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ARBEITSGEMEINSCHAFT MIT AKTUELLEM THEMA:

Mirror Symmetry

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Organizers: V. V. Batyrev (Tübingen) and D. van Straten (Mainz)

Mirror Symmetry is an unexpected duality between Calabi-Yau mainifolds of arbitrary dimension. This phenomenon has been descovered by physicists and during the last six years considerable progress has been made. However it is fair to say that most of the progress consists of lucky guesses for a correct mathematical formulation of mirror symmetry, mathematical conjectures about its properties and examples confirming these conjectures or consequences of them. For this reason mirror symmetry for Calabi-Yau manifolds still remains far from complete mathematical understanding. One of the main achievements of the last years is the rigorous mathematical foundation of *quantum cohomology*, which arose as an attempt to give a firm basis to the exciting calculation of Candelas et al. of the number of rational curves of fixed degree on a general quintic threefold.

According to the organizers it is difficult to arrange the recently published material on mirror symmetry in a logically connected series of eighteen lectures. Therefore they decided to select five topics (one for each day) and tried to keep logical connections between lectures mainly within one day. The five topics are the following:

- Gromov-Witten invariants, quantum cohomology and WDVV-equations (lectures 1-4).
- Rational curves on the quintic threefold (lectures 5-8).
- Mirror symmetry and physics (lectures 9-10).
- Mirror symmetry and toric geometry (lectures 11-14).
- Mirror symmetry "is" T-duality (lectures 15-18)

Abstracts

1 Spaces of stable maps

Fix $X|\mathbb{C}$ projective, and $\beta \in H_2(X, \mathbb{Z})$. Let $\overline{\mathcal{M}_{g,n}}(X, \beta)$ be the following stack over $(Sch/\mathbb{C})$:

$$\overline{\mathcal{M}_{g,n}}(X, \beta) : S \mapsto \text{category of all } (\pi : \mathcal{C} \to S; p_1 \ldots, p_n : S \to \mathcal{C}, \mu : \mathcal{M} \to X)$$

where

- π is a proper, flat morphism whose geometric fibres are curves of arithmetic genus g which are connected, reduced and have at most ordinary double points.
- The p_i are disjoint sections of $\pi_{sm} : \mathcal{C}_{sm} \to S$ (sm stands for "smooth part").
- In any geometric fibre $\mathcal{C}_s$, the following stability condition is satisfied:

$$\#\mathrm{Aut}(\mathcal{C}_s; p_i, \mu) < \infty.$$

(For $X = Spec\ \mathbb{C}$, this reduces to the usual concept of stabilty à la Knudsen-Mumford).

In the lecture, it was stated that for $g = 0$ the stack $\overline{\mathcal{M}_{g,n}}(X, \beta)$ admits a coarse moduli space $\overline{M_{0,n}}(X, \beta)$, which is actually projective over $\mathbb{C}$.

X is called *convex* if it is nonsingular, and if

$$H^1(\mathbb{P}^1_{\mathbb{C}}, \mu^*(T_X)) = 0 \quad \forall \mu : \mathbb{P}^1_{\mathbb{C}} \to X.$$

Under the assumption of convexity, it was shown that $\overline{M_{0,n}}(X, \beta)$ has at most quotient singularities, and that it is of pure dimension

$$\dim X + \int_\beta c_1(T_X) + n - 3.$$

(Jörg Wildeshaus)

2 Gromov-Witten invariants

In this lecture the Gromov-Witten invariants were introduced and discussed.

Definition: for convex, projective X and $\gamma_1, \ldots, \gamma_n \in H^{ev}(CX, \mathbb{Z})$ set

$$I_\beta(\gamma, \ldots, \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} p_1^*(\gamma_1) \cup \ldots \cup p_n^*(\gamma_n) \in \mathbb{Z}$$

(Gromov-Witten invariants); $\beta \in H_2(X, \mathbb{Z})$ fixed. The enumerative meaning of these numbers is contained in the following result if X is a homogenuous space:

