MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Convex Geometry

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This meeting was organized by P.Goodey (Norman, Oklahoma) and P.M.Gruber (Vienna). It dealt with convexity as related to analysis. There were lectures of survey character as well as presentations of more specialized problems and results. The topics ranged from valuation and dissection problems to approximation of convex bodies and affine and integral geometry, from the algebra of polytopes and the metric theory of polytopes to classical problems of convex bodies, including stability questions, and from the local theory of normed spaces to the geometry of numbers and inequalities. The different mathematical backgrounds of the participants strongly enlivened the exchange of ideas.

VORTRAGSAUSZÜGE

Rotation invariant continuous valuations on convex sets

Semyon Alesker

Let $\mathcal{K}^d$ denote the family of convex compact subsets of $\mathbb{R}^d$. A function $\phi : \mathcal{K}^d \to \mathbb{C}$ is called valuation if

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^d$. It follows from the Blaschke selection theorem that $\mathcal{K}^d$ equipped with the Hausdorff metric is locally compact complete space. We will be interested in valuations, which are continuous with respect to the Hausdorff metric.

Theorem. (Hadwiger 1957). *Every continuous translation invariant and $SO(d)$-invariant valuation ϕ has the form*

$$\phi(K) = \sum_{i=0}^{d} c_i V_i(K),$$

where V_i are intrinsic volumes, c_i are fixed constants, $K \in \mathcal{K}^d$.

We will describe continuous rotation invariant valuation without assumption of translation invariance. Description of all continuous valuation on the line is easy and non interesting, so we will assume $d \geq 2$.

Definition. *A valuation $\phi : \mathcal{K}^d \to \mathbb{C}$ is called polynomial valuation of degree at most l, if for every $K \in \mathcal{K}^d$ $\phi(K + x)$ is a polynomial in $x \in \mathbb{R}^d$ of degree at most l.*

Theorem 1. *Every continuous $SO(d)$(resp. $O(d)$)-invariant valuation can be approximated uniformly on compacts in $\mathcal{K}^d$ by $SO(d)$ (resp. $O(d)$)-invariant polynomial valuations.*

Polynomial continuous rotation invariant valuations can be described explicitly.

Theorem 2. *1) Let ϕ be a continuous polynomial valuation, which is $SO(d)$-invariant if $d \geq 3$ and $O(d)$-invariant if $d = 2$. Then there exist polynomials $p_0, \ldots, p_{d-1}$ in two variables such that*

$$\phi(K) = \sum_{j=0}^{d-1} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} p_j(|s|^2, (s,n)) d\Theta_j(K; s, n),$$

where $|\cdot|$ is the Euclidean norm, $(\cdot,\cdot)$ is the scalar product, $\Theta_j(K;\cdot)$ is the j-th generalized curvature measure of K. Moreover, any expression of the above form is a continuous polynomial $O(d)$-invariant valuation.

2) Let $d = 2$. Let ϕ be a continuous polynomial $SO(2)$-invariant valuation. Th[illegible] there exist polynomials q_0, q_1 in two variables sucht that

$$\phi(K) = \sum_{j=0}^{1} \int_{\mathbb{R}^2 \times \mathbb{S}^1} q_j((s,n),(s,n')) d\Theta_j(K; s, n)$$

with the above notation, where n' denotes vector n rotated to the angle $\pi/2$ counterclockwise. Moreover, any expression of the above form is a continuous polynomial $SO(2)$-invariant valuation.

Random polytopes and lattice polytopes in convex bodies: a survey

Imre Bárány

Let $K \subset \mathbb{R}^d$ be a convex body and $X_n = \{x_1, \ldots, x_n\} \subset K$ be a finite set. We consider the cases when X_n is a random sample, i.e., the x_i's are random, independent, and uniform points from K and when $X_n = K \cap L$ is the set of lattice points in K where L is a d-dimensional lattice in $\mathbb{R}^d$. We are interested in the properties of the polytope $K_n = \text{conv} X_n$. By and large, the behaviour of K_n is similar in the random and the lattice cases. For instance, when K is sufficiently smooth, the expectation of the number of vertices of K_n (random polytope) is const$(K) n^{\frac{-2}{d+1}}(1 + o(1))$ as n goes to infinity. The number of vertices of K_n (lattice polytope) as the lattice L gets finer a[illegible] finer is essentially the same. Similarly, if approximation is measured as missed volu[illegible] K_n approximates K in the same order which turns out to be almost as good as best approximation (with the same number of vertices).

Approximating general hypersurfaces

Karoly Böröczky, Jr.

Following the work of Rolf Schneider, Peter M. Gruber and Monika Ludwig, the theory of asymptotic approximation by polytopal hypersurfaces of a smooth hypersurface X with strictly positive curvature is basically complete.

The talk consist of two parts: first the long standing conjecture is verified that the results can be extended to smooth convex hypersurfaces where the Gauß curvature is allowed to be zero. This is the largest family for meaningful asymptotic results, as even differentiable hypersurfaces may show rather irregular behavior.

Next, a further generalization is considered if the facets of the approximating polytopal hypersurface touch the hypersurface; namely, the Gauß curvature is allowed to be negative.

The convex hull of random points in a tetrahedron

Christian Buchta and Matthias Reitzner

At first, let K be a convex polygon with r vertices and area one. Choose n points from K, independently and according to the uniform distribution on K. Clearly, their convex hull K_n is a polygon contained in K. Denote by $D_n(K)$ the difference of the area of K and the expected area of K_n. A classical result of Rényi and Sulanke (1963) implies that

$$D_n(K) = \frac{2}{3} r \frac{\log n}{n} + \frac{c_1(K)}{n} + o(\frac{1}{n}) \quad (n \to \infty),$$

with an explicitly given constant $c_1(K)$. More precise information about $D_n(K)$ follows in the following way: For a plane convex body K of area one consider all chords of K that divide K into two parts of areas s and $1-s$. The locus of the midpoints of these chords is a closed curve M_s, called equiaffine inner parallel curve of the boundary curve of K. The assumption that the chords rotate counter-clockwise implies an orientation of M_s. Put

$$K_{[s]} := 1 - \int_{z \in K \setminus M_s} w(z, M_s) dz,$$

where $w(z, M_s)$ is the winding number of the closed curve M_s about the point z. Then

$$D_n(K) = \frac{4}{3} n \int_0^1 s^{n-1} K_{[s]} ds,$$

whence it can be deduced in the particular case of a polygon with r vertices that

$$D_n(K) = \frac{2}{3} r \frac{\log n}{n} + \frac{c_1(K)}{n} + \frac{c_2(K)}{n^2} + \frac{c_3(K)}{n^3} + \dots .$$

The constants $c_1(K), c_2(K), c_3(K), \dots$ are all known explicitly. Furthermore, it is possible to give a simple explicit formula for $D_n(K)$ which extends an old result of Herglotz (1933) from the case $n = 3$ to arbitrary n and from a quadrilateral to a general polygon K.

Currently, we are working on the respective problem in dimension 3. In particular, we are able to give the asymptotic expansion for the difference of the volume of a tetrahedron and the expected volume of the of the convex hull of n random points in this tetrahedron as n tends to infinity. The structure of the asymptotic expansion turns out to be much more complicated as in the planar case.

Steiner type formulas for convex functions: applications and related topics

Andrea Colesanti

1. Steiner formulas

Let u be a convex function defined in a convex open set $\Omega \subset \mathbb{R}^d$. At each point x of Ω the *subgradient* (or *subdifferential*) of $u, \partial u(x)$, is defined. For a Borel subset η of Ω and for a nonnegative ρ we define the set

$$P_\rho(U;\eta) = \{x + \rho v : x \in \eta, v \in \partial u(x)\}.$$

It is known that $P_\rho(u;\eta)$ is Lebesgue measurable and its measure is a polynomial of degree (at most) d in the variable ρ:

$$\mathcal{L}^n(P_\rho(u;\eta)) = \sum_{j=0}^{d} \binom{n}{j} \rho^i F_j(u;\eta);$$

where $\mathcal{L}^d$ is the d-dimensional Lebesgue measure. The coefficients $F_j(u;\cdot), j = 0,\ldots,d$, are nonnegative Borel measures and they represent the analogue of the curvature measures $C_{d-j}(K;\cdot)$ of a convex body $K \subset \mathbb{R}^d$. In particular the numbers $F_j(u;\Omega)$ correspond to quermassintegrals $W_j(K)$. At this regards notice that if u is Lipschitz in Ω, with Lipschitz constant L and Ω is bounded, then the following sharp inequalities hold:

$$F_j(u;\Omega) \leq L^j W_j(\Omega), \quad j = 0,\ldots,d.$$

The measures $F_0(u;\cdot)$ do not depend on u; indeed it is simply the restriction of $\mathcal{L}^d$ to Ω. On the other hand the measure $F_d(u;\cdot)$ is the image measure of the subgradient map, i.e.

$$F_d(u;\eta) = \mathcal{L}^d(\{v : v \in \partial u(x), x \in \eta\}) = \mathcal{L}^d(\partial u(\eta)).$$

This corresponds to the fact that the curvature measure $C_0(K;\cdot)$ of a d-dimensional convex body K is the image measure of the Gauss map of K. A further analogy with the case of convex bodies arises when we consider the case of smooth functions; indeed if $u \in C^2(\Omega)$ we can write

$$F_j(u;\eta) = \int_\eta S_j(D^2u), \quad j = 0,\ldots,d,$$

where $S_j(D^2u)$ is the j-th elementary symmetric function of the Hessian of u. Similarly if K is a convex body with C^2 boundary, then its curvature measures can be written as integrals over ∂K of the elementary symmetric functions of the principal curvatures of ∂K.

2. An application

Steiner formulas are used to estimate the sizes of the singularities of a convex function u defined in a convex domain $\Omega \subset \mathbb{R}^d$. We considered the sets $\sum_i$ of singular points *of order* i of u:

$$\sum_i = \{x \in \Omega : \dim(\partial u(x)) \geq d - i\}, \quad i = 0 \ldots n-1.$$

The union of the $\sum_i$'s is the set of all singular points of u. The sets $\sum_i$ were studied in several papers; in particular it is well known that the Hausdorff dimension of $\sum_i$ is i at the most, for every $i = 0, \ldots, d-1$. Simple examples show that the i-dimensional measure of $\sum_i$ can be $+\infty$, even though, if we assign to each point x of $\sum_i$ a *weight* equal to the $(d-i)$-dimensional measure of $\partial u(x)$ and we integrate this weights over $\sum_i$, then we obtain a finite quantity. More precisely:

$$(1) \qquad \int_{\Sigma_i} \mathcal{H}^{d-i}(\partial u(x)) d\mathcal{H}^i(x) \leq L^{d-i} W_{d-i}(\overline{\Omega}), \quad i = 0, \ldots, d-1,$$

where $\mathcal{H}^s$ is the Hausdorff measure of order s and $\overline{\Omega}$ is the closure of Ω. These inequalities are sharp. This result corresponds to the estimates for weighted measures of sets of singular points of a convex body K. We remark that in order to establish estimates (1), we preliminarly proved integral respresentations of the coefficient measures $F_j(u;\cdot)$. Such respresentations parallel the ones obtained by Zähle for the curvature measures of a convex body. Furthermore, the proof of the integral respresentations for functions can be given independently of the corresponding one for bodies, by the use of the conjugate function of a convex function.

On the perimeter deviation of a convex disc from a polygon

August Florian

Let C_1 and C_2 be two compact convex subsets of the plane. We denote by $\varrho^P(C_1, C_2)$ the distance between C_1 and C_2 determined by the L_1 metric in the space of support functions. This distance can also be written in the form

$$2p([C_1, C_2]) - p(C_1) - p(C_2)$$

where $[C_1, C_2]$ denotes the convex hull of $C_1 \cup C_2$, and $p(C)$ is the perimeter of C. Let P_n be any convex polygon with at most n vertices. Given a convex set C, there is a polygon $P_n = P_n(C)$ minimizing the distance $\varrho^P(C, P_n)$. Let p be the perimeter of C. It is known that

$$\varrho^P(C, P_n(C)) \leq p\left(1 - \frac{2n}{\pi}\arcsin\left(\frac{1}{2}\sin\frac{\pi}{n}\right)\right)$$

with equality if C is a circle (Florian 1992). For this inequality I recently found an alternative proof which avoids limiting processes.

Affine inequalities and radial mean bodies

Richard J. Gardner (joint work with G. Zhang)

Two important objects in convex geometry are the difference body and the polar projection body of a convex body. The difference body $K + (-K)$ of a convex body K was studied by Minkowski, and is ubiquitous in geometry (and elsewhere, as the vector sum of a set and its reflection in the origin). The operation that forms the difference body is essentially that known as central or radial symmetrization and as such finds many applications in mathematical physics and partial differential equations. Projection bodies also originated in the work of Minkowski, and have found application in the theory of vector-valued measures (Liapounov's theorem), the local theory of Banach spaces, stochastic geometry, random determinants, Hilbert's fourth problem, mathematical economics and other areas. The projection body ΠK of a convex body K is defined for $u \in S^{n-1}$ by

$$h_{\Pi K}(u) = V(K|u^{\perp}),$$

where h denotes the support function, V the k-dimensional volume of a k-dimensional body, and $K|u^{\perp}$ the orthogonal projection of K on the $(n-1)$-dimensional subspace $u^{\perp}$ orthogonal to u. The polar projection body $\Pi^{*}K$, the polar body of the projection body of K, appears explicity (but very frequently) in the more recent literature; its behaviour under linear transformations often renders it more natural than the projection body itself.

Both the difference body and the polar projection body appear in known affine inequalities. The first is an ingredient in the famous Rogers-Shephard inequality:

$$V(DK) \leq \binom{2n}{n} V(K),$$

with equality if and only if the convex body K in $\mathbb{E}^n$ is a simplex. The second appears in another affine inequality, a reverse Petty projection inequality, first proved by the second author:

$$n^{-n}\binom{2n}{n} \leq V(K)^{n-1}V(\Pi^{*}K),$$

with equality if and only if K is a simplex.

