(U)$$

lead to estimates that are complementary (and sharper) to those obtained by symmetrization techniques. The p-harmonic radius $\rho(x)$ is defined by

$$G_x(y) = K(|y-x|) - K(\rho(x)) + o(1) \qquad \text{as } y \to x$$

where

$$K(r) = \begin{cases} c\, r^{\frac{p-n}{p-1}} & \text{if } 1 < p < n \\ -c \log(r) & \text{if } p = n \end{cases}$$

denotes the fundamental singularity of the p–Laplacian.
Reference: M. Flucher: Variational problems with concentration (Birkhäuser, to appear)

JENS FREHSE

Regularity for nonlinear mixed boundary value problems

A simple technique is presented to obtain $H^{3/2-\delta,2}$–regularity for mixed boundary value problems. (The Dirichlet and Neumann boundary may touch in corners with angles $\leq 180^0$.) This applies for equations and systems

$$-D_iF_i(x,\nabla u) = -D_if_i$$

where the F_i come from a variational integral $\int F(x,\nabla u)dx$ and uniform ellipticity is assumed.
For $n = 3$, also the stationary Navier–Stokes system is treated in a similar way. So it is for, say, $\Delta\Delta u = f$ where $H^{5/2-\delta,2}$ is achieved.
For Hencky's law, the tangential derivatives of the stresses are proven to be in $H^{-1/2}$, i.e.

$$\sup_h \int \frac{|\sigma(x+h)-\sigma(x)|^2}{|h|} dx \leq K,$$

h tangential. Imbedding theorems and weighted estimate yield L^q–properties for ∇u and $\nabla^2 u$. For example, $\nabla u \in L^{3+\delta}$ for $n = 3, u \in C^\alpha$.
This is joint work with Carsten Ebmeyer.

CLAUS GERHARDT

Closed hypersurfaces of prescribed mean curvature in conformally flat space

We prove the existence of closed hypersurfaces homeomorphic to S^n of prescribed mean curvature in a locally conformally flat space if $n \leq 6$.

Hans-Christoph Grunau

Positivity and critical dimensions in a semilinear polyharmonic eigenvalue problem

We are interested in the critical behaviour of certain dimensions in the semilinear polyharmonic eigenvalue problem

$$\begin{cases} (-\Delta)^m u &= \lambda u + |u|^{s-1}u,\ u \not\equiv 0 \quad \text{in} \quad B, \\ D^\alpha u|\partial B &= 0 \quad \text{for} \quad |\alpha| \le m-1. \end{cases} \tag{1}$$

Here $m \in \mathbf{N}$, $B \subset \mathbf{R}^n$ is the unit ball, $n > 2m$, $\lambda \in \mathbf{R}$; $s = (n+2m)/(n-2m)$ is the critical Sobolev exponent.

Pucci and Serrin (1990) raised the question in which so called "critical dimensions" the Dirichlet problem behaves "critically" with respect to the existence of "ground states". How does this phenomenon, which was in the case $m = 1$ discovered by Brezis & Nirenberg (1983), depend on the order $2m$ of the polyharmonic operator in (1), if m increases arbitrarily. Pucci and Serrin conjectured that if $n \in \{2m+1, \dots, 4m-1\}$ the Dirichlet problem (1) has no *radial* solution for λ close to 0. Their conjecture gains support from complementary existence results. Until now only parts of this conjecture could be proven, a full proof seems to be out of reach.

On the other hand if the Pucci-Serrin conjecture is (slightly?) modified, a surprisingly simple proof can be given. In the present talk I would like to show that if $n \in \{2m+1, \dots, 4m-1\}$ there is no *positive radial* solution to problem (1) for λ close to 0.

The proof relies upon some positivity and monotonicity properties of higher order Dirichlet problems. With respect to such properties only little seems to hold for higher order equations and still less seems to be known.