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Proposition: Let $X = G/P$, where G is a reductive group over $\mathbb{C}$, $P \subset G$ a parabolic subgroup. Let $V_1, \ldots, V_n, \subset X$ for $n \geq 3$ subvarieties such that

$$(*) \qquad \sum_i \text{codim } V_i = \dim \overline{M}_{0,n}(X, \beta).$$

Then for $g = (g_1, \ldots, g_n)$ in an open dense subset of G^n the (scheme theoretic) intersection:

$$p_1^{-1}(g_1 V_1) \cap \ldots \cap p_n^{-1}(g_n V_n) \subset M_{0,n}(X, \beta)$$

consists of $I_\beta(\gamma_1 \ldots \gamma_n)$ (reduced) points, where $\gamma_i \cap [X] = [V_i]$ via the isomorphism

$$- \cap [X] : H^k(X) \xrightarrow{\sim} H_{\dim_{\mathbb{R}} X - k}(X) \quad .$$

After stating basic properties of Gromov-Witten-numbers the Gromov-Witten invariants were introduced in general as linear maps for any smooth projective variety $X/\mathbb{C}$:

$$I^X_{g,n,\beta} : H^*(X)^{\otimes n} \longrightarrow H^*(\overline{M}_{g,n})$$

satisfying certain axioms GW 0) - GW 8) which were explained and motivated. Finally a construction of non-trivial Gromov-Witten-invariants was outlined based on the works of Manin, Kontsevich, Behrend and Fantecki. At the end of the lecture the boundary of $\overline{M}_{0,n}(X, \beta)$ was discussed in preparation for the next talk.

(Christopher Deninger)

3 Quantum cohomology

The purpose of this lecture was to prove associativity of the quantum cohomology ring of a homogeneous variety $X = G/P$, where G is a reductive group and P is a parabolic subgroup. Let $A^*(X) = \oplus A^i(X)$ denote the intersection ring with multiplication $\cup$. All $A^k(X)$ are free abelian groups of finite rank. We fix a basis $T_0 = 1$ of $A^0(X), T_1, \ldots, T_p$ of $A^1(X)$ and $T_{p+1} \ldots T_m$ of the rest (i.e. $\underset{i\geq 2}{\oplus} A^i(X)$). The matrix $(g_{ij} = \int_X T_i \cup T_j)_{i,j=0\ldots m}$ is invertible over $\mathbb{Z}$ (either by looking at the intersection of generalized Schubert cycles or by using that the cycle map to singular cohomology is an isomorphism). An easy calculation shows $T_i \cup T_j = \sum_{e,f} : g^{e,f} I_0(T_i T_j T_e)$ where $I_0(T_i T_j T_e)$ is the Gromov-Witten-invariant and $(g^{e,f})_{e,f}$ is the inverse matrix to $(g_{ij})_{i,j}$. Define a "quantum deformation" of this product as $T_i * T_j = \sum_{e,f} \phi_{ije} g^{e,f} T_f$, where ϕ_{ije} is the following formal power series in $y_0 \ldots y_m$: $\phi_{ije} = \sum_{n\geq 0} \sum_\beta \frac{1}{n!} I_\beta(\gamma^n T_i T_j T_e)$ for $\gamma = \sum_{i=0}^m y_i T_i$. This product defines the structure of a commutative, associative $\mathbb{Q}[[y_0 \ldots y_m]]$-algebra with unit T_0 on $A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}[[y_0 \ldots y_m]]$. The only difficult point is the associativity relation, which follows from a linear equivalence relation for divisors in the boundary of the moduli spaces $\overline{M_{0,n}}(X,\beta)$.

(Annette Werner)

4 Applications to enumerative geometry

In this lecture, the theory of Gromov-Witten-invariants and quantum cohomology was applied to questions of enumerative geometry. The system of differential equations (called WDVV-equations) coming from the associativity by relations of quantum cohomology are organized using Feynman diagrams. Then the special case of the quantum cohomology of the projective plane is discussed. For $d > 1$ let N_d denote the number of plane rational curves of degree d through $3d-1$ points in general position. The number N_d has an interpretation as a Gromov-Witten invariant. The corresponding WDVV-equation yields in this case Kontsevich's celebrated recursion formula

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_i>0}} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right]$$

which allows to compute the N_d as we have $N_1 = 1$. Finally, a presentation of quantum cohomology of $\mathbb{P}^2$ in terms of generators and relations was given which shows that quantum cohomology is in general not a deformation of usual cohomology.