We establish a strong new affine inequality that yields both the above inequalities as special cases. This involves a new body associated with a convex body, defined as follows. Let K be a convex body in $\mathbb{E}^n$. For $x \in K$, let

$$\varrho_K(x,u) = \max\{c : x + cu \in K\},$$

$u \in S^{n-1}$, be the radial function of K with respect to x. The *radial pth mean body* R_pK of K is defined for nonzero $p > -1$ by

$$\varrho_{R_pK}(u) = \left(\frac{1}{V(K)}\int_K \varrho_K(x,u)^p dx\right)^{1/p},$$

for each $u \in S^{n-1}$. We also define R_0K by

$$\varrho_{R_0K}(u) = \exp\left(\frac{1}{V(K)}\int_K \log \varrho_K(x,u) dx\right),$$

for each $u \in S^{n-1}$. Thus the radial function of R_pK is just the pth mean of the values of the radial function of K with respect to points inside K. Then $R_{\infty}K$ is the difference body of K, and the shape of R_pK tends to that of the polar projection body as p tends to -1. It turns out that R_pK is itself convex when $p > 0$.

If $-1 < p < q$, the new inequality states that

$$c^n_{n,q} V(R_q K) \le c^n_{n,p} V(R_p K),$$

with equality if and only if K is a simplex. Here

$$c_{n,p} = (nB(p+1,n))^{-1/p},$$

for nonzero $p > -1$, and $c_{n,0}$ is defined by continuity. When $p = n$ and $q \to \infty$, the new inequality becomes the Rogers-Shephard inequality, and when $p \to -1$ and $q = n$, it becomes the reverse Petty projection inequality.

Our proof of the new inequality requires a generalization due to C. Borell of a classical inequality of Berwald for the pth means of a concave function defined on a convex body. We find a new proof of Borell's inequality that yields exact equality conditions (not explicitly stated by Borell).

Integral geometry and boundary structure of convex bodies

Stefan Glasauer

Subjects of the talk were several new integral-geometric relations for mixed area measures and support measures (or generalized curvature measures) of convex bodies. The results concern the convex hull of the union of a fixed and a moved convex body, moved either by translations or by rigid motions. There are simple explicit results even in the case where one integrates with respect to a measure that is not invariant. The versions for support measures are closely connected with certain difficult questions about the boundary structure of convex bodies, which are related to investigations by Besicovitch, Ewald, Larman, Rogers, Zalgaller, B. A. Ivanov, and Schneider (among others).

Minkowski sums of projections of convex bodies

Paul Goodey

This work is motivated by questions which seek information about a convex body based on knowledge of its projections. We obtain results based on certain geometric averages of projections. To make this precise, we let K be a convex body (non-empty, compact convex set) in $\mathbb{E}^d$. For each $1 \le k \le d-1$, we denote by $\mathcal{L}^d_k$ the compact manifold of all k-dimensional subspaces of $\mathbb{E}^d$. The unique rotation invariant probability measure on this manifold is denoted by ν^d_k. For each $L \in \mathcal{L}^d_k$, $K|L$ denotes the orthogonal projection of K onto the subspace L of $\mathbb{E}^d$. Although this is typically a k-dimensional convex body, we find it convenient to think of it as a convex body in $\mathbb{E}^d$. The Minkowski sum $P_k(K)$ of these projections is defined, in terms of its support function $h(P_k(K),\cdot)$, by

$$h(P_k(K),u) = \int_{\mathcal{L}^d_k} h(K|L,u)\,\nu^d_k(dL), \quad \text{for each } u \in S^{d-1}.$$

Note that, although this can be thought of as an average of all the k-dimensional projections of K, the body $P_k(K)$ will typically be of dimension d. Our major objective

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is to obtain information about K based on knowledge of this Minkowski sum $P_k(K)$ for some particular value of k.

The operator P_{d-1} was introduced by Schneider who showed that if $P_{d-1}(K) = cK$ for some constant c, then K is a ball. More recently, Spriestersbach showed that P_{d-1} is injective. In fact she gave stability results which show that if $P_{d-1}(K)$ is close to $P_{d-1}(L)$ then K is close to L and that, if $P_{d-1}(K)$ is close to cK for some constant c then K is close to a ball. Interestingly, it transpires that a certain isoperimetric deficit of K is bounded above by a multiple of the distance between $P_k(K)$ and cK for a certain constant c depending only on the dimension d.

Here we obtain some analogous results for the operator P_k in the cases $1 \leq k \leq d-2$. We first show how Spriestersbach's techniques can be used to establish the injectivity of P_k in all the cases $k \geq d/2$. We then examine the operator P_2. Contrary to expectations, we find that this is injective in all dimensions except $d = 14$ where it is not injective. The principle techniques employed are those of integral geometry and harmonic analysis. The main results give circumstances under which a convex body is determined by sums of its projections.

Theorem 1. *Let K, M be convex bodies in $\mathbb{E}^d$ with $P_k(K) = P_k(M)$ for some $k \geq d/2$. Then $K = M$.*

Theorem 2. *a) Let K, M be convex bodies in $\mathbb{E}^d$ with $P_2(K) = P_2(M)$. Then, if $d \neq 14$, we have $K = L$.*

b) There are distinct bodies K, M in $\mathbb{E}^{14}$ with $P_2(K) = P_2(M)$.

Goodey and Weil previously carried out a similar investigation involving sums of sections, as opposed to projections, of convex bodies. They showed that convex bodies are determined by averages of their 2-dimensional sections, but not by averages of their 1-dimensional sections. Various averages of both sections and projections were investigated by Goodey, Kiderlen and Weil. They showed that certain apparently disparate averages have strong inter-relationships. Rather surprisingly, our operator P_k is very closely connected to another operator B_k which is defined in terms of Blaschke sums of sections instead of Minkowski sums of projections.

The proofs of the theorems make use of a continuous linear operator $p_k : L^2(S^{d-1}) \to L^2(S^{d-1})$. This is the functional equivalent of P_k and is defined in such a way that, for a convex body $K \in \mathbb{E}^d$, we have $h(P_k(K), \cdot) = p_k(h(K, \cdot))$. The operator p_k intertwines the group action of $SO(d)$ on S^{d-1} and is therefore susceptible to the methods of harmonic analysis. These intertwining properties imply that, when restricted to the space of spherical harmonics of degree n in dimension d, the operator p_k acts as a multiple $\alpha_{n,k,d}$ of the identity. The injectivity results described in the above theorems arise from analysing whether or not any of these multiples can be zero. This question is resolved by first using some integral geometry to find explicit integral representations for the $\alpha_{n,k,d}$ and then providing estimates which prove that, in most cases, they are not zero. The exceptional case $k = 2$, $d = 14$ is a consequence of the fact that

$$\alpha_{5,2,d} = -\frac{2(d-14)}{(d+2)(d+1)d(d-1)}$$

which is proved directly.

Volume formulas in L_p-spaces

Yehoram Gordon (joint work with M. Junge)

According to the definition of Firey, the Minkowski p-sum of m segments in $\mathbb{R}^n$ is

$$\sum_{i=1}^{m} \oplus_p [-x_i, x_i] = \left\{ \sum_{i=1}^{m} \alpha_i x_i \,\middle|\, \left(\sum_{i=1}^{m} |\alpha_i|^{p'} \right)^{\frac{1}{p'}} \le 1 \right\}$$

where $x_1, \ldots, x_m$ are m vectors in $\mathbb{R}^n$, $\frac{1}{p} + \frac{1}{p'} = 1$. The case $p = 2$ defines an ellipsoid, and $p = 1$ defines a zonotope in $\mathbb{R}^n$, i.e. the body $V(B_\infty^m)$, where $V(e_i) = x_i, i = 1, \ldots, m$, is the linear map $V : \mathbb{R}^m \to \mathbb{R}^n$.

We extend the classical volume formulas for ellipsoids and zonotopes to p-sums of segments and prove

$$\operatorname{vol}\left(\sum_{i=1}^{m} \oplus_p [-x_i, x_i] \right)^{\frac{1}{n}} \sim_{c_p} n^{-\frac{1}{p'}} \left(\sum_{\operatorname{card}(I)=n} |\det(x_i)_{i\in I}|^p \right)^{\frac{1}{pn}}.$$

More precisely,

Theorem. *Let $1 \le p \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and let $x_1, \ldots, x_m$ be m vectors in $\mathbb{R}^n$. The associated linear map $V : \mathbb{R}^m \to \mathbb{R}^n, V(e_i) = x_i$ satisfies*

$$\frac{\sqrt{2}}{\sqrt{\pi e^3 \min(p,n)}} \left(\frac{\operatorname{vol}(V(B_{p'}^m))}{\operatorname{vol}(B_{p'}^n)} \right)^{\frac{1}{n}} \le \left(\sum_{\operatorname{card}(I)=n} |\det(x_i)_{i\in I}|^p \right)^{\frac{1}{pn}} \le e^{\frac{1}{p'}} \left(\frac{\operatorname{vol}(V(B_{p'}^m))}{\operatorname{vol}(B_{p'}^n)} \right)^{\frac{1}{n}}.$$

Examples show, that equality only holds for $p \in \{1, 2\}$ and that it is necessary to take the n-th root.

We also describe related geometric properties of the Fritz John and Lewis maps associated to classical operator norms such as the p-summing, p-nuclear, p-factorizable ideal norms for $1 \le p \le \infty$. The results are then applied to yield estimates for the inner and external volume ratio of arbitrary convex bodies K with respect to the volumes of n-dimensional balls of quotients, of subspaces, and of subspaces of quotients, of $L_p(\mu)$ spaces.

Microlocal Aspects of Convex Bodies

Eric L. Grinberg

A number of geometric properties of convex bodies may be described by the action of integral transforms on functions associated to the bodies, especially radial and support functions. Notable such transforms include the Cosine and Funk-Radon transforms which describe k-projections and k-intersections. While a great deal is known about the analysis of the latter transform, less is known about the former. In particular, the Funk-Radon transform is, in typical contexts an elliptic real-analytic Fourier integral operator. This leads to some amusing continuation properties for convex bodies where analyticity is replaced by some standard property for cross-sections. We present a link between the Cosine and Funk-Radon transform which allows some of the microlocal properties of the latter to be transferred to the former.

The form of best approximating polytopes

Peter M. Gruber

1. Introduction

Let δ be a metric or some other notion of distance on the class of all proper convex bodies C in Euclidean d-space $\mathbb{E}^d$. For $n = d+1, d+2, \ldots$, let $\mathcal{P}_n$ be a class of convex polytopes with n vertices or n facets, respectively, or their subclasses of convex polytopes which are inscribed or circumscribed to C. Then the problems arise to determine or estimate

$$\delta(C, \mathcal{P}_n) = \inf\{\delta(C, P) : P \in \mathcal{P}_n\}$$

and to describe those polytopes $P_n \in \mathcal{P}_n$ for which the infimum is attained, the *best approximating polytopes of C in $\mathcal{P}_n$ with respect to δ.* These problems have numerous aspects.

It is out of reach to give precise descriptions of the best approximating polytopes or to specify algorithms for finding them. The best one may expect is to give a rough description of the form. The first weak results in these direction are due to Glasauer and Schneider for the Hausdorff metric δ^H and to Glasauer and Gruber for the symmetric difference metric δ^V and the mean width deviation δ^W. These results hold for convex bodies C of class C^2 with positive Gauss curvature κ_C. They say that the density of the distribution of the vertices of the inscribed polytopes, resp. the points where the facets of the circumscribed polytopes touch C, are proportional to appropriate powers of κ_C.

In the following we give more precise information in the case when $d = 3$.

2. The form of best approximating convex polytopes

Let C be a convex body in $\mathbb{E}^3$ of class C^2 with positive Gauss curvature κ_C. Let the boundary $\operatorname{bd} C$ of C be endowed with a Riemannian metric ϱ_C and let (P_n) be a sequence of circumscribed convex polytopes such that P_n has n facets.

We say that P_n has *asymptotically regular hexagonal facets of the same edgelengths* with respect to ϱ_C if the following hold: there are Landau symbols $o(n)$ and $o(1)$ and a positive real sequence (σ_n) such that each facet F of P_n, with a set of at most $o(n)$ exceptions, has 6 vertices $v_1, \ldots, v_6$ and

$$\|v_i - p\|_p, \|v_{i+1} - v_i\|_p = \sigma_n(1 \pm o(1)) \text{ for } i = 1, \ldots, 6, v_7 = v_1,$$

where p is the point where F touches $\operatorname{bd} C$ and $\|\ \|_p$ is the Euclidean metric induced by ϱ_C on the tangent plane of $\operatorname{bd} C$ at p.

Theorem 1. *Let* $\operatorname{bd} C$ *be endowed with the Riemannian metric ϱ_{II} of the second fundamental form. For $n = 4, 5, \ldots$, let P_n be a best approximating circumscribed convex polytope with n facets with respect to the Hausdorff metric δ^H. Then P_n has asymptotically regular hexagonal facets of the same edgelengths with respect to ϱ_{II}.*

Similar results hold for inscribed polytopes, the Banach–Mazur distance, and a notion of a distance due to Schneider.

Theorem 2. *Let* $\operatorname{bd} C$ *be endowed with the Riemannian metric ϱ_A of equi-affine differential geometry. For $n = 4, 5, \ldots$, let P_n be a best approximating circumscribed* *convex polytope with n facets with respect to the symmetric difference metric δ^v. Then P_n has asymptotically regular hexagonal facets of the same edgelengths with respect to ϱ_A.*

A similar result holds for the mean width deviation δ^W.

3. The form of convex polytopes with minimum isoperimetric quotient

Consider convex polytopes in $\mathbb{E}^3$ of given volume with n facets. If such a polytope has minimum surface area, a theorem of Lindelöf says that it is circumscribed to a Euclidean ball. Thus an application of Theorem 2 yields the following result.