Michael Grüter

Regularity at the free boundary of minimal surfaces and harmonic mappings

Parametric minimal surfaces in $\mathbf{R}^3$ are conformally parametrized mappings from a two-dimensional domain into $\mathbf{R}^3$ such that each coordinate is a harmonic function. Free boundary and partially free boundary problems for minimal surfaces have been studied since Courant's book appeared in 1950. In particular, the regularity problem has attracted the attention of a number of mathematicians ever since. Here I consider the corresponding regularity problem for harmonic mappings. For stationary harmonic mappings I show regularity up to the free boundary by reducing the problem to a two-dimensional minimal surface in $\mathbf{R}^5$ having its free boundary on a three-dimensional supporting surface. Earlier results by Hildebrandt, Nitsche and myself can then be applied and give the desired result. The construction can be generalized to the case of stationary points of conformally invariant integrals for mappings into Riemannian manifolds and for supporting surfaces with boundary. For weakly harmonic mappings I have so far only been able to exclude isolated singularities by using an idea of Sacks and Uhlenbeck.

Norbert Hungerbühler

Uniqueness for n–Laplace type systems

This talk was based on joint work with Georg Dolzmann and Stefan Müller.
We consider the nonlinear elliptic system

$$\begin{aligned} -\sigma(x,u,Du) &= \mu \quad \text{in} \quad \mathcal{D}'(\Omega), \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

for $u\colon \Omega \to \mathbf{R}^m$, where μ is an $\mathbf{R}^m$-valued Radon measure with finite mass on an open bounded domain $\Omega \subset \mathbf{R}^n$. We assume that σ has coercivity rate n and growth rate $n-1$ in the gradient variable and satisfies mild assumptions on the regularity and monotonicity together with a structure condition. Then, there exists a solution $u \in \mathrm{BMO}(\Omega)$ and $Du \in L^q(\Omega)$ for all $q < n$, provided Ω^c is of type A. We show that in fact $Du \in L^{n,\infty}(\Omega)$. In view of the Green's function for the n-Laplace equation these results are optimal.

Moreover, if Ω is a Lipschitz domain and if σ does not depend on u and is uniformly monotone, we prove the following uniqueness result:
If $u, v \in W^{1,1}(\Omega)$, $u - v \in W_0^{1,1}(\Omega)$ and $Du, Dv \in L^{n,\infty}(\Omega)$ then

$$\operatorname{div} \sigma(x, Du) = \operatorname{div} \sigma(x, Dv) \quad \text{in } \mathcal{D}'(\Omega),$$

implies $u \equiv v$. In fact, it is sufficient to assume $Dv \in L^q(\Omega)$ for $q < n$, close enough to n.

JOHN HUTCHINSON

Approximating H-surfaces of disc-type

In joint work with Gerd Dziuk (Univ. Freiburg) we consider the problem of obtaining finite element approximations to solutions of the Plateau Problem for disc-like surfaces in $\mathbf{R}^3$ having prescribed boundary Γ and prescribed mean curvature H.
The appropriate variational setting is to obtain stationary points for the energy functional

$$E(u) = \frac{1}{2}\int_D |Du|^2 + \frac{2H}{3}\int u \cdot u_x \wedge u_y$$

where D is the unit disc in $\mathbf{R}^2$ and

$$u : D \to \mathbf{R}^3$$
$$u|_{\partial D} : \partial D \to \Gamma$$

is a monotone parametrization (to rule out Jacobi fields corresponding to the conformal group, an integral version of the three point condition is also imposed).
We are interested in approximating both stable and unstable solutions, in particular the so-called small and large H-surfaces.
The natural idea of working directly with piecewise linear elements (functions) in a discretized disc D_h is not used, due to the nonlinearity of the class of competing functions. Instead one considers H-harmonic extensions of boundary functions $v : \partial D \to \Gamma$ and writes $v = \gamma \circ s$ for some fixed parametrisation $\gamma : S^1 \to \Gamma$.
Thus we take $C^0 \cap H^{\frac{1}{2}}$ - maps $s : \partial D \to S^1$ with winding number one and consider the functional $I(s) = E(\gamma \circ s)$.
Denoting solutions of the smooth and a certain corresponding discrete problem by u and u_h respectively, we have proved in case $H = 0$ that near any u (not necessarily stable) there is a unique u_h, and that moreover