(Klaus Künnemann)

5 The computations of Candelas et al.

We presented two ways of computing the number n'_d of (not necessarily smooth) rational curves of degree d on a general quintic 3-fold in $\mathbb{P}^4$. The first way consists in computing the top Chern class of the vector bundle $\mathcal{E}_d$ on the moduli stack $\overline{\mathcal{M}}_{00}(\mathbb{P}^4, d)$. Assuming the Clemens conjecture, and computing the contribution of curves of lower degree one finds:

$$\int_{\overline{\mathcal{M}}_{00}(\mathbb{P}^4,d)} C_{top}(\mathcal{E}_d) = \sum_{\substack{k \\ k|d}} k^{-3} n'_{d/k} \quad .$$

Physicists propose a different way: Starting from a mirror symmetry conjecture they compute hypergeometric functions - which might be interpreted as a normalized q-expansion on the moduli space of the mirror family - to obtain these numbers. The coincidence of both series of numbers gives rise to speculations about mirror symmetry but lacks a precise description.

(Georg Hein)

6 Enumeration of rational curves via torus action

After presenting the basic ideology of using torus actions to solve enumerative problems, we discussed two examples in detail. The first example was the counting of twisted cubics on general complete intersection Calabi-Yau manifolds following the work of Ellingsrud and Stromme. The second example consisted of the famous calculation of degree d curves on the quintic in $\mathbb{P}^4$ which was made by Kontsevich.

(Ralph Kaufmann)

7 Quantum differential systems

Consider the Picard-Fuchs equation

$$D^4 I(q) = 5q(5D+1)(5D+2)(5D+3)(5D+4)I(q) \quad ,$$

where $D := q\frac{d}{dq}$. A basis of solutions can be written down in the form

$$I(t) = I_0(t) + I_1(t)p + I_2(t)p^2 + I_3(t)p^3 \equiv e^{pt} \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d}(5d+m)}{\prod_{m=1}^{d}(p+m)^5} \quad \mathrm{mod}\ p^4,$$

where $q = e^t$. According to the computations of Candelas et al. the transformation $t' = I_1(t)/I_0(t)$ gives the new Picard-Fuchs equation

$$(D')^2 \frac{1}{K(q')} (D')^2 J(q') = 0 \quad ,$$

where

$$K(q') = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{(q')^d}{1-(q')^d}$$

is the generating function for the number of degree d rational curves on the generic quintic threefold. This claim allows to compute the numbers n_d (for any given d).

In the lecture the latter operator was interpreted in terms of the "Euler" quantum cohomology of the quintic, the quintic Y in fact being virtual, only represented by its fundamental class. The claim was thereby reduced to the statement that two elements S^Y and Φ^Y of the quantum homology are closely related.

$$S^Y = 1 + \sum_{d=1}^{\infty} q^d e_{1*} \left(\frac{E'_{1d}}{\hbar - c_1} \right)$$

$$\Phi^Y = 1 + \sum_{d=1}^{\infty} q^d e_{1*} \left(\frac{1}{\hbar - c_1} \right) \prod_{m=1}^{ld} (lp + m\hbar)$$

Here, $l = 5$, $e_1 : M_2(d) \to X = \mathbb{P}^4$ evaluates doubly marked stable rational curves into X at the first marking, and c_1 is the Euler class of the universal cotangent bundle at that point. The Euler class $E'_{1d} \in H^*(M_2(d))$ is constructed by pulling back $\mathcal{O}(l)$ to the universal curve and taking global sections along the curves, vanishing at the first marking.

(Klaus Wirthmüller)

8 Givental's Proof of $S^Y \sim \Phi^Y$

In this lecture, which directly continues the previous one, Givental's proof of the "mirror theorem" was sketched. The theorem states that the formal power series Φ^Y and S^Y, introduced at the end of the previous lecture, are equal up to a certain transformation.