Corollary of Theorem 2. *For $n = 4, 5, \ldots$, let P_n be a convex polytope in $\mathbb{E}^3$ with n facets of given volume and minimum surface area. Then P_n has asymptotically regular hexagonal facets of the same edgelengths.*

Note that the metric used here is the ordinary Euclidean metric in $\mathbb{E}^3$.

A four-vertex theorem for space curves

Erhard Heil

A regular closed simple curve in Euclidean 3-space, lying on the boundary of its convex hull and without zero curvature points, has at least four points where the torsion τ changes sign. Under minor additional assumptions this was shown by Bisctrizcky and in full generality by Sedykh. Admitting singular points and sign changes of the curvature κ, Romero Fuster and Sedykh showed that

$$V + 2K + 3S \geq 4$$

where S is the number of singular points, K the number of points where κ changes sign, and V the number of points where τ changes sign. Here we will consider regular closed space curves which may have double points and must not lie on the boundary of their convex hulls. We show that

$$V + K + D \geq 4 \qquad (*)$$

where D is the number of extrema of the conical curvature τ/κ. We call such points Darboux vertices because there the Darboux vector changes its sense of rotation within the rectifying plane.

Darboux vertices can be made visible in the following way: Wrap a rectangular strip symmetrically along the space curve. It then is the rectifying strip of the curve, and its straight generators, which have the directions of the Darboux vectors, can be seen on the paper strip.

The main idea in order to proof $(*)$ is to consider the unit tangent vector of the curve as a curve on the sphere. Its geodesic curvature is just τ/κ.

Rectifiability for curvature measures of convex sets

Daniel Hug

We investigate the connection between measure theoretic properties of curvature measures of convex sets and geometric properties of these sets in a d-dimensional Euclidean space. In a recent paper, we established explicit representations for the singular parts of the curvature measures of an arbitrary closed convex set with respect to the boundary measure of the set. This now leads to characterizations of absolute continuity for the curvature measures in terms of conditions on naturally defined generalized curvature functions which are defined on the unit normal bundle of a given convex set. For the curvature measure of order zero of a convex body, another characterization is obtained which involves the set of directions in which the convex body is touched from inside by a nondegenerate ball. By using a Crofton intersection formula and various integral-geometric transformations, we extend this result to curvature measures of any order. Another extension is given which uses the notion of a touching affine subspace and a certain lower-dimensional spherical supporting property. Dual results are obtained for the surface area measures of convex bodies. For the proofs we employ methods from convex and integral geometry and also some basic geometric measure theory.

One class of effective step-by-step algorithms for polyhedral approximation

George K. Kamenev

The class of step-by-step algorithms for approximating convex bodies C in $\mathbb{E}^d, d \geq 2$, by inscribed (circumscribed) polyhedra $\mathcal{P}_n^i(\mathcal{P}_n^c)$ are considered. These algorithms are based on the idea of the general adaptive schemes: the augmentation scheme and the cutting scheme (the second one was introduced by Button and Wilker 1978). Let $C \in \mathcal{C}$ with the supporting halp-space $H(C,u)$ and the support function $g(C,u)$.

Augmentation scheme. *Let* $P_n \in \mathcal{P}_n^i$.
Step 1. Choose the point $p \in \partial C$.
Step 2. Construct $P_{n+1} = \operatorname{conv}\{p, P_n\}$.

Cutting scheme. *Let* $P_n \in \mathcal{P}_n^c$.
Step 1. Choose the unit direction $u \in S^{d-1}$.
Step 2. Construct $P_{n+1} = P_n \cap H(C,u)$.

The particular algorithm is defined with the methods of choosing of the polyhedron P_0 and the point (the direction) on the step 1 of the scheme. Let us consider the schemes, that improve polyhedron approximately in the direction of the maximum deviation from the body. Formally we define the sequence $\{P_n\}$ as $H(\gamma)$-sequence for C with constant $\gamma(H(C,\gamma)$-sequence) if there exist a constant $\gamma > 0$ such that $\delta^H(P_n, P_{n+1}) \geq \gamma\delta^H(P_n, C)$. The corresponding adaptive schemes we define as H-schemes.

There are some examples of H-schemes with $\gamma < 1$. Let $T(C,u) = C \cap \partial H(C,u)$ and $U(P)$ be the set of external unit normals to hyperfaces of ∂P.

Algorithm A (Bushenkov 1981). *Let $P_n \in \mathcal{P}_n^i$ be constructed.*
Step 1. Find $u^* = \text{argmax}\{g(C,u) - g(P_n,u) : u \in U(P_n)\}$; *find* $p^* \in T(C,u^*)$.
Step 2. Construct $P_{n+1} = \text{conv}\{p^*, p_n\}$.

For the algorithm **A** it was proved (Kamenev 1986) that for $C \in \mathcal{C}$ it generates H-sequence with $\gamma \geq 1/\alpha$, where $\alpha = R/r$, where r is internal radius for P_0 and R is external radius for C, and for $C \in \mathcal{C}^2$ asymptotically $\gamma \approx 1$.

In case of $d = 2$ this algorithm coincides with the known "sandwich" algorithm with chord rule (Fruhwirth, Burkard and Rote 1989–1992). Obviously, in this case $\gamma = 1$.

Let $\{P_n\}$ be $H(C,\gamma)$-sequence for $C \in \mathcal{C}^2$. Then it was proved (Kamenev 1992) that asymptotically $\delta(P_n, C) \leq \text{const}/n^{2(d-1)}$ in Hausdorff and Nikodim metrics:

$$\delta^H(P_n,C) \leq \lambda/n^{2/(d-1)}, \text{ where } \lambda = (2/\rho)[d(d+1)\sigma(C)/((d-1)\gamma^d\pi_{d-1})]^{2/(d-1)},$$
$$\delta^S(P_n,C) \leq \lambda/n^{2/(d-1)}, \text{ where } \lambda = (2/\rho)[2d\sigma(C)^{(d+1)/2}/((d-1)\gamma^d\pi_{d-1})]^{2/(d-1)}$$

where $\sigma(C)$ — the "area" of $\partial C, \rho$ — the minimum curvature radius of ∂C (using Blaschke's rolling theorem (Brooks and Strantzen 1989) we do not imply positivity of Gaussian curvature k) and π_d — volume of the unit ball. Numerical computer experiments shows that for algorithm **A** in approximation of 2–6-dimensional ellipsoids the constant in the rate of convergence depends only of d and $\int k(x)^{1/2} d\sigma(x)$.

In nonsmooth case ($C \in \mathcal{C}$) there are more weak results for H-schemes: $\delta(P_n, C) \leq \text{const}/n^{1/(d-1)}$ (Kamenev 1986). Now it is proved that for augmentation H-schemes with some additional properties it follows that $\delta(P_n,C) \leq \text{const}/n^{2/(d-1)}, P_n \in \mathcal{P}_n^i$. More precisely let $\{P_n\}$ be the sequence of inscribed polyhedra generated by an augmentation scheme. We define $\{P_n\}$ as $H_1(C,\gamma)$-sequence if for each n there exists an external unit normal u in $p \in \partial C, P_{n+1} = \text{conv}\{p, P_n\}$, such that $g(C,u) - g(P_n,u) \geq \gamma\delta^H(P_n,C)$. Obviously $H_1(C,\gamma)$-sequence is $H(C,\gamma)$-sequence and $H(C,1)$-sequence is $H_1(C,1)$-sequence. Furthermore it is easy to see that the algorithm **A** generates H_1-sequence with the corresponding constant.

Let $C \in \mathcal{C}$ and $\{P_n\}$ — the $H_1(C,\gamma)$-sequence. Then it is proved (Kamenev 1997) that there exist n_0: for any $n \geq n_0$ if follow

$$\delta^H(P_n,C) \leq \lambda/n^{2/(d-1)}, \text{ where } \lambda = (2/\gamma)[\sigma(C+B)\alpha(C+B)^{d-2}/\pi_{d-1}]^{2/(d-1)}.$$

Here B is the unit ball in the origin, $\alpha(C)$ — the asphericity of C.

In some application there is a problem of polyhedral approximation using minimum calculations of the support function. For this reason step-by-step algorithm for reducing the number of calculations of the support function of the approximated body was developed. This algorithm uses augmentation and cutting adaptive schemes simultaneously.

Algorithm B (Kamenev 1986). *Let $P \in \mathcal{P}_n^i$ and $Q_n \in \mathcal{P}_n^c$ be constructed.*
Step 1. Find $u^* = \text{argmax}\{g(Q_n,u) - g(P_n,u) : u \in U(P_n)\}$; *find* $p^* \in T(C,u^*)$.
Step 2. Construct $P_{n+1} = \text{conv}\{p^*, P_n\}$ *and* $Q_{n+1} = Q_n \cap H(C,u^*)$.

In case of $d = 2$ this algorithm is similar to the "sandwich" algorithm with maximum error rule (Fruhwirth, Burkard and Rote 1989–1992). For the algorithm **B** it was proved (Kamenev 1994) that for $C \in \mathcal{C}^2$ asymptotically $\delta(P_n,C), \delta(Q_n,C) \leq \text{const}/n^{2/(d-1)}$ in

Hausdorff and Nikodim metrics. Note that on each iteration of this algorithm there is only one calculation of the support function of C. For this reason in case of $C \in \mathcal{C}^2$ we need $1 \leq \text{const}/\varepsilon^{(d-1)/2}$ calculations of $g(C,u)$ to approximate C with the deviation ε.

Kinematic formulas for finite lattices

Dan Klain

The essential link between convex geometry and combinatorial theory is the lattice structure of the collection of polyconvex sets; that is, the collection of all finite unions of compact convex sets in $\mathbb{R}^n$. In analogy to valuation characterizations and kinematic formulas of convex geometry, the author develops a combinatorial theory of invariant valuations and kinematic formulas on finite lattices.

Let P be a finite poset with minimum $\hat{o}$, and let $J(P)$ denote the lattice of order ideals of P. A theorem of Birkhoff states that every finite distributive lattice takes the form of $J(P)$ for some poset P. Let G be a finite group of automorphisms acting on P. The action of G partitions P into a family $\mathcal{U}$ of orbits U. The action of G on P also induces an action of G on the distributive lattice $J(P)$. In a recent paper the author showed that every G-invariant real-valued valuation φ on $J(P)$ must take the form

$$\varphi = \sum_{U \in \mathcal{U}} c_U \varphi_U, \tag{1}$$

where each $c_U \in \mathbb{R}$ is a constant and where the G-invariant valuation φ_U on $J(P)$ is defined by

$$\varphi_U(A) = |A \cap U|,$$

for each $U \in \mathcal{U}$. Here $|A|$ denotes the number of elements of a finite set A. This analogue of Hadwiger's characterization theorem (for rigid-motion invariant valuations on compact convex sets) yields kinematic formulas for the finite lattice $J(P)$, leading in some cases to new polynomial identities for the Whitney numbers (of the second kind) of a modular lattice P. In particular, the author develops the general kinematic formula for a G-invariant valuation φ on $J(P)$:

$$\frac{1}{|G|} \sum_{g \in G} \varphi(A \cap gB) = \sum_{U \in \mathcal{U}} \frac{1}{|U|} c_U \varphi_U(A) \varphi_U(B).$$

for all $A, B \in J(P)$. Here each c_U is the constant given by (1) for the valuation φ. By setting $\varphi = \chi$, the Euler characteristic of the lattice $J(P)$, one derives a combinatorial analogue of the *principal kinematic formula* of convex and integral geometry.

These kinematic formulas enable one of compute expectations of random valuations on $J(P)$. In many cases we are able to compute these expectations in more than one way, leading to identites such as the following identity for the Gaussian coefficients:

$$\sum_{i=0}^{n} (-1)^i q^{\binom{i}{2}} \binom{n}{i}_q^{-1} \binom{k}{i}_q \binom{l}{i}_q = q^{kl} \binom{n}{k}_q^{-1} \binom{n-l}{k}_q.$$

This particular identity results from the case of $P = L_n(q)$, the lattice of subspaces of a finite dimensional vector space over a finite field of order q. Setting $q = 1$ we obtain the analogous results for the case of $P = B_n$, the Boolean algebra of subsets of a finite set. Other examples to consider include the lattice of partitions of a finite set, the lattice of multisets over a finite set, and finite cell complexes exhibiting some degree of symmetry.

Report on recent research: Applications of the Fourier transform to convex geometry

Alexander Koldobsky

1. Inverse formula for the Blaschke-Levy representation

We say that an even continuous function H on the unit sphere Ω in $\mathbb{R}^n$ admits the Blaschke-Levy representation with $q > 0$ if there exists an even function $b \in L_1(\Omega)$ so that $H^q(x) = \int_\Omega |(x,\xi)|^q b(\xi)\, d\xi$ for every $x \in \Omega$. This representation has numerous applications in convex geometry, probability and Banach space theory. In this paper, we present a simple formula (in terms of the derivatives of H) for calculating b out of H. This formula leads to new estimates for the sup-norm of b that can be used in connection with isometric embeddings of normed spaces in L_q.

2. An application of the Fourier transform to sections of star bodies

We express the volume of central hyperplane sections of star bodies in $\mathbb{R}^n$ in terms of the Fourier transform of a power of the radial function, and apply this result to confirm the conjecture of Meyer and Pajor on the minimal volume of central sections of the unit balls of the spaces ℓ_p^n with $0 < p < 2$.

3. Intersection bodies, positive definite distributions and the Busemann-Petty problem

We prove that an origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if $\|x\|_K^{-1}$ is a positive definite distribution on $\mathbb{R}^n$, where $\|x\|_K = \min\{a > 0 : x \in aK\}$. We use this result to show that for every dimension n there exist polytopes in $\mathbb{R}^n$ which are intersection bodies (for example, the cross-polytope), the unit ball of every subspace of L_p, $0 < p \leq 2$ is an intersection body, the unit ball of the space ℓ_q^n, $2 < q < \infty$ is not an intersection body if $n \geq 5$. Using Lutwak's connection with the Busemann-Petty problem, we present new counterexamples to the problem for $n \geq 5$, and confirm the conjecture of Meyer that the answer to the problem is positive if the body with smaller sections is a polar projection body.