$$\|u - u_h\|_{H^1} \leq Ch\lambda^{-1}$$

where λ is the non-degeneracy constant for u.
Analogous results have also been obtained, if $H \neq 0$, for the Dirichlet problem.
We are currently applying the previous results to the $H \neq 0$ Plateau Problem.

TOM ILMANEN

The inverse mean curvature flow

We show long-term weak existence, compactness and partial regularity for the parabolic surface evolution

$$v = \frac{1}{H}, \qquad x \in N_t,\ t \geq 0$$

v = normal velocity, H = mean curvature, N_t a family of hypersurfaces flowing in an n-manifold.
The key ingredients are
(1) A variational level-set formulation whereby $N_t = \{u = t\}$, where $u : M \to [0, \infty)$ minimizes

$$J_u[v] := \int_{M \setminus E_0} |Dv| + v|Du|$$

where $\partial E_0 = N_0$.
(2) Elliptic regularization of the resulting level set equation

$$H_{\{u=t\}} = |Du|.$$

An application is the proof of the (Riemannian) Penrose Inequality of general relativity. This is joint work with Gerhard Huisken.

HITOSHI ISHII

Homogenization of the Cauchy problem for Hamilton-Jacobi equations

Asymptotic behavior of solutions of the Cauchy problem for Hamilton-Jacobi equations with periodic coefficients as the frequency of periodicity tends to infinity is discussed. The limit functions are characterized as unique solutions of Hamilton-Jacobi equations with the Hamiltonians determined by the corresponding cell problems. Our results apply to the case where the initial data oscillates periodically and so does the Hamiltonian both in the spatial and time variables.

NINA M. IVOCHKINA

The description of one class of fully nonlinear parabolic equations and some results concerning the solvability of the first initial boundary value problem

Evolutionary equations in question are described in terms of the pairs (G, D), $D \subset \mathbf{R}^1 \times \mathrm{Sym}(n), G : D \to \mathbf{R}^1$. The domain D has to be convex and G to be concave over D and also monotone, i.e. $G(s-\sigma, S-\eta) \geq G(s, S)$ if $(s, S) \in D(s)$, $\sigma \geq 0$, $\eta \geq 0$ are arbitrary.
We require $(s, BSB) \in D$ if $(s, S) \in D$ and B is an arbitrary orthogonal matrix. Here and below $D(s) = \{S : (s, S) \in D\}$. Let $\underset{\sim}{G}(s) = \sup_{\partial D(s)} \lim_{S \to S^0} G(s, S)$, $\bar{G}(s) = \inf_{D(s)} \lim_{\alpha \to \infty} G(s, \alpha S)$. We always assume $-\infty \leq \underset{\sim}{G}(s) < \bar{G}(s) \leq \infty$. Eventually the existence of $\nu(s, G)$ is assumed s.t. $\mathrm{tr}\, G^{ij}(s, S) \geq \nu(s, G) > 0$, $G^{ij} = \partial G / \partial S_{ij}$, $(s, S) \in D$.
Let $a \geq 0$, b, $A > 0$, $A^0 \geq 0$, W be given functions and matrices respectively defined on $\bar{Q} \times \mathbf{R}^1 \times \mathbf{R}^n$, $Q = \Omega \times (0, T)$, $\Omega \subset \mathbf{R}^n$. We introduce the operators $s[u] = -au_t + b$, $S[u] = u_{(xx)} - u_t A^0 + W, \mathbf{G}[u] = G(s[u], S[u])$, $u_{(xx)} = A^{\frac{1}{2}} u_{xx} A^{\frac{1}{2}}$. By definition a function u is admissible iff $(s[u], S[u])(z) \in D$, $z = (x, t) \in \bar{Q}_+$.
Theorem. Let $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q}) \cap C^{4,2}(Q)$ be an admissible solution to the problem

$$(*) \qquad \mathbf{G}[u] = g, \; u(x, 0) = \Psi(x), \; u|_{\partial'' Q} = \Phi(x, t)$$