We consider the standard $T = (\mathbb{C}^*)^{n+1}$ action on $\mathbb{P}^n = X$ and work with equivariant cohomology $H^*(X, \mathbb{Q})$. The base ring of this theory is $H_T^* = \mathbb{Q}[\lambda_0, \ldots, \lambda_n] =: \mathbb{Q}[\lambda]$. By $p \in H_T^*(X, \mathbb{Q})$ be denoted the equivariant first Chern class of $\mathcal{O}_X(1)$.

For the proof we introduce the set $\mathcal{P} \subset H_T^*(X, \mathbb{Q})[[\hbar^{-1}, q]]$ of power series fulfilling the so-called "recursion relation" and a "polynomial" condition. In the lecture we have shown the *uniqueness Lemma:* If $Z, Z' \in \mathcal{P}$ fulfill $Z \equiv Z' \equiv 1 \mod q$ and $Z \equiv Z' \mod \hbar^{-2}$, then $Z = Z'$. The Lemma cannot be applied directly to S^Y and Φ^Y, because S^Y is not equivalent to Φ^Y mod $\hbar^{-2}$. Therefore we study the following transformations:

1. $Z(q, \hbar) \mapsto f(q)Z(q, \hbar)$ with $f(q) \in \mathbb{Q}[[q]] \setminus \{0\}$
2. $Z(q, \hbar) \mapsto \exp(g(\lambda, q)/\hbar)Z(q, \hbar)$ with $g(\lambda, q) \in q \cdot H_T^*(X)[[q]] \setminus \{0\}$
3. $Z(q, \hbar) \mapsto \exp(f(q)p/\hbar)Z(q \cdot \exp(f(q)), \hbar)$ with $f(q) \in \mathbb{Q}[[q]] \setminus \{0\}$

It is straightforward to show that $\mathcal{P}$ is closed under these transformations. The key result which enables one to apply this theory to S^Y and Φ^Y, is a Theorem of Givental which says that S^Y belongs to $\mathcal{P}$. The proof uses fixed point localization (Attiyah-Bott) on moduli spaces of stable maps and a detailed analysis of the fixed point sets of the actions on the moduli spaces. From the definition of Φ^Y it is not hard to deduce as a Corollary that Φ^Y also belongs to $\mathcal{P}$. Writing $\Phi^Y(q, \hbar) = \sum_{\nu \geq 0} \Phi^\nu(q)\hbar^{-\nu}$, one can show $\Phi^{(0)}(q) \in 1 + q \cdot \mathbb{Q}[[q]]$ and $\Phi^{(1)}(q)/\Phi^{(0)}(q) = p \cdot f(q) + g(\lambda, q) \in q \cdot H_T^*(X)[[q]]$. This is used to transform S^Y in the following way:

$$\tilde{S}^Y(q, \hbar) := \Phi^{(0)}(q) \cdot \exp(g(\lambda, q)/\hbar) \cdot \exp(f(q)p/\hbar) \cdot S^Y(qe^{(f(q))})$$

and we have $\tilde{S}^Y \in \mathcal{P}$. Using the uniqueness Lemma, we arrive at the main result (a Theorem of Givental): $\tilde{S}^Y = \Phi^Y$.

Finally, by descending to the non-equivariant setting (i.e. replacing λ by 0), we obtain a precise description of the transformation from $J(t_0, t) := \exp((t_0 + pt)/\hbar) \cdot E \cdot S^Y(q, \hbar)$ to $I(t_0, t) := \exp((t_0 + pt)/\hbar) \cdot E \cdot \Phi^Y(q, \hbar)$. Namely $t_0 \mapsto t_0 + \varphi(q)\hbar$ and $t \mapsto t + f(q)$ where $q = \exp(t)$ and $\Phi^{(0)}(q) = \exp(\varphi(q))$. This proves the statement at the beginning of the previous lecture.