4. Intersection bodies in $\mathbb{R}^4$

We prove that the unit cube in $\mathbb{R}^n$ is an intersection body if and only if $n \leq 4$, and give precise expressions for generating measures (signed measures).

5. Second derivative test for intersection bodies

We use the connection between intersection bodies and positive definite distributions, established in an earlier paper, to give a necessary condition for intersection bodies in terms of the second derivative of the norm. This result allows us to produce a variety of counterexamples to the Busemann-Petty problem in $\mathbb{R}^n$, $n \geq 5$. For example, the unit ball of the q-sum of any finite dimensional normed spaces X and Y with $q > 2$, $dim(X) \geq 1$, $dim(Y) \geq 4$ is not an intersection body, as well as the unit balls of the Orlicz spaces ℓ_M^n, $n \geq 5$ with $M'(0) = M''(0) = 0$.

6. An analytic solution to the Busemann-Petty problem on sections of convex bodies (joint work with R. J. Gardner and Th. Schlumprecht)
We derive a formula connecting the derivatives of parallel section functions of an origin-symmetric star body in $\mathbb{R}^n$ with the Fourier transform of powers of the radial function of the body. (A parallel section function gives the ($(n-1)$-dimensional) volumes of all hyperplane sections of the body orthogonal to a given direction.) This formula provides a new characterization of intersection bodies in $\mathbb{R}^n$ and leads to a complete analytic solution to the Busemann-Petty problem. In conjunction with earlier established connections between the Busemann-Petty problem, intersection bodies, and positive definite distributions, our formula shows that the answer to the problem depends on the behavior of the $(n-2)$-nd derivative of the parallel section functions. The affirmative answer to the Busemann-Petty problem for $n \leq 4$ and negative answer for $n \geq 5$ now follow from the fact that convexity controls the second derivatives, but does not control the derivatives of higher orders.

7. A short proof of Schoenberg's conjecture on positive definite functions (joint work with Y. Lonke)
In 1938 I. J. Schoenberg asked for which positive numbers p is the function $\exp(-\|x\|^p)$ positive definite, where the norm is taken from one of the spaces ℓ_q^n, $q > 2$. The solution of the problem was completed in 1991, by showing that for every $p \in (0, 2]$, the function $\exp(-\|x\|^p)$ is not positive definite for the ℓ_q^n norms with $q > 2$ and $n \geq 3$. We prove a similar result for a more general class of norms, which contains some Orlicz spaces and q-sums, and, in particular, present a simple proof of the answer to the original Schoenberg's question. Some consequences concerning isometric embeddings in L_p spaces for $0 < p \leq 2$ are discussed as well.

Old and new aspects of the affine geometry of convex bodies

Kurt Leichtweiß

Issuing from the equiaffine differential geometry and equiaffinely associated sets as centroids, difference body, floating body, projection body, random simplices etc. the classical affine geometry of convex bodies was established in the early twenties. Here the (old) aspects like

- inequalities and discussions of equality
- other characterizations of special curves and hypersurfaces
- affine rigidity
- analogy to the euclidean case

were determining and led to a list of typical results.

In the last years new aspects like

- generalization to n dimensions

- relaxation of smoothness assumptions
- consideration of affine evolutions

appeared. The aim of the survey lecture is to explain the progress at the first mentioned results and to indicate new ones.

Infinite-dimensional convexity

Joram Lindenstrauss

This is a report on some joint work with V. Fonf. It involves three results concerning the structure of convex sets in infinite dimensional Banach space.

(i) A polytope is a closed bounded convex set which intersects every finite-dimensional subspace in a usual polytope.

Theorem 1. *If C is a polytope in a separable Banach space X then the affine span of C is closed and C has an interior point in this affine span.*

(ii) **Theorem 2.** *There is no discrete proximinal net in a separable infinite dimensional $L_p(\mu)$ space $1 < p < \infty$.*

For $p = 2$ this is related to (a still open) problem on existence of a nice tiling of l_2 by convex bodies.

(iii) Approximation of convex sets by sets in which the extreme points are dense.

Theorem 3. *Let X be a separable infinite dimensional Banach space. A closed bounded convex set C can be approximated (in the Hausdorff distance) by closed convex sets whose extreme points are dense if and only if C does not intersect any affine space with finite codimension by a set with non empty interior.*

If the approximation is possible the approximating sets can be chosen to actually have a dense set of strongly exposed points.

Zonoids

Yossi Lonke

1. Zonoids whose polars are zonoids

Some time ago I have found examples of non-smooth zonoids whose polars are zonoids. They were of the form $B_2^n + rB_2^{n-1}$ where B_2^n is the Euclidean unit ball in $\mathbb{R}^n$ and $0 \leq r \leq 1$. However, these examples do not work for $n \geq 6$. The question is whether there exist at all examples of non-smooth zonoids whose polars are zonoids in any dimension. As a first step, the search is restricted to rotation-bodies. Some partial results in this direction are the following.

Proposition. *Assume $n \geq 6$. If K is an n-dimensional rotation body, such that $K + B_2^{n-1}$ is a polar of a zonoid, then K is not a polar of a zonoid.*

Corollary. *Assume $n \geq 6$. If K is an n-dimensional rotation body then there exists a number $r(K) > 0$, such that for every $0 \leq r < r(K)$, the body $K + rB_2^{n-1}$ is not a polar of a zonoid.*

2. Isometric embeddings into L_p spaces, $0 < p \leq 2$ (joint work with Alexander Koldobsky)

Theorem. *Assume $X = (\mathbb{R}^3, \| \ \|)$ is a 3-dimensional normed space. Assume that for each fixed $(y,z) \in \mathbb{R}^2 \backslash \{0\}$, the function $x \to \|(x,y,z)\|$ is in $C^2(\mathbb{R})$, and that the following two conditions are satisfied:*

(i) For every $(y,z) \in \mathbb{R}^2 \backslash \{0\}$,

$$\frac{d}{dx}\|(x,y,z)\|\Big|_{x=0} = \frac{d^2}{dx^2}\|(x,y,z)\|\Big|_{x=0} = 0,$$

(ii) There exists a constant $C > 0$, such that if $(y,z) \in \mathbb{R}^2$ and $\|(0,y,z)\| = 1$, then

$$(\forall x)\frac{d^2}{dx^2}\|(x,y,z)\| \leq C.$$

Then the space X is not linearly isometric to a subspace of L_p, when $0 < p \leq 2$.

Examples of spaces satisfying the hypothesis of the theorem are ℓ_q^3 for $q > 2$. In this case the theorem provides an answer to a question posed by Schoenberg in 1938 about positive definite functions. The answer was known before, but its proof was more complicated than the proof of the theorem here, which is very simple.

A characterization of affine length and asymptotic approximation of convex bodies

Monika Ludwig

Let $\mathcal{K}$ be the set of planar convex bodies (compact, convex sets). For $K \in \mathcal{K}$ affine length λ is defined as

$$\lambda(K) = \int_0^l \kappa_K(t)^{\frac{1}{3}}\, dt$$

where $\kappa_K(t)$ is the curvature of K given as a function of arclength t and l is the length of bd K. Since κ_K exists a.e. and is an integrable function, this functional is well defined for a general (not necessarily smooth) convex body. Affine length has the following properties:

- it is equiaffine invariant:

  $$\lambda(\phi(K)) = \lambda(K)$$

  for every affine map ϕ with determinant 1

- it is upper semicontinuous:

  $$\lambda(K) \geq \limsup_{n\to\infty} \lambda(K_n)$$

  for $K_n \to K$

- it is a valuation:

$$\lambda(K \cup L) + \lambda(K \cap L) = \lambda(K) + \lambda(L)$$

  for K, L, $K \cup L \in \mathcal{K}$

Besides $\lambda(K)$, the area $\mathrm{A}(K)$ and the Euler characteristic have these properties.

Theorem 1. *Let $\mu : \mathcal{K} \to I\!R$ be an upper (or lower) semicontinuous and equiaffine invariant valuation. Then, there are constants c_0, c_1, and c_2 such that*

$$\mu(K) = c_0 + c_1\,\mathrm{A}(K) + c_2\,\lambda(K)$$

for all $K \in \mathcal{K}$. If μ is upper semicontinuous, then $c_2 \geq 0$, if it is lower semicontinuous, then $c_2 \leq 0$.

This theorem can be used to obtain results on asymptotic approximation of convex bodies. Let $\delta(K, L)$ denote the area of the symmetric difference of K and L and let $\mathcal{P}_n^i(K)$ denote the set of polygons with at most n vertices which are contained in K. Define

$$\delta(K, \mathcal{P}_n^i) = \inf\{\delta(K, P) : P \in \mathcal{P}_n^i(K)\},$$

i.e., $\delta(K, \mathcal{P}_n^i)$ is the distance of K from its best approximating polygon with at most n vertices. It was shown by L. Fejes Tóth, McClure and Vitale that for a convex body K with boundary of class C^2 and positive curvature

$$\delta(K, \mathcal{P}_n^i) \sim \frac{\lambda(K)^3}{12\,n^2}$$

as $n \to \infty$. This can be extended to general convex bodies.

Theorem 2. *For every $K \in \mathcal{K}$*

$$\lim_{n\to\infty} n^{\frac{2}{3}}\delta(K, \mathcal{P}_n^i)^{\frac{1}{3}} = \left(\frac{1}{12}\right)^{\frac{1}{3}} \lambda(K).$$

In a joint work with Matthias Reitzner, Theorem 1 is extended to general dimensions. It follows from results of K. Leichtweiß, E. Lutwak, C. Schütt and E. Werner that affine surface area can be defined for general convex bodies and is an equiaffine invariant and upper semicontinuous valuation. Our result says that every equiaffine invariant and semicontinuous valuation can be written as a linear combination of Euler characteristic, volume and affine surface area.

Each symmetric convex body in $I\!R^3$ admits an inscribed cube

Endre Makai, Jr. (joint work with T. Hausel and A. Szücs)

Answering a question of Klee-Wagon, we prove that each convex body in $I\!R^2$ admits inscribed rectangles with any given ratio of the side lengths. We prove that each centrally symmetric convex body in $I\!R^3$ admits an inscribed cube, and, more generally, an inscribed similar copy of any given rectangular parallelepiped. For the case of the cube we give simple arguments using elements of (equivariant) algebraic topology, via the solution of a special case of Knaster's problem. Connected with this, we investigate the existence of equivariant maps $SO(3) \to S^2$ for certain group actions of subgroups of S_4 on $SO(3)$ and S^2. The statement for the general rectangular parallelepiped follows from a rather technical recent theorem of Griffiths, about another special case of Knaster's problem, that includes the above one. Answering a question of Bodlaender-Gritzmann-Klee-van Leeuwen, we prove that for n large enough there exists in $\mathbb{R}^n$ a (centrally symmetric) convex body, admitting no inscribed parallelepiped; moreover, a typical (centrally symmetric) convex body in $\mathbb{R}^n$ has this property. We prove that in $\mathbb{R}^3$ any set of diameter at most 1 can be included to a rhombic dodecahedron, with distance of opposite faces equal to 1. Possible application of this theorem to the Borsuk problem in $\mathbb{R}^3$ is pointed out.

On inner illumination of convex bodies

Horst Martini (joint work with V. Boltyanski and V. Soltan)

Due to P. Soltan (1962), a set $F \subset \operatorname{bd} K$ *illuminates* a convex body $K \subset \mathbb{E}^d$ ($d \geq 2$) *from within* if for each point $x \in \operatorname{bd} K$ there is some $y \in F$ ($x \neq y$) such that $]x, y[\subset \operatorname{int} K$. For example, any $K \subset \mathbb{E}^d$ is illuminated from within by at most $d+1$ points, with equality if K is a simplex (Soltan 1962). Moreover, $F \subset \operatorname{bd} K$ is said to be a *primitive inner illuminating system* of K if no proper subset of it will illuminate K from within. Although even a proof for the existence of the maximum number of points of a primitive inner illuminating syste was lacking for $d \geq 3$, B. Grünbaum (1964) conjectured this number to be 2^d. For $d = 3$, this was confirmed by V. Soltan (1995), with equality if K is combinatorially equivalent to the 3-cube. We show that for $d \geq 4$ Grünbaum's conjecture is wrong: for any positive integer m, there is a convex body $K \subset \mathbb{E}^d$, $d \geq 4$, with a primitive inner illuminating system of at least m points. Also we show that any such system is finite. H. Hadwiger (1972) asked whether a convex d-polytope $P \subset \mathbb{E}^d$ *illuminated by its vertices* (i.e., for any vertex x of P there is another vertex y of P such that $]x, y[\subset \operatorname{int} P$) has at least $2d$ vertices. P. Mani (1974) proved that for $d \leq 7$ the answer is affirmative, while for $d > 7$ there is such a polytope having about $d + 2\sqrt{d}$ vertices. We show that any convex d-polytope primitively illuminated by its vertices has at least $2d$ vertices.

Polytopes: valuations, dissections and combinatorics

Peter McMullen

The purpose of this talk is to survey recent developments in the area of the title. The emphasis will be firmly on the algebraic aspects of the subject. An initial motivation for studying valuations in the abstract was to investigate the extent to which equality of volume of ordinary polyhedra is characterized by equidissectability, as is the case for area of polygons. Indeed, some of the problems still being looked at were first raised by the Greeks.

Rather than develop the subject historically, the starting point here will be the most recent approach to the polytope ring and algebras. The polytope ring, which is

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the abstract algebraic object corresponding to valuations on polytopes (with no translational properties as yet assumed) is identified with the ring of piecewise exponential functions. Completion and quotients then provide connexions with polytope algebras, piecewise polynomials and tensor weight algebras. Another natural quotient then describes the conditions for equidissectability of polytopes by translation (when any rigid motions are allowed, the general problem is still open).