$g = g(z, u, u_x)$, Ψ, Φ_z are given smooth functions. Assume that either a or A is strictly positive and G is uniformly monotone over D, i.e.

$$\nu_0((\xi^0)^2 + \xi^2)\ \mathrm{tr}\, G^{ij} \leq -\frac{\partial G}{\partial s}(\xi^0)^2 + G^{ij}\xi^i\xi^j \leq \frac{1}{\nu_c}((\xi^0)^2 + \xi^2)\ \mathrm{tr}\, G^{ij} \qquad (s, S) \in D,\ \xi^0 \in \mathbf{R}^1, \xi \in \mathbf{R}^n.$$

Then $\|u\|_{C^{2,1}(Q)} \leq c\ (\nu_0;\ \|u\|_{C^{1,0}(Q)})$. The conditional solvability of problems $(*)$ sequels the Theorem.

ROBERT JERRARD

Vortex dynamics for the Ginzburg-Landau Schroedinger equation

We characterize the asymptotic behavior in the limit $\varepsilon \to 0$ of solutions of the equation

$$iu_t^\varepsilon - \Delta u^\varepsilon + \frac{1}{\varepsilon^2}(|u^\varepsilon|^2 - 1)\, u^\varepsilon = 0 \qquad \text{in } \mathbf{T}^2 \times [0, T]$$

for appropriate initial data h^ε. In particular, we show that solutions exhibit "vortices" and that these vortices evolve by an ODE. If $\xi_1(t), \ldots, \xi_n(t)$ are the limiting vortex locations, then

$$\frac{\partial \xi_i}{\partial t} = \mathbf{J}\ \nabla_{\xi_i} W(\xi_1(t), \ldots, \xi_n(t)), \qquad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where W is the renormalized energy identified by Bethuel, Brezis and Helein. This resolves a wellknown open problem.
Main ingredients in the proof are a new weighted energy estimate

$$\frac{d}{dt}\int \eta\ Ju^\varepsilon\ dx = \int \eta_{x_i x_j}\ \mathrm{J}_{jk}\ u^\varepsilon_{x_i}\cdot u^\varepsilon_{x_k}\ dx$$

for $Ju^\varepsilon := det\ Du$, η smooth; and results which show that, if ξ_i are vortex locations,

$$\mu^\varepsilon := \frac{1}{|\ln\ \varepsilon|}\ [\frac{1}{2}|Du^\varepsilon|^2 + \frac{1}{4\varepsilon^2}(|u^\varepsilon|^2 - 1)^2]\ dx \quad , \quad d_i = \pm 1$$

are vortex degrees, then

$$Ju\ dx \approx \pi\ \Sigma\ d_i\ \delta_{\xi_i}$$

$$\mu^\varepsilon \approx \pi\ \Sigma\ \delta_{\xi_i}$$

This is joint work with J. Colliander.