(Bernd Kreußler)

9 $N = 2$ superconformal field theories

In the first part of the talk we tried to motivate from a physicist's point of view that the representation theory of the Virasoro algebra plays an important role in (2-dim) conformal QFT. After recalling some facts about unitary irreducible highest weight representations of this algebra, we focused on the discrete series obtained for $c < 1$, the so-called minimal models. In the second part we proceeded along the same lines for $N = 2$ superconformal QFT, namely we studied the representation theory of the $N = 2$ superconformal algebra, where one again gets discrete series for the central charge c being smaller than 3, also called minimal models. Finally we introduced the notions of the formal character of a representation (for convenience in the ordinary Virasoro case) and the modular invariant so-called partition function of minimal models. It is worth mentioning that the latter are classified according to the ubiquous $A - D - E$ pattern.

(Christian Adler)

10 Green-Plesser Mirror Construction for Fermat hypersurfaces

For any Calabi-Yau threefold X one constructs a conformal field theory called "non-linear sigma model", with central charge g and only integral $U(1)$-charges. Gepner has conjectured that this gives all such CFT's. The $N = 2$ superconformal algebra allows an involution, which permutes its representations. By Gepner, this should induce a mirror symmetry among 3-dimensional Calabi-Yau manifolds. Green and Plesser have given examples confirming this conjecture. They consider $U(1)$-projections of tensor products of five minimal models whose central charges add up to g and on which products of five cyclic groups act. Their mirror theories are constructed by "orbifolding"; calculation of the partition functions shows when one theory is the mirror of the other. For the corresponding Calabi-Yau's, which are hypersurfaces of Fermat type in weighted projective 4-spaces, orbifolding also makes sense, and the Hodge diamonds of (crepant resolutions of) the corresponding mirror paires appear to be related by a rotation of 90^o.

(Joseph Steenbrink, Nijmegen)

11 Periods of Calabi-Yau hypersurfaces

A Laurent polynomial $s = \sum_{j=1}^{N} v_j u^{\mathfrak{a}_j} \in \mathbb{C}[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ defines a hypersurface Z_s in the torus with coordinates $u_2, \dots, u_n$ if the exponents $\mathfrak{a}_j \in \mathbb{Z}^n$ all have first coordinate

1. If $\mathfrak{a}_j = (1,0,\ldots,0)$, then on easily checks:

$$\int_{|u_2|=\cdots=|u_n|=1} \frac{u_1}{s}\frac{du_2}{u_2} \wedge \cdots \wedge \frac{du_n}{u_n} = (2\pi i)^{n-1} v_1^{-1} \sum \frac{(-l_1)!}{l_2! \ldots l_N!} v_1^{l_1} \cdot \cdots \cdot v_N^{l_N}$$

where the sum runs over all $(l_1,\ldots,l_N) \in \mathbb{Z}^N$ which satisfy $l_1\mathfrak{a}_1 + \cdots + l_N\mathfrak{a}_N = 0$ and $l \leq 0$, $l_2,\ldots,l_N \geq 0$. Via the Poincaré residue this can be interpreted as a period of a holomorphic $(n-2)$-form on Z_s. The series is an example of a Gelfand-Kapranov-Zelevinsky hypergeometric function in a "resonant case". Other periods and solutions of GKZ hypergeometric differential equations with parameters $(\mathfrak{a}_1,\ldots,\mathfrak{a}_N)$ and $\beta \in \mathbb{Z}\mathfrak{a}_1 + \cdots + \mathbb{Z}\mathfrak{a}_N$ are obtained from the following series $\Phi_{\mathcal{T},\beta}$ (which also needs a triangulation $\mathcal{T}$ of the polytope $\Delta = \mathrm{conv}\{\mathfrak{a}_1,\ldots,\mathfrak{a}_N\}$):

$$\Phi_{\mathcal{T},\beta}(v) = \sum_{\substack{\lambda \in A^{-1}\beta \\ \lambda \in \mathbb{Z}^N}} Q_\lambda(c) \prod_{j=1}^{N} v_j^{\lambda_j} \prod_{j=1}^{N} v_j^{c_j}$$

where A is th $n \times N$-Matrix with columns $\mathfrak{a}_1,\ldots,\mathfrak{a}_N$, $c = (c_1,\ldots,c_N)$,