A striking application of scalar weights (which can be thought of as the algebra of mixed volumes of polytopes) was to the purely combinatorial problem of characterizing the f-vectors of simple polytopes. Part of the structure of the weight space of a simple polytope, established by non-analytic methods, is a family of quadratic inequalities, among which is Minkowski's second inequality. From this (as is well known) the Brunn-Minkowski theorem can be deduced; of interest, perhaps, is that the equality conditions, which are lost in proceeding to a limit, can be recovered for polytopes by an inductive argument.

Affine surface area and p-affine surface area

Mathieu Meyer and Elisabeth Werner

For a convex body K in $\mathbb{R}^n$ and $t \in \mathbb{R}$, $t > 0$, we define the Santaló-bodies $S(K,t)$ as

$$S(K,t) = \{x \in K : \frac{|K||K^x|}{v_n^2} \leq t\},$$

where $|K|$ denotes the n-dimensional volume of the convex body K and v_n is the volume of the n-dimensional Euclidean unit ball $B(0,1)$. K^x is the polar of K with respect to x.

Those bodies are related to the affine surface area

$$O_1(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} d\sigma(u) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_K(x),$$

where $f_K(u)$ is the Gauss curvature function, that is the reciprocal of the Gauss curvature $\kappa(x)$ at this point $x \in \partial K$ that has u as outer normal. μ_K is the usual surface measure on the boundary ∂K of K and σ is the spherical Lebesgue measure.

It was shown that the connection between $O_1(K)$ and the Santaló-bodies is as follows

$$\lim_{t\to\infty} t^{\frac{2}{n+1}}(|K| - |S(K,t)|) = \frac{1}{2}(\frac{|K|}{v_n})^{\frac{2}{n+1}} \ O_1(K).$$

Lutwak introduced for a convex body K in $\mathbb{R}^n$ with positive continuous curvature function the p-affine surface area $O_p(K)$.

$$O_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{< x, N(x) >^{\frac{n(p-1)}{n+p}}} d\mu_K(x),$$

where h_K is the support function of K and $N(x)$ is the outer normal in $x \in \partial K$.

We show that for bodies with sufficiently smooth boundary there is a geometric interpretation for the p-affine surface area (for $-n < p$) comparable to the one for affine surface area in terms of a generalization of the notion of Santaló-bodies.

On isometric embedding of subspaces of L_p into l_p

Aleksander Pelczynski (joint work with F. Delbaen and H. Jarchow)

The following results answer a question of A. Pietsch:

Theorem 1. *Let $0 < p < \infty$, $p \notin 2\mathbb{N}$. Let E be a (closed linear) subspace of $L_p = L_p([0;1])$. Then E is isometric to a subspace of l_p iff every unital subspace of L_p isometric to E consists of functions having discrete distributions.*

It is well known that every subspace of L_p is isometric to a unital one.

Corollary 1. *A subspace E of L_p $(0 < p < \infty,\ p \notin 2\mathbb{N})$ is isometric to a subspace of l_p iff every 2-dimensional subspace of E has the same property.*

Corollary 2. *The 2-dimensional Euclidean space is not isometric to a subspace of l_p $(0 < p < \infty,\ p \notin 2\mathbb{N})$.*

Proof. Otherwise by Corollary 1 l_2 would be isometric to a subspace of l_p.

Theorem 2. *If $p \in 2\mathbb{N}$ then every finite dimensional subspace of L_p is isometric to a subspace of l_p. Moreover for every $n \in \mathbb{N}$ there exists an $N = N(p, n, \mathbb{K}) \in \mathbb{N}$ ($\mathbb{K}$ is the field of scalars, either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) such that every n-dimensional subspace of L_p embedds isometrically into l_p^N.*

An isoperimetric inequality for hyperplane sections of a convex body

Carla Peri

Consider the euclidean space $\mathbb{R}^n$ with the canonical inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. Denote $S^{n-1} := \{x \in \mathbb{R}^n \mid \| x \| = 1\}$ the unit euclidean sphere, $V(\cdot)$ the Lebesgue n- dimensional measure and $A(\cdot)$ the Lebesgue $(n-1)$-dimensional measure defined on Borel sets lying in hyperplanes of $\mathbb{R}^n$.

Let us suppose that a convex body $K \subset \mathbb{R}^n$ is divided by a hyperplane H into two sets K_1, K_2. We want to find a sharp upper bound for the product $V(K_1)V(K_2)$ in terms of the area $A(K \cap H)$ of the intersection of K with the hyperplane H. This problem has been studied by Bokowski and Sperner (1979), Bokowski (1980), Gysin (1986), Mao (1993) and Santalò (1983) who obtained upper bounds where $A(K \cap H)$ is multiplied by a constant depending on the diameter of K and on the dimension n. It should be noted that a more general version of this problem, where H is replaced by a measurable surface, has been studied by many authors in connection with randomized algorithms for approximating the volume of a convex body. Relative isoperimetric inequalities are also related to immersion theorems for Sobolev spaces.

We prove that if $K \subset \mathbb{R}^n$ is a convex body divided into two parts K_1, K_2 by a hyperplane H orthogonal to $u \in S^{n-1}$ then

$$V(K_1)V(K_2) \leq \frac{1}{\ln 2}\Big(\min_{y\in\mathbb{R}^n} \int_K \mid \langle u, x-y\rangle \mid dx\Big)A(K \cap H).$$

This inequality is asymptotically tight, for sufficiently large n, and implies the following:

$$V(K_1)V(K_2) \leq \frac{1}{\ln 2}(\int_K \mid \langle u, x - c(K)\rangle \mid dx)A(K \cap H),$$

where $c(K)$ denotes the centroid of K. We also show that if K is centrally symmetric then the constant $1/\ln 2$ can be dropped. Moreover, the inequality we obtain in this case is exact as equality holds for a cylinder and a hyperplane cut, through the center, parallel to the base of the cylinder.

The main ingredient in the proofs is a "Localization Lemma" due to Kannan, Lovász and Simonovits (1995).

The constant $\int_K \mid \langle u, x - c(K)\rangle \mid dx$ can be given a geometric interpretation: it is the support function in the direction u of the centroid body ΓK of K, multiplied by the volume of K. Thus, the previous results imply

$$\min\{V(K_1), V(K_2)\} \leq \frac{1}{\ln 2} D_{\Gamma K} A(K \cap H)$$

($D_{\Gamma K}$ being the diameter of ΓK), where the constant $1/\ln 2$ can be dropped when K is centrally symmetric.

We think that the last inequality can be generalized as follows.

Conjecture 1. *Let K be a convex body in $\mathbb{R}^n$ such that $c(K) = o$ and let $E \subset K$ be a measurable set. Let $A(K, E)$ denote the Minkowski content of the part of the boundary of E which is contained in the interior of K. Then*

$$\min\{V(E), V(K \backslash E)\} \leq \frac{1}{\ln 2} D_{\Gamma K} A(K, E)$$

and the constant $1/\ln 2$ can be dropped when K is centered.

In the case where K is the unit cube this inequality was proved by Hadwiger (1972); in the general case it would improve an inequality due to Kannan, Lovász and Simonovits (1995).

The subindependence of coordinate slabs in ℓ_p^n balls

Irini Perissinaki (joint work with K. Ball)

It is proved that if the probability P is normalised Lebesgue measure on one of the ℓ_p^n balls in $\mathbb{R}^n$, then for any sequence $t_1, t_2, \ldots, t_n$ of positive numbers, the coordinate slabs $\{|x_i| \leq t_i\}$ are subindependent, namely,

$$P\left(\bigcap_{i=1\ldots n} \{|x_i| \leq t_i\}\right) \leq \prod_{i=1\ldots n} P(\{|x_i| \leq t_i\})$$

A consequence of this result is that the proportion of the volume of the unit ℓ_1^n ball which is inside the cube $[-t, t]^n$ is less than or equal to $f_n(t) = (1 - (1 - t)^n)^n$

It turns out that this estimate is remarkably accurate over most of the range of values of t. A reverse inequality, demonstrating this, is the second major result of this work.

1. The two Theorems and their relation

Theorem 1 Subindependence of coordinate slabs. *If the probability P is normalised Lebesgue measure on one of the ℓ_p^n balls in $\mathbb{R}^n$, then for any sequence $t_1, \ldots, t_n$ of positive numbers,*

$$P\left(\bigcap_{i=1\ldots n} \{|x_i| \le t_i\}\right) \le \prod_{i=1\ldots n} P(\{|x_i| \le t_i\}).$$

The particular case $p = 1,\ t_1 = \ldots = t_n$ of Theorem 1 gives an upper bound for the proportion of the volume of the unit ℓ_1^n ball which is inside the cube $[-t, t]^n$. Since the proportion of the volume of the unit ℓ_1^n ball which is inside a coordinate slab of width $2t$ is $1 - (1 - t)^n$ when $t \le 1$, the result in this case is given by the following Corollary.

Corollary 1.1. *If $F_n(t)$ is the proportion of the volume of the unit ℓ_1^n ball inside the cube $[-t, t]^n$ then*

$$F_n(t) \le f_n(t) = (1 - (1 - t)^n)^n$$

Although $F_n(t)$ is the function $\sum_0^{[1/t]}(-1)^j\binom{n}{j}(1-jt)^n$ (an indirect result of the proof of Theorem 1), which is a spline with many knots, we prove in Theorem 2 that the polynomial $f_n(t) = (1 - (1 - t)^n)^n$ is an astonishingly good approximation to $F_n(t)$, at least when $F_n(t)$ is not too small.

Theorem 2 (An estimate in the reverse direction). *With $F_n(t)$ as above,*

$$\frac{1 - F_n(t)}{1 - f_n(t)} = 1 + O\left(\frac{(\log n)^3}{n}\right)$$

as $n \to \infty$ uniformly in t.

Theorem 2 enables us to describe the threshold behaviour of $F_n(t)$ quite accurately. For example, if $t = \frac{\log n - \log c}{n}$ then the information we get from Theorem 2 is that $F_n(t)$ should be something like $f_n(t)$, which in turn is something like

$$(1 - \exp(-\log n + \log c))^n = \left(1 - \frac{c}{n}\right)^n \simeq \exp(-c).$$

2. A brief account for the method of the proofs of the two Theorems

We shall briefly explain the crucial points of the proofs of the two Theorems, for the simplest case $p = 1$ and $t_1 = \cdots = t_n = t$. The general case is treated in a very similar way, so we shall not examine this here.

As above, we write $F_n(t)$ for $P([-t, t]^n)$, where P is now normalised Lebesgue measure on the unit ℓ_1^n ball. We also write $f_n(t)$ for $(1 - (1 - t)^n)^n$ which is a function dominating F_n according to Corollary 1.1.

Proof of Theorem 1. The proof of Theorem 1 (the upper bound for F_n) depends on a very convenient interaction between two different equations expressing F_n and its derivative in terms of F_{n-1}. Each of these equations is proved using a simple geometric argument: they can readily be combined to give a differential inequality for F_n which integrates up to the stated result.

These equations are:

$$F_n(t) = n \int_0^t (1-u)^{n-1} F_{n-1}\left(\frac{t}{1-u}\right) du$$

$$\frac{d}{dt} F_n(t) = n^2(1-t)^{n-1} F_{n-1}(\frac{t}{1-t})$$

The upper bound is extremely precise as long as $F_n(t)$ is not too small. The easiest way to state this is to write it as an estimate for the volume outside the cube, namely for $1 - F_n(t)$. This is what we do in Theorem 2.

Proof of Theorem 2. The proof of Theorem 2 (a lower bound for F_n) is technically more complicated although it is much less delicate. The crucial point is to show that at its maximum, the function $\frac{1-F_n}{1-f_n}$ is dominated by the value of a related function, which in turn can be shown to be small by means of the (rather precise) upper bound already proved.

Selection measures

Krzysztof Przeslawski

Let $\mathcal{K}^n$ be the family of all convex bodies in $\mathbb{R}^n$. An $\mathbb{R}^n$-valued Borel measure μ over S^{n-1} is said to be a *selection measure* if it is of finite variation, and if for every $A \in \mathcal{C}^n$

$$\int h_A(x) d\mu(x) = s(A) \in A,$$

where h_A is the support function of A. We denote by $\mathcal{M}^n$ the family of all selection measures over S^{n-1}. It is easily seen that s is Lipschitz continuous with respect to the Hausdorff metric, and linear, i.e. Minkowski additive, and homogeneous with respect to the multiplication by nonnegative scalars. Conversely, if $s : \mathcal{K}^n \to \mathbb{R}^n$ is a Lipschitz continuous and linear selection, then a standard application of the Riesz representation theorem shows that there exists a unique selection measure μ such that the above equation is satisfied. The family of all such selections will be denoted by $\mathcal{S}^n$.

It would be interesting to have effective methods that enable one to determine whether a given measure over S^{n-1} is a selection measure. A little appears to be known in this direction even in case of $n = 2$. The following properties of selection measures have been established in a joint work of G. Rote and the present author.

1° If $n \geq 3, \mu \in \mathcal{M}^n$, then for every $n-2$-dimensional subspace X of $\mathbb{R}^n$,

$$|\mu|(S^{n-1} \cap X) = 0,$$

where $|\mu|$ denotes the variation measure of μ. In particular, there is not a discrete selection measure if $n \geq 3$.

2° If $\mu \in \mathcal{M}^2$ is singular with respect to the arc length measure, then μ is odd. (Equivalently, if s is the selection corresponding to μ, then s is centrally symmetric, i.e. $s(-A) = -s(A)$ for every $A \in \mathcal{K}^2$.)

3° If $\mu \in \mathcal{M}^2$ is odd, then for every $x, y \in S^1$, if $x \wedge y > 0$, then

$$0 \geq \int_{\{u:\langle x,u\rangle\langle y,u\rangle \leq 0\}} \langle y, u\rangle x \wedge d\mu(u) \leq x \wedge y$$

and

$$0 \geq \int_{\{u:\langle x,u\rangle\langle y,u\rangle \geq 0\}} \langle y, u\rangle x \wedge d\mu(u) \leq x \wedge y,$$

where $v \wedge w$ is the signed area of the parallelogram determined by the two vectors v and w.

