MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 15/1998

Numerical Methods for Singular Perturbations

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The meeting was organized by P. Hemker (Amsterdam), H.-G. Roos (Dresden) and M. Stynes (Cork). A significant number of differential equations in the applied sciences — for example in fluid dynamics — are singularly perturbed, but unfortunately standard numerical methods often fail to provide satisfactory solutions for such problems. Since the early 1970s, many different numerical techniques have evolved with the aim of overcoming the drawbacks of standard methods. This proliferation of ideas galvanized the organizers into gathering together various researchers at a first Oberwolfach meeting in January 1995.

In recent years, the development of numerical techniques for singular perturbation problems has become one of the focal points of numerical analysis. Three new books recently appeared on this subject, and several new groups of numerical analysts have begun research into singular perturbation problems. In fact when organizing the 1998 workshop, the organizers were unable to accommodate all colleagues who wished to attend. At the meeting, 49 participants from 15 countries discussed their work and 29 lectures were presented. It is worth noting that more than 50% of the participants did not attend the previous Oberwolfach meeting in 1995 and that most of the participants wished to present a lecture on their research.

The aims of the meeting were:

1. The dissemination of up-to-date information about current research and open questions in the numerical and asymptotic analysis of singular perturbation problems;

2. The comparison and evaluation of several approaches and methods for the solution of such problems;

3. An examination of promising future research directions in this rapidly-changing area.

The lectures presented covered key aspects of the discretization and numerical analysis of singular perturbation problems: exponential fitting, layer-adapted meshes, stabilization techniques, etc. The variety of techniques represented (including, for example, iterative methods for the solution of the discrete linear problems generated), meant that aim (1) above was succesfully achieved. Towards aim (3), the organizers devoted one day of the meeting to adaptive mesh approaches, because the analysis of adaptive procedures for singularly perturbed problems is important but is still in its infancy.

To achieve aims (2) and (3), it was very important to schedule ample discussion time both after lectures and for long periods during the day. Researchers with differing points of view and varied approaches could then meet and profit from each other's experience. The warm and stimulating atmosphere of the Forschungsinstitut was of course a positive factor in fostering such mixing. The week's workshop laid the foundation for several future collaborations between various groups and individuals, which testifies to the attainment of aims (2) and (3).

All participants agreed that the workshop was extremely beneficial and that contact between new groups should be established and maintained in the years to come. Consequently, the organizers hope to organize a further meeting on this topic three years from now (as we observed in the period since the last meeting in 1995, three years allows adequate time for the development of new research, to the extent that another workshop then will be desirable and successful).

The alphabetical list below of contributions could be classified under the following headings: exponential fitted uniformly convergent methods, layer-adapted meshes for convection-diffusion problems, adaptive procedures and a posteriori error estimation, anisotropic problems, stabilisation methods for finite element and finite volume approaches, iterative methods, $h-p$ finite element methods, numerical methods for the Navier-Stokes equations and other problems in fluid mechanics, numerical methods for the drift-diffusion equations in semiconductor device modelling.

Dank einer Unterstützung im Rahmen des EU-Programmes TMR (Training and Mobility of Researchers) konnten zusätzlich einige jüngere Mathematiker zu der Tagung eingeladen werden. Dies ist einerseits eine hervorragende Förderung des wissenschaftlichen Nachwuchses und gibt andererseits den etablierten Kollegen die Gelegenheit, besonders begabte junge Mathematiker kennenzulernen.