$$Q_\lambda(c) := \frac{\prod_{\lambda_j<0} \prod_{k=0}^{-\lambda_j-1}(c_j - k)}{\prod_{\lambda_j>0} \prod_{k=1}^{\lambda_j}(c_j + k)}$$

and $c_1,\ldots,c_N$ are the classes of $C_1,\ldots,C_N$ in the ring $R_{A,\mathcal{T}} := \mathbb{Z}[D^{-1}][C_1,\ldots,C_N]/J$ with $D =$(product of the volumes of the simplices of the triangulation $\mathcal{T}$) and J denotes the Ideal generated by the linear forms $a_{i1}C_1 + \cdots + a_{iN}C_N$, $(A = (a_{ij}),\ i = 1,\ldots,n)$ and the products $C_{i_1} \cdot \cdots \cdot C_{i_r}$ with $\mathrm{conv}\{\mathfrak{a}_{i_1},\ldots,\mathfrak{a}_{i_r}\}$ not a simplex in $\mathcal{T}$. The ring $R_{A,\mathcal{T}}$ is a free $\mathbb{Z}[D^{-1}]$-module of rank= $\sharp$(max. simpl. in $\mathcal{T}$).

View on A-side of mirror symmetry: If the triangulation $\mathcal{T}$ is such that all maximal simplices have volume = 1 and the intersection of all maximal simplices is nonempty and not contained in the boundary, then $R_{A,\mathcal{T}}$ is the cohomology of a toric variety $\mathbb{P}_\mathcal{T}$, and better even, naturally constructed from A and $\mathcal{T}$ is a vector bundle $\mathbb{E}_\mathcal{T} \to \mathbb{P}_\mathcal{T}$ and the zero locus of a generic section of the dual vector bundle $\mathbb{E}_\mathcal{T}^\vee \to \mathbb{P}_\mathcal{T}$ is a Calabi-Yau complete intersection in $\mathbb{P}_\mathcal{T}$.

View on B-side of mirror symmetry: solutions of GKZ differential equations *may* give periods of differential forms. More precisely: expand $\Phi_{\mathcal{T},\beta}$ for an appropriate β in terms of a (linear) basis of $R_{A,\mathcal{T}}$; candidates with respect to this basis are functions (of $v_1,\ldots,v_N$) which generate the period lattice of an appropriate differential form on $\mathbb{T} \setminus Z_s$.

(Jan Stienstra, Utrecht)

12 Hypersurfaces in toric varieties

Any lattice polytope $\Delta \subset M_\mathbb{R} = \mathbb{R}^n$ gives rise to a projective toric variety $\mathbb{P}_\Delta$; it contains the torus $T := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^\times)$ as an open subset. This construction allows to assign to every Laurent polynomial $f \in \mathbb{C}[M]$ with support supp$f \subseteq \Delta$ a hypersurface $\overline{Z_f} \subseteq \mathbb{P}_\Delta$ obtained as compactification of the zero set $Z_f := \{f = 0\} \subseteq T$. If Δ is a reflexive polytope (i.e. if Δ contains 0 as an interior point and both Δ and $\Delta^* := \{a \in M^* \mid \langle a, \Delta\rangle \geq -1\}$ are lattice polytopes), then its dual Δ^* provides another family of hypersurfaces $\overline{Z_g} \subseteq \mathbb{P}_{\Delta^*}$. This is the ultimate candidate for being the mirror of $\overline{Z_f}$; at least the resolutions of both families satisfy the relation $h^{1,1}(\widehat{Z_f}) = h^{n-2,1}(\widehat{Z_g})$.

(Klaus Altmann)