BERND KAWOHL

Maximum and Comparison Principles for Anisotropic Diffusion

Image processing leads to many interesting variational and diffusion problems. Among these problems N. Kutev and I considered the so called Perona-Malik model, in which a given gray-scale $g(x)$ is supposed to evolve into a sharper image under the flow described by $u_t - \mathrm{div}\,(a(|Du|^2)Du) = 0$ in $\Omega \times \mathbf{R}^+$, with no flux condition on $\partial\Omega \times \mathbf{R}^+$ and initial datum $u(0, x) = g(x)$. A model case for a is $a(s) = (1+s)^{-1}$. If ν denotes $-Du(x)/|Du(x)|$ and η all the orthogonal directions to ξ, the differential equation turns into

$$u_t = b(|Du|^2)u_{\nu\nu} + a(|Du|^2)u_{\eta\eta}$$

where $b(s) = (1-s)(1+s)^{-2}$ changes sign at a threshhold $s_0 = 1$. Therefore the spatial operator on the right hand side is of mixed elliptic-hyperbolic type. In one space dimension the equation becomes a forward-backward parabolic equation, for which maximum and comparison principles have previously been dismissed as hopeless.
Assuming existence of weak C^1-solutions we prove a maximum principle, and for the case of one space dimension we furthermore derive a number of qualitative results such as the preservation and enhancement of "edges". Moreover we give a restricted comparison principle and suggest a way to prove the existence of solutions.

GARY LIEBERMANN

Parabolic equations with exponential growth

It is well-known that solutions of parabolic equations of the form $-u_t + \mathrm{div}\,A(Du) = 0$ have good regularity properties provided A acts sufficiently like a polynomial. I showed that the same is true if $A(p) = \exp(\theta|p|^2)p$ for some positive constant θ. In this talk, we discuss the corresponding equation when $A(p)$ grows even faster, for example $A(p) = \exp(\exp(1+|p|^2))p$. For the analogous elliptic equation $\mathrm{div}\,A(Du) = 0$, such behavior is also well-understood.

THOMAS NEHRING

Hypersurfaces with prescribed Gauss curvature and boundary in Riemannian manifolds

Consider a strongly convex, strictly locally convex compact subset $B \subset N$, (N, γ) a Riemannian manifold of dimension $n+1$; assume ∂B is smooth and $\Omega \subset \partial B$ is a connected subset with smooth boundary $\Gamma = \partial\Omega$. If there exists a strictly convex function $\Psi : \bar{B} \to \mathbf{R}$, and if $K : \bar{B} \to \mathbf{R}_{>0}$ is a smooth function with $K|_{\partial B} \leq K_{\partial B}$ (the Gauss curvature of ∂B), it is shown that there exists a hypersurface $M \subset B$ such that

- $K|_M = K_M$ and M is strictly locally convex

• $\partial M = \Gamma$
• M is diffeomorphic to Ω
The proof is based on *mod* 2 degree theory and the main ingredients for this are C^2-estimates at the boundary. These are derived by looking at N locally as a perturbation of Euclidean space and by applying methods which are known to work in Euclidean space.

MICHAEL RUZICKA

Existence results for electrorheological fluids

The steady motion of a stear dependent electrorheological fluid is governed by the system

$$
\begin{aligned}
-div\, T + u \cdot \nabla u + \nabla \pi &= f + \chi_E E \cdot \nabla E & & \\
\operatorname{div}\, u &= 0 & \text{in } \Omega \subseteq \mathbf{R}^3 & \qquad (1)\\
u &= 0 & \text{on } \partial\Omega &
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{div}\, E &= 0 & & \\
\operatorname{curl}\, E &= 0 & \text{in } \Omega & \qquad (2)\\
E \cdot n &= E_0 \cdot n & &
\end{aligned}
$$

where the stress tensor T is given by

$$T = \alpha_2\, E \otimes E + \alpha_3\, D + \alpha_5\, (DE \otimes E + E \otimes DE)$$

$$
\begin{aligned}
\alpha_2 &= \alpha_{21}\, ((1 + |D|^2)^{\frac{p(|E|^2)-1}{2}} - 1) \\
\alpha_3 &= (\alpha_{31} + \alpha_{33}\, |E|^2)(1 + |D|^2)^{\frac{p(|E|^2)-2}{2}} \qquad (3)\\
\alpha_5 &= \alpha_{51}\, (1 + |D|^2)^{\frac{p(|E|^2)-2}{2}} \\
p &\in C^1(\mathbf{R}^+),\ 1 < p_\infty \le p(s) \le p_0 < \infty.
\end{aligned}
$$