Abstracts

Mark Ainsworth: *A Posteriori Error Estimates for Singularly Perturbed Reaction Diffusion Problems.*
A posteriori error estimates are considered for the reaction diffusion problem

$$-\Delta u + \kappa^2 u = f$$

based on solving local Neumann problems. The main difficulty is in selecting the boundary fluxes for the local problems. The classical method of Bank-Weiser and Demkowicz et al. is shown to be non-robust in the limit $h \to 0$ (or equivalently $\kappa \to 0$). The reason for the non-robustness lies in the fact that the fluxes do not satisfy the compatibility condition for the limiting pure Neumann problem for the Laplacian. The Equilibrated Residual Method (ERM) proposed by the author with J.T. Oden seeks to select the fluxes so that the compatibility condition is satisfied. The resulting estimator gives an upper bound on the true error that is robust in the limiting case $\kappa \to 0$. However, the estimator is not robust in the singularly perturbed limit and will over-estimate by at most $1 + \mathcal{O}(\sqrt{\kappa h})$. Counterexamples show this estimate to be sharp. A new modification of the ERM due to the author and I. Babuska is described and shown to lead to an estimator that bounds the true error and that is robust in all limiting cases $\kappa \to \infty$ and κ or $h \to 0$.

Vladimir B. Andreev: *Application of condensing grids for solving singularly perturbed problems.*
The difference schemes on "smoothly" condensing grids are considered for two singularly perturbed problems. In the first problem it is the equation — the one-dimensional steady-state convection-diffusion equation — that is singularly perturbed. In the second problem it is the domain where the equation is defined that is singularly perturbed — the second-order ordinary differential equation with regular singularity is considered on a segment disposed at a small distance from the singular point. For both problems condensing grids depending on a small parameter ε are constructed, use of which makes it possible to obtain approximate solutions with L_∞^h-accuracy $O(N^{-2})$ uniformly with respect to ε, where N is the meshpoints number.

Lutz Angermann: *Numerical solution of convection-dominated anisotropic diffusion equations.*
The proposed method is designed to handle the case of a full-tensor diffusion coefficient. It is based on an additive decomposition of the differential operator and on a fitted discretization of the resulting components. For standard situations, the derived stability and error estimates in the energy norm qualitatively coincide with wellknown estimates. In the case of small diffusion, a uniform error estimate with reduced order is obtained.

Carmelo Clavero:* *Finite difference methods of high order on Shishkin meshes.*
In this talk we present some numerical methods to solve one dimensional convection-diffusion problems with dominated convection term. These methods are finite difference schemes of classical type constructed on a piecewise Shishkin mesh, which condense the nodes in the boundary layer zone. To obtain the schemes we will impose that the local error, associated to the method, be zero for a polynomial basis. When we consider the polynomial of degree less or equal two, the scheme use a combination of the values of the second term of differential equation in two consecutives points of the mesh (x_{j-1} and x_j, $j = 1, \ldots, N-1$, when we discretize in the point x_j, where $N+1$ is the number of points in the mesh). So, we can deduce a method of order $N^{-2} \log^2 N$ uniformly with respect the diffusion parameter ε. If the polynomial have degree less or equal three, a similar idea permit us to construct a method having a uniform order of convergence $N^{-3} \log^3 N$. Some numerical examples confirming the theoretical results are shown for both schemes.

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*joint work with J. L. Gracia

Alan Craig:* *Exact difference formulas for linear differential operators.*
A difference approximation to a linear differential operator can, in many cases be represented as an integral. If the integral is evaluated exactly one obtains an exact method for obtaining a table of values of the solution to an ordinary differential equation. Many well-known methods for ODEs, including upwinded finite element methods can be derived by approximating the integral by a quadrature formula. Such exact representations can in principle be obtained for any nth order linear differential operator and any set of n+1 nodes under mild technical conditions. Our construction involves a generalised B-spline that, instead of being peicewise polynomial is a piecewise solution of the homogeneous adjoint equations.

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*joint work with Dirk Laurie

Manfred Dobrowolski: *On a posteriori error estimators on anisotropic meshes.*
We consider the standard piecewise linear finite element approximation of Poisson's equation. The usual residual based a posteriori error estimator is derived and it is shown that the true error will in general be overestimated if the mesh becomes anisotropic. Based on a paper of Bank and Weiser, a nonlocal error estimator is constructed which avoids the drawback with the standard estimators. Moreover, the analysis of the nonlocal estimator indicates that local a posteriori error estimation in energy is impossible on anisotropic meshes.