13 Mirror construction for Calabi-Yau complete intersections

Let V be a Calabi-Yau complete intersection in a toric Fano-variety $\mathbb{P}_\Delta$ associated with a reflexive polytope Δ. Denote by $\Delta_1, \ldots, \Delta_r$ the Newton polyhedra of equations for the complete intersection V. One has $\Delta = \Delta_1 + \cdots + \Delta_r$. We define a reflexive Gorenstein cone $C := \{(\lambda_1, \ldots, \lambda_r, x) \in \mathbb{R}^r_{\geq 0} \times M_\mathbb{R} \mid x \in \sum_{i=1}^r \lambda_i \Delta_i\}$ and denote by $C^\vee$ the dual cone in $\mathbb{R}^r_{\geq 0} \times N_\mathbb{R}$. If there exist lattice polyhedra $\nabla_1, \ldots, \nabla_r$, such that $C^\vee = \{(\mu_1, \ldots, \mu_r, y) \in \mathbb{R}^r_{\geq 0} \times N_\mathbb{R} \mid y \in \sum_{i=1}^r \mu_i \nabla_i\}$ then the polyhedra define another Calabi-Yau complete intersection in a toric variety $\mathbb{P}_\nabla$ associated with the reflexive polytope $\nabla = \nabla_1, \ldots, \nabla_r$. Moreover, the duality between reflexive Gorenstein cones works for the case of rigid Calabi-Yau varieties.

(V. Batyrev)

14 Mirror symmetry and string-theoretic Hodge numbers

This was the continuation of Batyrev's lecture. Having as starting-point the *dual* nef-partitions of Δ, ∇: $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\nabla = \nabla_1 + \cdots + \nabla_r$ respectively, we gave a sketch of the proof of the mirror-duality identity $E_{st}(V; u, v) = E_{st}(W; u, v)$ between V and W by using the "correction terms" of the E-polynomials. Several examples and comments for the explicit computation of the "string-theoretic" Hodge numbers were also included in the talk.

(D. I. Dais)

15 Mirror symmtery for lattice polarized $K3$-surfaces

First we gave a summary about basic results on $K3$-surfaces, i.e. structur of the lattice $H^2(X, \mathbb{Z})$, global Torelli, surjectivity of the periode map, existence of Kähler metrics, Ricci flat metrics and hyper-Kähler structure. Then the notion of M-polarized $K3$-surfaces and the moduli-spaces for M-polarized $K3$-surfaces were discussed. Following Dolgachev's paper, "Mirror symmetry for lattice polarized $K3$-surfaces", the notion of admissible vector and resulting mirror symmetric families were discussed, including the fact that the moduli space of one family is related to a tube domain in $Pic(X^*) \otimes \mathbb{C}$ of a generic member of the mirror family. Also examples were discussed.

(Herbert Kurke)

16 Special Lagrangian submanifolds

First I discussed the notion of a calibration and calibrated submanifolds. As the most important examples I showed that holomorphic submanifolds of a Kähler manifold and special Lagrangian submanifolds in a Calabi-Yau space are calibrated manifolds. Examples of special Lagrangian submanifolds are fixed point sets of an antiholomophic involution on a CY, and holomophic submanifolds of a $K3$ surface for a rotated complex structure. A theorem due to McLean tells us that the moduli space of special Lagrangian submanifolds is smooth and that the tangent space can be identified with the space of harmonic 1-forms on the submanifold. The p-branes were discribed as pairs of a submanifold on which strings are allowed to end and a line bundle defined on this submanifold. p-branes preserving some supersymmetry correspond to calibrated manifolds (this follows from the BPS-condition) together with a flat line bundle. This gives two classes of branes corresponding to holomorphic submanifolds or to special Lagrangian submanifolds. Mirror symmetry exchanges these two, while preserving their moduli spaces. These muduli spaces are fibrations with as fibres real tori of dimension the first Betti numbers of the submanifold. Following Strominger, Yau and Zaslow, we consider a mirror pair of Calabi-Yau spaces X and Y. Starting with a holomorphic 0-brane on X consisting of a single point x, we find by considering the mirror of this brane in Y that X should have a fibration by tori of dimension n ($n = \dim_{\mathbb{C}} X$). This allows us to apply T-duality to X. By dualising the torus fibration we found a T-dual manifold $\widehat{X}$. Now we can construct an r-brane on X in two ways. First as the T-dual of a point $\widehat{x} \in \widehat{X}$ and second as the mirror image of a point $y \in Y$. If one supposes that these branes coincide, one finds that $\widehat{X} = Y$. This leads to a description of mirror symmetry as T-duality and to some natural conjectures about Calabi-Yau spaces X. Namely first that they should have a fibration by spacial Lagrangian tori, and second that the mirror can be found by dualising this fibration.