PROBLEMS

1. Does there exists a measure $\mu \in \mathcal{M}^n, n \geq 3$, which is singular with respect to the surface area measure?

2. Let $B \subset S^{n-1}$. We say that B has the *intersection property* if for every function k which is the restriction to B of a support function, the set $\cap\{K \in \mathcal{K}^n : h_K|_B = k\}$ is nonempty. Clearly, if μ is a selection measure, then suppμ has the intersection property. The question is whether there exists a finite subset $B \subset S^{n-1}, n \geq 3$, wich has the intersection property. Observe that for $n = 2$, B has the intersection property iff $\# B \geq 4$.

3. Let $x, y \in S^{n-1}$ be linearly independent. Let $s_{xy}(K)$ be the center of the smallest parallelogram containing $K \in \mathcal{K}^n$ which has one pair of the sides perpendicular to x and the other to y. It is easily seen that $s_{xy} \in \mathcal{S}^2$, and that the selection measure μ which corresponds to s_{xy} is of the form:

$$\mu = \frac{1}{2}\left(\delta_x \frac{y'}{y \wedge x} + \delta_{-x} \frac{y'}{y^\wedge - x} + \delta_y \frac{x'}{x \wedge y} + \delta_{-y} \frac{x'}{x \wedge -y}\right).$$

where z' is obtained by revolving z about 0 counter-clockwise through a right angle. It can be shown that each centrally symmetric selection $s \in \mathcal{S}^2$ is an affine, possibly infinite, combination of such *parallelogram* selections. Is s a convex combination of parallelogram selections? A slightly weaker question reads as follows: Is it true that for any triangle $T, s(T)$ belongs to the triangle with vertices at the midpoints of the sides of T?

A special case of Mahler's conjecture

Shlomo Reisner (joint work with M. Lopez)

A special case of Mahler's conjecture on the volume-product of symmetric convex bodies in n-dimensional Euclidean space is treated here. This is the case of polytopes with at most $2n + 2$ vertices (or facets). Mahler's conjecture is proved in this case for $n \neq 8$ and the minimal bodies are characterized.

On the algorithmical solution of the main problems in the metric theory of polyhedra

I. Kh. Sabitov

The following two problems in the metric theory of polyhedra are known as main ones: the first is the problem of isometric realization in $\mathbb{R}^3$ of a given polyhedral metric and the second is the problem of recognition of the flexibility of any given polyhedron. For the first problem there are many different settings; we suppose that the polyhedral metric is given as one of a metric simplicial complex K and we require that the combinatorial structure of K should be carry on the sought polyhedron P isometric to K so that the faces of K must serve for P as its natural development. Up to now in such a formulation there has been no result either positive or negative nature (even the famous Alexandrov's theorem on the existence of the isometric realization of any convex polyhedral metric as a convex polyhedron in $\mathbb{R}^3$ doesn't guarantee that the combinatorial structure of K would be carry over to P). It turns out that the solution of the "bellows conjecture" admits to indicate an algorithmical approach to the solution of the both problems above.

Indeed it is known that the volume V of any polyhedron P in $\mathbb{R}^3$ may be calculed as a root of a polynomial equation

$$Q(V^2) = 0 \tag{1}$$

with the coefficients depending only on the P's combinatorial structure (defined as one of a simplicial complex K) and the metric of P i.e. for the calcul of $V = vol(P)$ we have a generalization of the Heron's formule. It is essential that the polynomial $Q(V^2)$ may be found by the application of an algorithm (the proof affirms only the existence of such a polynomial equation for V but it doesn't give any method to find it). First we note that the equation (1) gives the following necessary conditions for the solution of the both problems:

1. *Let $|K|$ be the body of a simplicial metric complex K. Then for the existence of a simplicial isometric map $P : |K| \to \mathbb{R}^3$ it is necessary that the equation (1) composed on the base on $|K|$ should have at least a root $V^2 \geq 0$.*

2. *Let P be a flexible polyhedron then its volume is a multiple root of the corresponding equation (1).*

The algorithm for seaking of isometric realizations of the given polyhedral metric is based on the following

Lemma. *Let A be a vertex in a polyhedron P of degree $m \geq 4$ and let $p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_m$ be the vertices of $\partial(Star A)$ enumerated in a cyclic ordre. Suppose that the segment $< p_{i-1}p_{i+1} >$ is not an edge of P. Denote the length of $< p_{i-1}p_{i+1} >$ by d_i. Then the volume V of P and d_i are related by a polynomial equation of the form*

$$Q_i(V, d_i) = a_0^{(i)}(l, V^2)d_i^{2M_i} + \ldots + a_{M_i}(l, V^2) = 0 \tag{2}$$

where $l = (l_1^2, \ldots l_e^2)$ and e is the number of P's edges.

If P is in the general position then it is guaranteed that $2M_i \geq 2$. This means that the equation (2) has the leading coefficient $a_0^{(i)} \neq 0$ so taking V^2 as one of the non-negative roots of (1) we find *a finite* number of the d_i's values. After this for the construction of polyhedra having the given values of edge lengths and diedral angles it is easy to indicate a finite algorithm which will give either a required polyhedron or show the non-existence of such polyhedron. As a consequence of this construction we have an other proof of the affirmation that almost all polyhedra in $\mathbb{R}^3$ are rigid, a result due to Gluck (for polyhedra homeomorphe to a sphere), Whiteley and Graver (for the toroidal polyhedra) and Fogelsanger (in the general case).

When in the equation (2) for the given values of l and V all coefficients $a_j^{(i)}, 0 \leq j \leq M_i - 1$, are zero (by the way it is *a necessary* condition for the flexibility of P) we have an algorithm for the construction of P too but for the moment it gives a result only under some supplementary suppositions.

As to the verification of flexibility of a given polyhedron P we can indicate an algorithm based on the following idea: in the star of A (in notations of the lem[illegible]) we remove the edge $< Ap_i >$ with adjacent faces and consider the new polyhedro[illegible] with the edge $< p_{i-1}p_{i+1} >$ and the faces $< Ap_{i-1}p_{i+1} >$ and $< p_{i-1}p_ip_{i+1} >$. Now we apply to P' the lemma; if it is applicable then P' is rigid so P is flexible for the values of d_i near of the one given in initial position of P. If the lemma is not applicable we can repeat analogical consideration with P' and so on.

Adapted convex bodies

Jane R. Sangwine-Yager

The support function and supporting hyperplane of a convex body K with non-empty interior are denoted by $h(K,\cdot)$ and $H(K,\cdot)$. For $x \in \operatorname{bd} K$, the boundary of K, $N(K,x)$ is the *normal cone* at x and it is defined by $\{u \neq o | x \in H(K,u)\} \cup \{o\}$, where o is teh origin of $\mathbb{E}^n$. The *touching cone* of K containing u is $T(K,u) = N(K,x)$ for any $x \in \operatorname{relint} H(K,u) \cap K$. $E(K)$ is the set of unit extreme directions of K, and B is the unit ball.

Adapted convex bodies were introduced by Schneider (1990). A is *adapted* to C if for all $x \in \operatorname{bd} C$, there is $y \in \operatorname{bd} A$ such that $N(C,x) \subset N(A,y)$.

Schneider (1990) uses the first result below to obtain the second.

If A is adapted to C and $u \in S^{n-1}$, then

$$h'_r(C \sim \tau A, u)|_{r=0} + h(A,u) = 0,$$

where h'_r denotes the right-hand derivative and $C \sim \tau A$ is the Minkowski differe[illegible] $\tau > 0$.

Let K, L and C_i be convex bodies for $\mathcal{C} = (C_1, \ldots, C_{n-2})$. Suppose equality holds in the Aleksandrov–Fenchel inequality, $V(K,L,\mathcal{C})^2 \geq V(K,K,\mathcal{C})V(L,L,\mathcal{C})$. If $\mathcal{A} = (A_1, \ldots, A_{n-2})$ where A_i is adapted to C_i and $i = 1, \ldots, n-2$, then equality also holds when $\mathcal{A}$ replaces $\mathcal{C}$.

It is a corollary to this result that equality holds in $A-F$ when $\mathcal{C}$ consists of smooth bodies if and only if K and L are homothets.

To prove strengthened characterizations of zonoids, Goodey and Zhang [1996] establish the denseness of differences of surface area measures using the following result.

If K is a smooth convex body and $u \in S^{n-1}$, then

$$h'_\tau(K \sim \tau B, u)|_{\tau=0} + 1 = 0,$$

and the convergence of the difference quotient to the derivative is uniform on S^{n-1}.

We show

Theorem. *If K and A are any convex bodies and $u \in S^{n-1}$, then*

$$-h'_\tau(K \sim \tau A, u)|_{\tau=0} = \max \sum \lambda_i h(A, v_i),$$

where the maximum is over all sets $\{v_i\}$ such that $u = \sum_{i=1}^{k} \lambda_i v_i$ for some integer $k, \lambda_i > 0$, and each $v_i \in E(K) \cap T(K, u)$.

Corollary. *A is adapted to K if and only if $h'_\tau(K \sim \tau A, u)|_{\tau=0} + h(A, u) = 0$ for all $u \in S^{n-1}$.*

Lemma. *If the boundary of K is smooth, the difference quotient for the derivative in the Theorem converges uniformly to $h(A, u)$ on S^{n-1}.*

Sketch of the Proof of the Theorem and Lemma. Fix $u \in S^{n-1}$. For each $\tau, 0 \le \tau < r(K, A)$, the relative inradius, there exists a set $\{v_i(\tau, u)\}$, $i = 1, \ldots, k$, of elements of $E(K \sim \tau A) \cap T(K \sim \tau A, u)$ such that

$$u = \sum_{i=1}^{k} \lambda_i(\tau, u) v_i(\tau, u), \;\; \lambda_i(\tau, u) \ge 0.$$

It follows that $h(K \sim \tau, A, v_i(\tau, u)) = h(K, v_i(\tau, u)) - \tau h(A, v_i(\tau, u))$ for $i = 1, \ldots, k$, and

$$h(K \sim \tau A, u) = \sum \lambda_i(\tau, u)(h(K, v_i(\tau, u)) - \tau h(A, v_i((\tau, u))).$$

Let $DQ(\tau, u) = (h(K, u) - h(K \sim \tau A, u))/\pi$. These results and the sublinearity of the support function lead to

$$\sum \lambda_i(0, u) h(A, v_i(0, u)) \le DQ(\tau, u) \le \sum \lambda_i(\tau, u) h(A, v_i(\tau, u)),$$

for all sets $\{v_i(0, u)\} \subset E(K) \cap T(K, u)$ and $\{v_i(\tau, u)\} \subset E(K \sim \tau A) \cap T(K \sim \tau A, u)$ with u in their posivite hulls. Next we show that the $v_i(\tau, u) \to v_i(u) \in T(K, u)$ as $\tau \to 0$.

If bd K is smooth, the derivative is $h(A, u)$. Suppose the convergence is not uniform. For $\varepsilon > 0$ there must exist $\ell \to \infty$, such that $\tau_\ell \to 0, u_\ell = \sum \lambda_i(\tau_\ell, u_\ell), \lambda_i(\tau_\ell, u_\ell) > 0, v_i(\tau_\ell, u_\ell) \in E(K \sim \tau_\ell A) \cap T(K \sim \tau_\ell A, u_\ell)$, and $\varepsilon < DQ(\tau_\ell, u_\ell) - h(A, u_\ell)$. If $u_\ell \to u$, then $v_i(\tau_\ell, u_\ell) \to v_i \in T(K, u)$ which implies that each $v_i = u$, and we have a contradiction.

A rearrangement inequality & applications

Michael Schmuckenschläger (joint work with A. Burchard)

Let M_k denote the simply connected d dimensional manifold with constant sectional curvature k. We study functionals J of the form

$$J(f_1,\dots,f_n) := \int\cdots\int \prod_{1\le i\le n} f_i(x_i) \prod_{1\le i<j\le n} K_{ij}(x_i,x_j)\,dx_1\dots dx_n$$

where for each pair i,j, the kernel $K_{ij}(x,y)$ is a nonincreasing nonnegative function of the distance between x and y and $f_1,\dots f_n$ are nonnegative measurable functions on M_k which vanish at infinity (so that the spherically decreasing rearrangements $f_1^*,\dots,f_n^*$ can be defined.

Theorem 1. *The functional J (with fixed nonincreasing kernels K_{ij}) never decreases under spherically decreasing rearrangement of the f_i, that is,*

$$J(f_1,\dots,f_n) \le J(f_1^*,\dots,f_n^*)$$

for all nonnegative measurable functions $f_1,\dots,f_n$ on M_k so that the rearrangements $f_1^*,\dots,f_n^*$ can be defined.

Applications: 1. Suppose we are given a regular domain $A \subseteq M_k$ of finite volume $v(A)$, i.e. A is an open connected subset with smooth boundary. Denote by $u_A(t,x)$ the solution of the Dirichlet problem

$$-\Delta u_A = \partial_t u_A \quad \forall x\in A:\ u_A(x,0)=1 \text{ and } \forall x\in\partial A\,\forall t>0:\ u_A(x,t)=0,$$

where the signe of the Laplacian Δ is choosen in order to make Δ a positive operator. Then $u_A(t,x)$ is always bounded from above by $u_B(x_0,t)$, where B is a gedodesic ball centered at x_0 such that $v(B)=v(A)$.

2. Let V be a potential on M_k satisfying suitable growth conditions at infinity (if $k\le 0$). Then the trace of $e^{-t(\Delta+V)}$ can only increase under spherically decreasing rearrangement of V, that is

$$tr(e^{-t(\Delta+V)}) \le tr(e^{-t(\Delta+V_*)})\ .$$

3. A particular case of the Theorem implies the isoperimetric inequality on M_k: Let P_t be the heat semigroup on M_k with generator $-\Delta$, A a measurable subset of M_k and B a geodesic ball in M_k such that $v(B)=v(A)$. By the Theorem we have for all $t>0$: $\int_A P_tI_A \le \int_B P_tI_B$. In this form the rearrangement inequality has already been proved by A. Baernstein and Taylor. Following an idea of M. Ledoux we prove a formula that relates $\int_A P_tI_A$ to the volume of the boundary of ∂A.