Willy Dörfler: *Uniformly Convergent Finite-Element Methods for Singularly Perturbed Convection-Diffusion Equations.*

We consider the singularly perturbed boundary value problem for bounded $\Omega \subset I\!R^n$. We prove uniform a priori error estimates which allow to prove stability of the weak form on a suitable pair of Banach spaces $X \times Y$. This also shows uniform quasi-optimality for conforming space approximations $X_h \times Y_h$ in case of discrete stability.

In one space dimension ($\Omega = [0, 1]$) we construct X_h and Y_h by means of exponentially fitted basis functions and prove stability and the optimal approximation property of this discretisation. Moreover, we show that a residual error estimator gives uniform efficiency and reliability in L^p-norms. Our theory covers not only the case of strictly positive b, but also problems with turning points and vanishing b are included.

In two space dimension ($\Omega = [0, 1]^2$) we consider tensor product exponentially fitted basis functions on uniform rectangular grids. We show uniform error estimates in L^p-norms for 3 case studies: b has two strictly positive components, $b = [1, y - 1/2]$, and $b = [1, 0]$, $c > 0$.

Michael Eiermann:* *On Some Recurrent Theorems Concerning Krylov Subspace Methods.*

The recent development of Krylov subspace methods for the solution of operator equations has shown that two basic construction principles, the orthogonal residual (OR) and minimal residual (MR) approaches, underlie the most commonly used algorithms. It is shown that these can both be formulated as techniques for solving an approximation problem on a sequence of nested subspaces of a Hilbert space, a problem not necessarily related to an operator equation. Most of the familiar Krylov subspace algorithms result when these subspaces form a Krylov sequence. The well-known relations among the iterates and residuals of OR/MR pairs are shown to hold also in this rather general setting. We further show that a common error analysis for these methods involving the canonical angles between subspaces allows many of the recently developed error bounds to be derived in a simple manner. We illustrate these results by discretized convection-diffusion problems.

*joint work with Oliver Ernst

Joseph E. Flaherty: *Adaptive and Parallel hp-Refinement Methods for Conservation Laws.*

We describe a discontinuous Galerkin procedure for solving hyperbolic systems of conservation laws. The procedure is adaptive and combines mesh refinement/coarsening (h-refinement), method order variation (p-refinement), and mesh motion (r-refinement). New limiting procedures preserve monotonicity through discontinuities and a high order of accuracy in smooth regions. A posteriori error estimation procedures utilize superconvergence at Radau points. Three-dimensional meshes are automatically generated from CAD descriptions of the domain. A parallel meshdata base distributes data across the memories of cooperating processors. Software tools maintain a balanced computation and migrate data between processors. Applications involving compressible flows are presented.

David F. Griffith:* *The "No Boundary Condition" Outflow Boundary Condition.*
Many situations in fluid dynamics are posed on unbounded domains and, when solving such problems numerically, it becomes necessary to truncate the domain. Some boundary condition must be devised for the artifical (outflow) boundary that will not seriously affect the solution in the interior.
A new form of boundary condition for this purpose was introduced by Papanastasiou and colleagues (1992) specifically for finite element methods. We shall describe its properties in the context of convection dominated diffusion problems. The boundary condition is unusual because, in the weak formulation, it appears to be impose no boundary condition at all at the outflow, thus making the problem ill-posed.
Reference: D. F. Griffiths, The "No Boundary Condition" Outflow Boundary Condition, Inter. J. Numer. Methods Fluids. Vol. 24 (1997) 393-411.