Within the framework of Sobolev-spaces T defines an operator with non-standard growth conditions and thus monotonicity arguments cannot work. Under certain conditions ensuring that T defines a coercive, monotone operator we show that for $p_\infty \in (\frac{9}{5}, 6)$, $p_0 < f(p_\infty)$ there exist strong solutions $u \in W^{2,r}(\Omega) \cap W_0^{1,p_\infty}(\Omega), r < p_0'$, $E \in W^{2,q}(\Omega)$, $q > 3$ of the system (1) – (3). Also in the time dependent situation we obtain the existence of weak solutions if $p_\infty \in [2, 6)$, $p_0 < f(p_\infty)$.

FRIEDRICH SAUVIGNY

Minimal surfaces in a wedge

This is a report on my joint investigations together with Professor Stefan Hildebrandt (Univ. Bonn) on minimal surfaces solving a singular mixed boundary value problem (for minimal surfaces). We observe that minimal surfaces creep along the singular line of a support surface which consists of two halfplanes meeting there. Furthermore, we derive an oscillation estimate for the unit normal of a minimal surface having a wedge-type free boundary. This implies a result of Bernstein-type for minimal surfaces in a wedge.

BARBARA STOTH

A free boundary problem in the mean field theory of superconductivity

We prove a partial regularity result for a free boundary problem in the mean field theory of superconductivity. It consists in determining $\psi \in C^{0,1}(\bar{\Omega})$ and $q \in H^1_{loc}(\mathbf{R}^2)$ with

$$
\begin{cases}
|\nabla\psi|(q - \psi) = 0 & \text{in } \Omega \\
-\Delta q + (q - \psi)\, \chi_\Omega = 0 & \text{in } \mathbf{R}^2 \\
\nabla(q - q_\infty) \in L^2(\mathbf{R}^2). &
\end{cases}
$$

Here $\Omega \subseteq \mathbf{R}^2$ is a bounded domain and q_∞ is given with $-\Delta q_\infty + q_\infty \chi_\infty = 0$ in $\mathbf{R}^2$. This is a two dimensional superconductivity model where $\vec{\omega} = \nabla^\perp \psi$ is the vortex density and $\vec{H} = \nabla^\perp g$ is the magnetic field.

We show the

Theorem: Either $q \equiv const$ in Ω or $A := \{x \in \Omega | \; q(x) = \psi(x), \; \nabla q(x) = 0\}$ consists locally in Ω of finitely many points.

To prove this theorem we extend a regularity result of Caffarelli/Friedman for the zero-set of functions satisfying a bound $|\Delta v(x)| \le G \; |v(x)|$ to the case of functions satisfying the weaker bound $|\Delta v(x)| \le G \; \hat{v}(x)$, where $\hat{v}(x)$ is the radially maximal function

$$\hat{v}(x) := \sup_{t \in [0,1]} |v(tx)|$$

In addition we extend the unique continuation theorem of Aronszajn and Cordes to cover this case. This is joint work with R. Schätzle.

MICHAEL STRUWE

Uniqueness of harmonic maps with small energy

Let N be a smooth, compact k-manifold with $\partial N = \emptyset$, isometrically embedded in $\mathbf{R}^n$ and let $B = B_1(0; \mathbf{R}^3)$ with $\partial B = S^2$. For $g \in H^{1,2}(S^2; N)$ denote

$$H_g^{1,2}(B; N) = \{u \in H^{1,2}(B; N); \; u = g \text{ on } \partial B\}.$$

Theorem: There exist constants $\varepsilon_0 = \varepsilon_0(N) > 0$, $C = C(N)$ such that for any $g \in H^{1,2}(S^2; N)$ with

$$E(g; S^2) = \frac{1}{2} \int_{S^2} |\nabla g|^2 do < \varepsilon_0 \tag{1}$$

there is a unique harmonic map $u \in H_g^{1,2}(B; N)$ such that

$$\sup_{x_0, r > 0} r^{-1} \int_{B \cap B_r(x_0)} |\nabla u|^2 dx < CE(g) \tag{2}$$