Theorem 2. *Let A be an open relatively compact subset of a complete Riemannian manifold M with Ricci curvature bounded below. Assume A has smooth boundary. Then*

$$\lim_{t\to 0}\sqrt{\frac{\pi}{t}}\int_{A^c} P_tI_A\,dv = v(\partial A)\ .$$

Corollary. *For all measurable subsets A of M_k, $k > 0$, with smooth boundary ∂A and all $t > 0$:*

$$\frac{1}{v(\partial A)} \int_{A^c} P_t I_A \, dv \leq \frac{1}{v(\partial H)} \int_{H^c} P_t I_H \, dv \, .$$

where H is a hemisphere.

Conjecture: Suppose the Ricci curvature of M is bounded from below by $(d-1)k$, $k > 0$. Let μ, μ_k be the normalized Riemannian measures on M and M_k respectively and P_t, P_t^k the corresponding heat semi groups. Let A be any measurable subset of M and B a geodesic disk in M_k such that $\mu(A) = \mu_k(B)$, then for all $t > 0$:

$$\|P_t I_A\|_2^2 \leq \|P_t^k I_B\|_2^2,$$

If both sides coincide for some $t > 0$, then $M = M_k$ and $A = B$.

Stability results in convex geometry

Rolf Schneider

A stability result, as it is understood here, answers questions of the following type: If some condition enforcing uniqueness of a geometric object is satisfied "only upt o ϵ", can uniqueness be ascertained "up to $f(\epsilon)$", in a precise and explicit sense? The aim of this survey is a report on stability results in convex and dicscrete geometry from the last decade. The described stability results are grouped in four sections.

(1) **Inequalities.** Examples are stability versions of the isoperimetric inequality and the general Brunn-Minkowski theorem, a sharpening of the difference body inequality due to Böröczky jr, Gruber's recent results on the stability of the regular hexagonal pattern in the plane with respect to extremum properties.

(2) **Inverse problems.** Stability results by Campi, Goodey-Groemer, Bourgain-Lindenstrauss on the determination of convex bodies from projections, new extensions of these results, other inverse problems related to intertwining operators on the sphere, mean section and mean projection bodies.

(3) **Curvature conditions.** Almost umbilical surfaces, recent stability results of Kohlmann on the general Liebmann-Süss theorem involving curvature measures.

(4) **Geometric conditions.** Stabilized versions of some classical characterizations of ellipsoids, as the ones due to Brunn or Blaschke, obtained by Groemer, Gruber, and others.

Floating body, illumination body, and polytopal approximation

Carsten Schütt

The convex floating body K_t of a convex body K is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume t from K.

The illumination body K^t of a convex body K is

$$\{x \in \mathbb{R}^d | \text{vol}_d([x, K] \setminus K) \leq t\}$$

Theorem. *Let K be a convex body in $\mathbb{R}^d$. Then we have for every $t, 0 \leq t \leq \frac{1}{4}e^{-4}\mathrm{vol}_d(K)$, that there are $n \in \mathbb{N}$ with*

$$n \leq e^{16d}\frac{\mathrm{vol}_d(K\backslash K_t)}{t\,\mathrm{vol}_d(B_2^d)}$$

and a polytope P_n that has n vertices and such that

$$K_t \subset P_n \subset K.$$

Theorem. *Let K be a convex body in $\mathbb{R}^d$ such that*

$$\frac{1}{c_1}B_2^d \subset K \subset c_2 B_2^d.$$

Let $0 \leq t \leq (5c_1c_2)^{-d-1}\mathrm{vol}_d(K)$ and let $n \in \mathbb{N}$ with

$$(\frac{128}{7}\pi)^{\frac{d-1}{2}} \leq n \leq \frac{1}{32\,\mathrm{edt}}\mathrm{vol}_d(K^t\backslash K).$$

Then we have for every polyotpe P_n that contains K and has at most n $d-1$ dimensional faces

$$\mathrm{vol}_d(K^t\backslash K) \leq 10^7 d^2 (c_1c_2)^{2+\frac{1}{d-1}}\mathrm{vol}_d(P_n\backslash K).$$

Convex bodies instead of needles in Buffon's experiment

Marius Stoka

The idea of repeating Buffon's experiment using other objects instead of a needle is not new. Various special planar convex bodies have been investigated in the literature; we shall consider here the general case. Of course, to go beyond convexity makes no sense, because only the convex cover is relevant to our problem (if the object is supposed connected).

Among the particular cases alread treated in the literature we mention those of a circular disc, a sector thereof, a segment thereof, a (not necessarily symmetric) lens, and an ellipse.

Also, we follow the idea of considering not only one family of parallel lines but two, and of studying the hitting probability for the resulting lattice and the independence case of the two hitting events.

Consider a convex body (which means here a compact convex set) $K \subset \mathbb{R}$. Let $\mathcal{R}_a$ be a set of equidistant parallel lines in $\mathbb{R}$ (at distance a) and $\mathcal{R}_b$ another such set of lines (at distance b), the two directions making an angle $\alpha \in (0, \pi)$. The objects of our investigation are the events I_a, I_b that the random convex body K — more precisely the random congruent copy of K — meets some line in $\mathcal{R}_a, \mathcal{R}_b$ respectively.

Let $L(\phi)$ be the width of K in direction ϕ. We consider the following natural condition:

$$\max_{0\leq\phi<\pi} L(\phi) < \min\{a, b\}.$$

All cells of the lattice $\mathcal{R}_a \cup \mathcal{R}_b$ are congruent to a parallelogram Π.

Let $\mathcal{K}$ be the set of all convex bodies congruent with K and their centroids (just to make a choice) inside Π. We consider our convex bodies as uniformly distributed, in the sense that the centroid as a random variable is uniformly distributed in Π and the random variable ϕ (the rotation angle) is uniformly distributed in the interval $[0, 2\pi]$.

Various solutions of the isoperimetric problem

Anthony C. Thompson

In Minkowski planes (two-dimensional normed spaces) the isoperimetric problem is well-defined and the solution well-known. However, there is a variety of descriptions of that solution. If B denotes the unit ball then, up to homothety, the isoperimetrix $\mathbb{I}$ (the solution to the isoperimetric problem) may be described in one of the following three ways: up to homothety,

$$\mathbb{I} = I(B)^\circ = \Pi(B^\circ) = \Lambda(B^\circ)$$

where I denotes intersection body, Π denotes projection body and Λ denotes the mapping from X^* to X defined by $g(\Lambda f) := \det[f, g]$.

In higher dimensions, area ($n-1$ dimensional content) may be defined in a variety of ways. Depending on the definition one now has: $\mathbb{I} = I(B)^\circ$ (Busemann), $\mathbb{I} = \Pi(B^\circ)$ (Holmes-Thompson) or $\mathbb{I} = \Lambda(B^\circ)$ (Benson). Moreover, these bodies are now different although for certain symmetric balls B they have similar shapes. Note that Λ is now a mapping from $(X^*)^{n-1}$ to X defined by $g(\Lambda(f_1, f_2, \dots f_{n-1}) := \det[g, f_1, f_2, \dots f_{n-1}]$ and $\Lambda(B^\circ)$ is the image under Λ of all $n-1$-tuples from B°. The first two of these mappings are injective (on centrally symmetric balls) the third is not; it is suspected that the first two have only ellipsoids as fixed points ($n > 2$) the third has the rhombic dodecahedron as a fixed point.

One would like either to show that one definition is clearly preferable to the others or, failing that, to impose some structure on the variety.

Recent results in convex tomography

Aljoša Volčič

P.C. Hammer posed in 1961 the following question:

The X-ray problem. Suppose there is a convex hole C in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the lenght of a chord in C along an X-ray line. (No diffusion, please.) How many pictures must be taken to permit exact reconstruction of the body if:

a. The X-rays issue from a finite point source?

b. The X-rays are assumed parallel?

Note that the X-ray problem makes sense also if the finite point is taken in the interiorof K. A device which rotates around the solid so that all the X-rays pass through a given point $p \in \mathrm{int}K$ permits to determine the length of all the chords through p.

Let us denote by λ_i the i-dimensional Lebesgue measure. By $\mathcal{G}(n,i)$ we denote the set of all i-dimensional subspaces of $\mathbb{R}^n$. If $G \in \mathcal{G}(n,i)$ and $p \in \mathbb{R}^n$, let us denote by by $\mathcal{G}(n,i,p)$ the family of all the sets $F = p + G = \{x : x = p + y,\ y \in G\}$.

The i-section function at a point p of a convex body $K \subset \mathbb{R}^n$ is the function which assigns, to every $F \in \mathcal{G}(n,i,p)$, the measure $\lambda_i(K \cap F)$.

If $u \in S^{n-1}$ identifies a direction (a point at infinity p), and $\mathcal{G}(n,i,u)$ is the set of all the affine i-dimensional subspaces containing a translate of u, the i-section function of K in direction u is the function which assigns the value $\lambda_i(K \cap F)$ to every $F \in \mathcal{G}(n,i,u)$.

In particular, if $i = 1$, the i-section function reduces to the X-ray function of K at p.

A natural generalization of the Hammer's X-ray problem is the following:

The generalised Hammer X-ray problem. Suppose that K is a convex body in $\mathbb{R}^n$ and let $1 \leq i \leq n-1$. How many i-section functions must be taken in order to permit the exact reconstruction of K if:

a. The i-section functions are taken at finite points?

b. The i-section functions are taken at points at infinity?

Sample problem. Suppose $K \subset \mathbb{R}^3$ and let p_1, p_2 and p_3 be three non collinear points not belonging to K such that the plane P determined by them intersects the interior of K. Suppose we are given the 2-section functions of K at p_1, p_2 and p_3. Does this data determine K uniquely?

We prove that the answer to this question is affirmative.

We also consider the following further generalization:

The mixed Hammer X-ray problem. Suppose that K is a convex body in $\mathbb{R}^n$. How many points $p_1, p_2, \dots, p_k$ must be taken in order to permit the exact reconstruction of K if the i_j-section function is known, with $1 \leq i_j \leq n-1$, at points p_j, $1 \leq j \leq k$?

Sample problem. Let K be a three-dimensional convex body, let p_1 and p_2 be two interior points and suppose that we know

a) the lengths of all the chords of K through p_1 and

b) the areas of the intersections of K with all the planes through p_2.

Is K uniquely determined?

The question is open in all its generality. We only know a partial answer. Denote by $[a,b]$ the chord of K containing p_1 and p_2, with a, p_1, p_2, b in that order. Then K is uniquely determined if p_2 is closer to b than to a.

Translative integral geometry

Wolfgang Weil

Iterated translative intersection formulae for curvature measures and intrinsic volumes involve mixed measures and functionals. These can be introduced directly for polytopes and extended to arbitrary convex bodies by continuity. Various integral formulae for mixed measures and functionals are presented and a number of consequences are discussed. They include a kinematic formula for projection functions, a formula for mean section bodies and a translative formula for support functions. The investigations are

motivated by applications in Stochastic Geometry and are finally used to estimate the intensity of a non-isotropic Boolean model in $\mathbb{R}^3$.

Mixed volumes and packings

Jörg M. Wills

A family of convex bodies K_i, $i \in I$ in euclidean d-space $\mathbb{E}^d$ is called a packing, if

$$\operatorname{int}(K_i \cap K_j) = \emptyset \quad \text{for } i \neq j \qquad (*)$$

If I is finite, the packing is called finite, otherwise infinite. A basic property of packings is the density, and for important special cases, e.g. for lattice packings or bin packings there are appropriate and useful density definitions since long.

For general packings $(*)$ there is A. Thue's density (1892), which leads to a general theory in $\mathbb{E}^2$ (Rogers, Bambah, Zassenhaus, Groemer, Graham), but not in $\mathbb{E}^d$, $d \geq 3$. The introduction of a parameter in 1992/93 led to application of mixed volumes and to parametric density, which permits a joint theory of finite and infinite packings. We describe the simplest and most relevant case:

Let $K_i = K + c_i$, $i = 1, \ldots, n$ be a packing, $C_n = \{c_1, \ldots, c_n\}$ and $\rho > 0$. Let V denote the volume. Then

$$\delta(K, C_n, \rho) = nV(K) \,/\, V(\operatorname{conv} C_n + \rho K)$$

is the parametric density of the packing $C_n + K$ with respect to ρ. Here

$$V(\operatorname{conv} C_n + \rho K) = \sum_{i=0}^{d} \binom{d}{i} V_i[\operatorname{conv} C_n, K]\, \rho^i$$

is a polynomial in ρ, and its coefficients are the mixed volumes. The density of densest packing of n translates of K is given by

$$\delta(K, n, \rho) = \max\{\delta(K, C_n, \rho) \mid C_n + K \text{ packing}\}.$$

Any C_n with $\delta(K, C_n, \rho) = \delta(K, n, \rho)$ is denoted by $C_{n,\rho}(K)$, and $C_{n,\rho} + K$ is called a best or densest packing (of n translates of K with respect to ρ).

In this basic concept of packings the mixed volumes are the essential tool for a joint theory of finite and infinite packings. So this is an application of convex geometry to discrete geometry. The interaction between finite and (classical) infinite packings, finite analogues of classical theorems (Minkowski–Hlawka, Blichfeldt, Rogers, Rankin), applications to strange packing phenomena (sausages, sausage catastrophes), and to crystallography (Wulff shape, online packings, quasicrystals, microclusters) are shown.

Cut locus and ambiguous locus

Tudor Zamfirescu

The notion of a *cut locus* is well-known to differential geometers. The cut locus of a point x in a Riemannian manifold M is the set of those points $y \in M$ such that no shortest path from x to y can be extended (as a shortest path) beyond y.

The notion of an *ambiguous locus* belongs to Analysis. The points without a unique nearest point in a given closed set form its ambiguous locus.