*joint work with Luwai Wazzan

Raphaèle Herbin: *Finite volume schemes for convection diffusion reaction equations with non-admissible refinement of the mesh.*
The topic of this presentation is the discretization of convection diffusion reaction by the finite volume method in one or several space dimensions on general unstructured meshes. These grids may consist of polygonal control volumes which are not necessarily ordered in a cartesian grid, but need to be such that there exists a family of points associated with the grid cells such that the line segments joining the points associated with two neighbouring grid cells intersect the edge between these two cells at a right angle.
Error estimates for finite volume methods have recently been proved by finite difference or direct finite volume approaches or by finite element approaches.
We consider here the classical cell centered finite volume scheme, with an upwind choice for the convection flux. Assuming C^2 or H^2 regularity of the solution to the equation, error estimates of order h where h is the "size" of the mesh are proven, when a discretization for the flux over an interface by a 1st order finite difference scheme. Discrete Poincaré-type inequalities are used for this estimate. We also show that if the mesh is locally refined using some atypical nodes, then the error estimate is of order $h + m(A)$ where A is the area of the cells with atypical nodes. In typcial cases, the area of these cells is of order h, and the error estimate is tehrefore the same.
The convergence of the scheme without any assumption on the regularity of the exact solution may also be proven using some compactness results which are shown to hold for the approximate solutions. This is in particular the case for nonlinear problems.

Paul Houston: *Local Mesh Design for the Numerical Solution of Hyperbolic and Nearly-Hyperbolic Problems.*
We consider the design and implementation of an adaptive mesh refinement algorithm for the numerical solution of hyperbolic and nearly–hyperbolic problems. In particular, emphasis will be given to the design of a local error indicator to identify regions in the computational domain where the error is locally large. In practice, one of the most common approaches is to design the mesh according to the size of the local residual of the underlying partial differential operator. Clearly, the success of this approach greatly depends on the relationship between the size of the *local* error and the size of the *local* residual. We show that for hyperbolic problems, the local residual on an element κ only controls a portion of the local error on κ, referred to as the *cell error*. Moreover, due to error propagation effects, the error on an element κ is not only influenced by the size of the residual on κ, but on the size of the residual calculated on the *domain of dependence* of the element κ. Thus, an adaptive mesh refinement algorithm driven by residual–based error indicators will only refine elements with large cell error. Moreover, in regions of the computational domain where error propagation effects are important, the local residual may give a very *poor* estimate of the local error. Consequently, localised structures in the solution may not be accurately resolved, even when global control of the error has been achieved. To overcome these difficulties we design a local error indicator based on solving the partial differential equation for the error with the residual as the right–hand side function. Preliminary numerical experiments will be presented to demonstrate the performance of this error indicator on both uniformly and adaptively refined meshes.

Juan Carlos Jorge:* *Uniformly Convergent Scheme on a Nonuniform Mesh for Convection-Diffusion Parabolic Problems.*
A general framework for the construction and analysis of numerical methods for evolutionary singular perturbation problems is proposed. This consists of splitting the totally discrete scheme in two discretization stages. In first one, only the time variable is discretized by means of a convenient implicit method; secondly, we propose the use of standard finite difference methods on special nonuniform meshes for discretizing the spatial variables. Using this technique, a uniformly convergent scheme is deduced and analyzed for a onedimensional evolutionary convection-diffusion problem. Some numerical experiences are presented, confirming the good approximation properties predicted by the theoretical results.

*joint work with C. Clavero and F. Lisbona

Bruce Kellogg:* *n-widths and a self-adjoint singular perturbation problem.*
We consider the problem (1): $-\epsilon\Delta u + u = f$ in Ω, $u = 0$ on $\Gamma =$ the boundary of Ω, where Ω is a bounded domain in R^2 with smooth boundary. Let $E : f \mapsto u$ denote the solution operator of (1). Let B_s be the unit ball in $H^s(\Omega)$. Let $d_n = d_n(EB_s, L_2(\Omega))$ denote the Kolmogorov n-width of the set EB_s in $L_2(\Omega)$. We show that there are constants C_1 and C_2, independent of ϵ, such that

$$\frac{C_1 n^{-1-s/2}}{1+\epsilon n} \leq d_n \leq \frac{C_2 n^{-1-s/2}}{1+\epsilon n}. \tag{2}$$