Moreover, u minimizes the Dirichlet energy in $H_g^{1,2}(B; N)$, $u \in C^\infty \cap H^{\frac{3}{2},2}(B; N)$, and $\|\nabla u\|_{H^{\frac{1}{2},2}} \le CE(g)$.

Remarks. (i) The result is related to the partial regularity results of Evans and Bethuel for stationary harmonic maps.

(ii) In view of an example of Rivière of a nowhere continuous weakly harmonic map $u \in H^{1,2}(B; S^2)$, condition (2) is essential for uniqueness.

(iii) The global regularity gain of half a derivative is best possible, even for harmonic functions.

VLADIMIR SVERAK

Periodic equilibria of a polyconvex functional

We give an example of a strictly polyconvex function $f : M^{2 \times 6} \to \mathbf{R}$ such that the Euler–Lagrange equation

$$\frac{\partial}{\partial x^\alpha} \frac{\partial f}{\partial x_{i_\alpha}}(Du) = 0$$

admits a non-trivial doubly periodic solution $u : \mathbf{R}^2 \to \mathbf{R}^6$.

FRIEDRICH TOMI

Existenz konvexer Hyperflächen mit vorgeschriebener Krümmung und gegebenem Rand

Es wird die Existenz lokal strikt konvexer n-dimensionaler Hyperflächen untersucht, die einen vorgegebenen Rand haben und deren Hauptkrümmungen $\lambda_1, ..., \lambda_n$ eine Relation $\frac{S_n}{S_k}(\lambda_1, ..., \lambda_n) = c$ erfüllen. Hierbei bezeichnet S_k, $0 \leq k \leq n$, die elementarsymmetrische Funktion vom Grad k in n Veränderlichen, und c ist eine positive Funktion des Orts. Es handelt sich um gemeinsame Ergebnisse mit N. Ivochkina (St. Petersburg).

TATIANA TORO

Geometry of measures

We study the relationship between the geometries of a set in $\mathbf{R}^{n+1}$ and the doubling properties of the measures it supports. In particular we show that the boundary of a domain in $\mathbf{R}^{n+1}$ is well approximated by n-affine spaces if and only if asymptotically the doubling constant of its harmonic measure coincides with the doubling constant of the n-dimensional Lebesgue measure.
This is joint work with C. Kenig.

NEIL S. TRUDINGER

Hessian measures

The *k-Hessian operator* F_k acting on functions $u \in C^2(\Omega)$, Ω open, $\subset \mathbf{R}^n$, $k = 1, .., n$, is defined by

$$F_k[u] = [D^2u]_k = \text{ sum of } k \times k \text{ principal minors of Hessian matrix } D^2u.$$

An upper–semicontinuous function $u : \Omega \to [-\infty, \infty)$ is called *k-convex* in Ω if $F_k[q] \geq 0$ for all quadratic polynomials q for which $u - q$ has a local maximum in Ω.
$\Phi^k(\Omega)$ denotes the class of proper k–convex functions in Ω (i.e. those $\not\equiv -\infty$ on a component of Ω).
Properties. (a) $u \in C^2(\Omega) \cap \Phi^k(\Omega) \Leftrightarrow F_j[u] \geq 0$, $j = 1, .., k$.
(b) $\Phi^k(\Omega) \subset \Phi^j(\Omega)$, $j \leq k \Rightarrow k$–convex functions are subharmonic.
(c) $u \in \Phi^k(\Omega) \Rightarrow$ mollifications $u_h \downarrow u$, $\in \Phi^k(G')$, $h < dist(\Omega', \partial\Omega)$.
Main theorem. For any $u \in \Phi^k(\Omega)$ there exists a Borel measure $\mu_k[u]$ such that:
(i) $\mu_k[u](e) = \int_e F_k[u]$ for Borel $e, u \in C^2(\Omega)$
(ii) if $u_m \to u$ in measure, $\{u_m\} \subset \Phi^k(\Omega)$, $\mu_k[u_m] \to \mu_k[u]$ weakly.
This is joint research with X.T. Wang.