Apparently quite distant from each other, the two notions share a common soul. Why this is so and some of their joint properties will be revealed in my talk.

Sobolev-type inequalities with best constants

Gaoyong Zhang

The Sobolev inequality in the Euclidean space $\mathbb{R}^n$ states that for any C^1 function $f(x)$ with compact support in $\mathbb{R}^n$ there is

$$\int_{\mathbb{R}^n} |\nabla f|\, dx \geq n\omega_n^{1/n} \|f\|_{\frac{n}{n-1}}, \tag{1}$$

where $|\nabla f|$ is the Euclidean norm of the gradient of f, $\|f\|_p$ is the usual L_p norm of f in $\mathbb{R}^n$, and ω_n is the volume of the unit ball in $\mathbb{R}^n$. The constant in the inequality is sharp. It is attained at the characteristic functions of balls.

It is known that the Sobolev inequality (1) is equivalent to the classical isoperimetric inequality. Let M be a compact domain in $\mathbb{R}^n$ with piecewise C^1 boundary. If M has volume $V(M)$ and surface area $S(M)$, then the isoperimetric inequality is

$$S(M) \geq n\omega_n^{1/n} V(M)^{\frac{n-1}{n}}, \tag{2}$$

with equality if and only if K is a ball. One can consider (1) as the analytic form of (2).

The aim of this paper is to establish Sobolev-type inequalities with best constants. Our first result is the following inequality.

Theorem 1. *If f is a C^1 function with compact support in $\mathbb{R}^n$, then*

$$\left(\int_{S^{n-1}} \left\| \frac{\partial f}{\partial u} \right\|_1^{-n} du \right)^{-1/n} \geq \frac{2\omega_{n-1}}{n^{1/n}\omega_n} \|f\|_{\frac{n}{n-1}}, \tag{3}$$

where $\frac{\partial f}{\partial u}$ is the partial derivative of f in direction u.

The constant in (3) is best. It is attained at the characteristic functions of ellipsoids. Applying the Hölder inequality and Fubini's theorem to the left-hand side of (3), one can easily see that inequality (3) is stronger than the Sobolev inequality (1). We prove inequality (3) by using an affine isoperimetric inequality, which is called the Petty projection inequality.

Our second result is a generalization of the Gagliardo-Nirenberg inequality.

Theorem 2. *Let $\{u_i\}_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$, and let $\{c_i\}_1^m$ be a sequence of positive numbers for which*

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n,$$

where $u_i \otimes u_i$ is the rank-1 orthogonal projection onto the span of u_i and I_n is the identity on $\mathbb{R}^n$. If f is a C^1 function with compact support in $\mathbb{R}^n$, then

$$\prod_{i=1}^{m} \left\| \frac{\partial f}{\partial u_i} \right\|_1^{\frac{c_i}{n}} \geq 2 \|f\|_{\frac{n}{n-1}}. \tag{4}$$

If $m = n$ and $\{u_i\}_1^n$ is an orthonormal basis of $\mathbb{R}^n$, then inequality (4) becomes the Gagliardo-Nirenberg inequality. Inequality (4) is proved by using an isoperimetric-type inequality of Keith Ball, which is a generalization of the Loomis-Whitney inequality.

Finally, we prove a generalization of the Sobolev inequality.

Theorem 3. *If h_K is the support function of a convex body K in $\mathbb{R}^n$, then, for $1 \leq p < n$ and for every C^1 function $f(x)$ with compact support,*

$$\|h_K(\nabla f)\|_p \geq c(n,p) V(K)^{\frac{1}{n}} \|f\|_q, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}, \tag{5}$$

where the constant $c(n,p)$ is best and is given by

$$c(n,p) = n^{\frac{1}{p}} \left(\frac{n-p}{p-1} \right)^{1-\frac{1}{p}} \left(\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(n)} \right)^{\frac{1}{n}}. \tag{6}$$

When K is the unit ball in $\mathbb{R}^n$, (5) was proved by Aubin and Talenti, and in particular, when $p = 1$, it further reduces to (1). When K is origin-symmetric and $p = 1$, (5) was shown by Gromov.

Berichterstatter: P.M.Gruber

ALESKER Semyon	semyon@math.tau.ac.il
BARANY Imre	barany@math-inst.hu
BÖRÖCZKY Karoly, Jr.	carlos@math-inst.hu
BUCHTA Christian	cbuchta@pop.tuwien.ac.at
COLESANTI Andrea	colesant@udini.math.unifi.it
FLORIAN August	august.florian@sbg.ac.at
GARDNER Richard J.	gardner@baker.math.wwu.edu
GLASAUER Stefan	glasauer@mathematik.uni-freiburg.de
GOODEY Paul	pgoodey@ou.edu
GORDON Yehoram	gordon@tx.technion.ac.il
GRINBERG Eric L.	grinberg@math.poly.edu grinberg @math.temple.edu
GRUBER Peter M.	pmgruber@pop.tuwien.ac.at
HEIL Erhard	heil@mathematik.tu-darmstadt.de
HUG Daniel	hug@sun1.mathematik.uni-freiburg.de
KAMENEV George	kamenev@ccas.ru
KLAIN Dan	klain@math.gatech.edu
KOLDOBSKY Alexander	koldobsk@math.utsa.edu
LARMAN David	d.larman@ucl.ac.uk
LEICHTWEISS Kurt	leichtw@magnix.mathematik.uni-stuttgart.de
LINDENSTRAUSS Joram	joram@math.huji.ac.il
LONKE Yossi	lini@math.huji.ac.il
LUDWIG Monika	mludwig@pop.tuwien.ac.at
MAKAI Endre	makai@math-inst.hu
MARTINI Horst	horst.martini@mathematik.tu-chemnitz.de
MCMULLEN Peter	p.mcmullen@ucl.ac.uk
MEYER Mathieu	meyer@math.univ-mlv.fr mam@ccr.jussieu.fr
MILMAN Vitali	vitali@math.tau.ac.il
PAJOR Alain	pajor@math.univ-mlv.fr
PELCZYNSKI Aleksander	olek@impan.impan.gov.pl
PERI Carla	cperi@mi.unicatt.it
PERISSINAKI Irini	irin@math.ucl.ac.uk
PRZESLAWSKI Krzysztof	k.przeslawski@im.pz.zgora.pl
REISNER Shlomo	reisner@mathcs2.haifa.ac.il
REITZNER Matthias	mreitzne@pop.tuwien.ac.at
SANGWINE-YAGER Jane	sangwine@stmarys-ca.edu
SCHLUMPRECHT Thomas	schlump@math.tamu.edu
SCHMUCKENSCHLÄGER Michael	100277.361@compuserve.com
SCHNEIDER Rolf	rschnei@ruf.uni-freiburg.de
SCHÜTT Carsten	carsten.schuett@compuserve.com
STOKA Marius	stoka@dip.mat.utorin.it
THOMPSON Tony	tony.thompson@dal.ca
VOLCIC Aljosa	volcic@univ.trieste.it
WEIL Wolfgang	weil@math.uni-karlsruhe.de
WERNER Elisabeth	emw2@po.cwru.edu
WILLS Jörg M.	wills@hrz.uni-siegen.d400.de
ZAMFIRESCU Tudor	zamfi@mathematik.uni-dortmund.de
ZHANG Gaoyong	gzhang@math.poly.edu

Tagungsteilnehmer

Dr. Semyon Alesker
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, P.O. Box 39040

Tel Aviv 69978
ISRAEL

Prof.Dr. Imre Barany
Mathematical Institute of the
Hungarian Academy of Sciences
P.O. Box 127
Realtanoda u. 13-15

H-1364 Budapest

Prof.Dr. Karoly Böröczky Jr.
Mathematical Institute of the
Hungarian Academy of Sciences
P.O. Box 127
Realtanoda u. 13-15

H-1364 Budapest

Dr. Christian Buchta
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstr. 8-10

A-1040 Wien

Dr. Andrea Colesanti
Dipt. di Matematica "U.Dini"
Universita di Firenze
Viale Morgagni 67/A

I-50134 Firenze

Prof.Dr. August Florian
Institut für Mathematik
Universität Salzburg
Hellbrunnerstr. 34

A-5020 Salzburg

Prof.Dr. Richard J. Gardner
Dept. of Mathematics
Western Washington University

Bellingham ,WA 98 225-9063
USA

Dr. Stefan Glasauer
Mathematisches Institut
Universität Freiburg
Eckerstr. 1

79104 Freiburg

Prof.Dr. Paul R. Goodey
Dept. of Mathematics
University of Oklahoma
601 Elm Avenue

Norman , OK 73019-0315
USA

Prof.Dr. Yehoram Gordon
Department of Mathematics
Technion
Israel Institute of Technology

Haifa 32000
ISRAEL

Prof.Dr. Eric L. Grinberg
Dept. of Mathematics
Polytechnic University
333 Jay Street

Brooklyn , NY 11291
USA

Prof.Dr. Peter M. Gruber
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstr. 8-10

A-1040 Wien

Prof.Dr. Erhard Heil
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7

64289 Darmstadt

Daniel Hug
Mathematisches Institut
Universität Freiburg
Eckerstr. 1

79104 Freiburg

Prof.Dr. George K. Kamenev
Computing Center
Russian Academy of Sciences
Vavilova 40

117967 Moscow GSP-1
RUSSIA

Prof.Dr. Daniel Klain
Department of Mathematics
Georgia Institute of Technology

Atlanta , GA 30332
USA

Prof.Dr. Alexander Koldobsky
Division of Mathematics
University of Texas at San Antonio

San Antonio , TX 78249
USA

Prof.Dr. David G. Larman
Department of Mathematics
University College London
Gower Street

GB-London , WC1E 6BT

Prof.Dr. Kurt Leichtweiß
Mathematisches Institut B
Universität Stuttgart

70550 Stuttgart

Prof.Dr. Joram Lindenstrauss
The Israel Academy of Sciences
Albert Einstein Square
P.O.Box 4040

91040 Jerusalem
ISRAEL

Yossi Lonke
Institute of Mathematics
The Hebrew University
Givat-Ram

91904 Jerusalem
ISRAEL

Prof.Dr. Mathieu Meyer
Departement de Mathematiques
Universite de Marne la Vallee
2 rue de la Butte Verte

F-93160 Noisy-le-Grand Cedex

Dr. Monika Ludwig
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstr. 8-10

A-1040 Wien

Prof.Dr. Vitali D. Milman
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, P.O. Box 39040

Tel Aviv 69978
ISRAEL

Prof.Dr. Endre Makai
Mathematical Institute of the
Hungarian Academy of Sciences
P.O. Box 127
Realtanoda u. 13-15

H-1364 Budapest

Prof.Dr. Alain Pajor
Departement de Mathematiques
Universite de Marne la Vallee
2 rue de la Butte Verte

F-93160 Noisy-le-Grand Cedex

Prof.Dr. Horst Martini
Fakultät für Mathematik
Technische Universität
Chemnitz-Zwickau
Reichenhainer Str. 41

09126 Chemnitz

Prof.Dr. Aleksander Pelczynski
Institute of Mathematics of the
Polish Academy of Sciences
P.O. Box 137
ul. Sniadeckich 8

00-950 Warszawa
POLAND

Prof.Dr. Peter McMullen
Department of Mathematics
University College London
Gower Street

GB-London , WC1E 6BT

Prof.Dr. Carla Peri
Istituto di Matematica generale,
finanziaria ed economica
Universita Cattolica d.Sacro Cuore
Largo Gemelli 1

I-20123 Milano

Irini Perissinaki
Department of Mathematics
University College London
Gower Street

GB-London , WC1E 6BT

Prof.Dr. Jane R. Sangwine-Yager
Department of Mathematics
Saint Mary's College of California
P.O.Box 3517

Moraga , CA 94575
USA

Dr. Krzysztof Przeslawski
Instytut Matematyki
Politechnika Zielonogorska
ul. Podgorna 50

65-246 Zielona Gora
POLAND

Prof.Dr. Thomas Schlumprecht
Department of Mathematics
Texas A & M University

College Station , TX 77843-3368
USA

Prof.Dr. Shlomo Reisner
Department of Mathematics and
Computer Science
University of Haifa

31905 Haifa
ISRAEL

Prof.Dr. Michael Schmuckenschläger
Institut für Mathematik
Universität Linz
Altenbergerstr. 69

A-4040 Linz

Dr. Matthias Reitzner
Abteilung für Analysis
Technische Universität Wien
Wiedner Hauptstr. 8-10

A-1040 Wien

Prof.Dr. Rolf Schneider
Mathematisches Institut
Universität Freiburg
Eckerstr. 1

79104 Freiburg

Prof.Dr. Idjad Kh. Sabitov
Dept. of Mathematics and Mechanics
Moscow State University
Lenin Hills

119899 Moscow
RUSSIA

Dr. Carsten Schütt
Mathematisches Seminar
Universität Kiel

24098 Kiel

Prof.Dr. Marius Stoka
Ist. Geometria
Universita di Torino
Via Principio Amadeo 8

I-10123 Torino

Prof.Dr. Jörg M. Wills
Fachbereich 6 Mathematik
Universität Siegen

57068 Siegen

Prof.Dr. Tony Thompson
Department of Mathematics
Dalhousie University

Halifax , N.S. B3H 3J5
CANADA

Prof.Dr. Tudor Zamfirescu
Fachbereich Mathematik
Universität Dortmund

44221 Dortmund

Prof.Dr. Aljosa Volcic
Dipartimento di Scienze Matematiche
Universita di Trieste
Piazzale Europa 1

I-34100 Trieste (TS)

Prof.Dr. Gaoyong Zhang
Dept. of Applied Math. and Physi
Polytechnic University
6 Metrotech Center

Brooklyn , NY 11201
USA

Prof.Dr. Wolfgang Weil
Mathematisches Institut II
Universität Karlsruhe

76128 Karlsruhe

Dr. Elisabeth Werner
Dept. of Mathematics and Statistics
Case Western Reserve University
10900 Euclid Avenue

Cleveland , OH 44106-7058
USA

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