(Christian van Enckwort)

17 Special Lagrangian fibrations on $K3$'s and Fourier Mukai functors

In a recent approach of Strominger, Yau and Zaslow, the phenomenon of mirror symmetry on Calabi-Yau threefolds admitting a T^3 fibration is interpreted as T-duality on the T^3 fibres. In two dimensions this means that we have to consider a $K3$ surface elliptically fibred over a projective line $p : X \to \mathbb{P}^1$. A mirror dual to X can be identified with the component M of the moduli space of simple sheaves on X having Mukai vector $(0, \mu, 0) \in H^\bullet(X, \mathbb{Z})$, where μ is the cohomology class defined by the fibres of p. The mirror map between the Hodge lattices of X and M should be given by a suitable Fourier-Mukai transform. In order to define this functor, one can identify M with a suitable compactification $\widehat{X}$ of the relative jacobian of X; $\widehat{X}$ is the variety representing the relative Picard functor $\mathrm{Pic}_{X/\mathbb{P}^1}$. One must assume that X has a section, and one finds that X is isomorphic to $\widehat{X}$. If $\mathcal{P} \to X \times_{\mathbb{P}^1} \widehat{X}$ is the Poincaré bundle (suitably normalized), then the Fourier-Mukai functors are defined as $S^i(\mathcal{F}) = R^i\widehat{\pi}_*(\pi^*\mathcal{F} \otimes \mathcal{P})$, $(i = 0, 1)$. for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ (π, $\widehat{\pi}$ are the natural projections of $X \times_{\mathbb{P}^1} \widehat{X}$ onto X and $\widehat{X}$). In this way one gets an equivalence at the level of derived categories of coherent sheaves and we described the induced map between cohomology lattices, $f : H^\bullet(X, \mathbb{Q}) \to H^\bullet(\widehat{X}, \mathbb{Q})$. In particular we proved that $f(\mu) = -\widehat{w}$, where $\widehat{w}$ is the fundamental class of $\widehat{X}$. So, Fourier-Mukai formulation gives the correspondence between special Lagrangian 2-tori in X and 0-branes in $\widehat{X}$ predicted by physicists. Relations between this "mirror symmetry" construction and Dolgachev-Nikulin's definition were also discussed.

(Claudio Bartocci)

18 T-duality for Borcea-Voisin mirror pairs

The first confirmation for threefolds of the recently proposed T-duality construction is given in the recent work of M. Gross and P. Wilson. The starting point is a $K3$ surface S whith a holomorphic involution i acting as (-1) on holomorphic 2-forms. The fixed point locus of i is a union of N holomorphic curves of total genus N'; investigating the action of i on $H^2(S, \mathbb{Z})$ it is possible to find another $K3$ with involution (S_1, i_1) such that i_1 has N' fixed curves of totoal genus N, and S_1 is the mirror of S according to the "mirror of lattice polarized $K3$" construction. The next step is to compare the above with the T-duality construction. Rotating the complex structure of S gives and ellipic fibration $f : S_K \to S$ which is then a SLAG fibration on S. Dualizing this fibration gives another $K3$ (topologically) and it can be checked that this dual can be identified with S_1 such that the involutions become homomorphisms of torus fibrations.

Choosing an elliptic curve with involution (E, j), we can build threefolds a la Borcea-Voisin: $X \xrightarrow{\tau} Y = S \times E/(i, j)$ where τ is the blowup. Choosing a degenerate metric on X, the fibration $f : S \to S^2$ and $g : E \to S^1$ (trivial) give a SLAG fibration $h : X \to S^2 \times S^1/(i', j') \simeq S^3$ whith torus fibres. The main theorem is that dualizing this fibration gives $X_1 = (S_1 \times E/(i_1, j))\widehat{}$ (blowup) which is the mirror of X. So the T-duality construction is compatible with previous mirror symmetry constructions in this case. The talk concluded with a topological discussion of singular fibres in the various fibrations, and a direct confirmation of the fact that $12(N - N') = e(X) = -e(X_1) = 12(N' - N)$ as predicted by physics.

(Balázs Szendrói)

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