LIHE WANG

Weakly harmonic maps and biharmonic maps

Consider maps between manifolds

$$u : N^n \to M^m \subset \mathbf{R}^k$$

with energy:

$$\int_N |\Delta u|^2 \, dVol.$$

Any weak solution of the Euler-Lagrange equation is called weakly biharmonic map.
We show the following:
Theorem 1: If $n = 4$, then any weakly biharmonic map is smooth.
Theorem 2: For general dimensions the stationary biharmonic is regular up to a set of $(n-4)$-dimensional Hausdorff measure 0.
Our techniques also yield an easy proof for the regularity of weakly harmonic maps.
Theorem 2 was proved using a monotonicity formula.
This is a joint with A. Chang and P. Yang.

DANIEL WIENHOLTZ

A method to exclude branch points of minimal surfaces

Let $B \subset \mathbf{R}^2$ denote the unit disc. Assume $\gamma \in C^{r,\mu}(S^1, \mathbf{R}^n)$ is a regularly parametrized closed Jordan curve and let $w \in C^0(\bar{B}, \mathbf{R}^n) \cap C^2(B, \mathbf{R}^n)$ be a minimal surface monotonically bounded by γ, i.e.
(1) $\Delta w = 0$,

(2) $\|w_x\|^2 = \|w_y\|^2$, $\langle w_x, w_y \rangle = 0$ in B,
(3) there exists $\varphi \in C^0(\mathbf{R}, \mathbf{R})$ monotone, such that $w(e^{it}) = \gamma(e^{i\varphi(t)})$ and $\varphi(t+2\pi) = \varphi(t) + 2\pi$ for $t \in \mathbf{R}$.
Assume w has a branch point $\zeta \in \bar{B}$ of order $\nu \in \mathbf{N}$, i.e. $w_z(z) = A(z-\zeta)^\nu + c(|z-\zeta|^\nu)$ as $z \to \zeta$ with $A \in \mathbf{C}^n \setminus \{0\}$.
If the function $g(z) := w_{zz}(z) \circ w_{zz}(z) := \sum_{j=1}^n (w^j_{zz}(z))^2;\ \bar{B} \to \mathbf{C}$ behaves like $g(z) = a(z-\zeta)^L + o(|z-\zeta|^L)$ as $z \to \zeta$ with $a \in \mathbf{C} \setminus \{0\}$ and $0 \leq L < 3\nu$, and if (i) $\zeta \in B$ or (ii) L odd or (iii) $Re\{a(i\zeta)^{L+4}\} < 0$, then there exists a C^{r-s}–family $w_\lambda \in C^1(\bar{B}, \mathbf{R}^n)$, $0 \leq \lambda < \varepsilon$, of mappings monotonically bounded by γ (in the sense of (3)) with $w_0 = w$ and

$$D(w) := \frac{1}{2} \int_B \|\nabla w\|^2 dx\, dy \; > \; D(w_\lambda) \text{ for } 0 < \lambda < \varepsilon \text{ (} D \text{ is the Dirichlet energy).}$$

So w is not a local minimizer of D.
Reference: D. Wienholtz, Zum Ausschluß von Randverzweigungspunkten bei Minimalflächen. Dissertation, Bonn 1996. Also in: Bonner Math. Schriften 298, Bonn 1997

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