The upper bound in (2) means that there is a subspace X_n of dimension n such that if u is the solution of (1), there is a $u_n \in X_n$ such that

$$\|u - u_n\|_{L_2(\Omega)} \leq \frac{Cn^{-1-s/2}}{1+\epsilon n}\|f\|_{H^s(\Omega)}.$$

The subspace X_n is constructed in the form $X_n = V_{n_i} + W_{n_b}$, where the functions in V_{n_i} approximate the smooth part of u and the functions in W_{n_b} approximate the boundary layer part of u. The dimensions n_i and n_b in this optimal subspace satisfy $n_b \sim \sqrt{n_i}$.

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*joint work with Martin Stynes

Rajco Lazarov:* *Stream-Line Diffusion Least-Squares Mixed Finite Element Method for Convection-Diffusion Problems.*
We discuss the least-squares finite element approximations of nonsymmetric and/or indefinite problems as stabilization of the classical Galerkin method by using an inner product in the minus one Sobolev norm. This concept has been developed by Bramble, Lazarov and Pasciak for both second order elliptic equations and for their mixed. We apply a simpler version of the least-squares method, which involves only L^2-norms, to the modified mixed system for convection dominated diffusion equation. This modification allows us to derive an error estimate in a norm which together with the standard energy norm $\epsilon\|\nabla u\|^2 + \|u\|^2$ contains the norm of the stream-line derivative $\delta\|\beta \cdot \nabla u\|^2$. The small parameter δ is proportional to the mesh size and can be chosen locally. The derived finite element method is stable and yields symmetric and positive definite matrix. Finally, we discuss some numerical experiments which confirm the uniform stability of the method and its accuracy.

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*joint work with L. Tobiska, and P. Vassilevski

William Layton: *Adaptive finite element methods for highly convection dominated problems.*
This talk will describe research done in collaboration with V. Ervin and J. Maubach on solving convection dominated, convection-diffusion problems and high Reynolds number flow problems in an adaptive fashion. This includes algorithms, error estimation, grid adaptation and design and some (so far) unexplained difficulties which occurred in practical computations. Our approach is based on: defect correction discretizations using higher

order elements, subgrid scale modelling, a posteriousing higher order elements, subgrid scale modelling, a posteriori error estimation and adaptation of conforming meshes using a bisection type local refinement algorithm.

Torsten Linß:* *Convection-diffusion problems, Shishkin meshes, SDFEM.*
We consider linear convection-diffusion problems on the unit square. We give sufficient conditions that guarantee the existence of a Shishkin-type decomposition of the exact solution. This decomposition can be used for the analysis of a number of numerical methods on Shishkin meshes. We give a survey of convergence results for numerical methods on Shishkin meshes (simple upwind FDM and (bi)linear Galerkin FEM).
Finally we derive a variant of the streamline diffusion finite element method using linear elements on Shishkin meshes. We prove local and global estimates in the L_∞ norm and in the L_2 norm.

*joint work with Martin Stynes

Gerd Lube:* *On the reliability of a non-overlapping domain decomposition method for elliptic problems.*
The application of a non-overlapping domain decomposition method (DDM) to the solution of a stabilized finite element method for elliptic boundary value problems - with emphasis on the singularly perturbed case - is considered. A properly chosen Robin type transmission condition at the interface yields in numerical experiments for the coarse granular case a linear convergence rate which is independent on the fine mesh width h and is even more favourable in the singularly perturbed case. A main problem of this iteration-by-subdomains method is a lacking numerical analysis for the discrete case, hence an appropriate stopping criterion is not available.
Here we derive an a-posteriori error estimate which bounds the error on the subdomains by the interface trace of the subdomain solutions. As a by-product, some foundation is given to the design of the interface transmission condition. Numerical results support the theoretical results. Furthermore we combine the new result and recent results on a-posteriori estimates for singularly perturbed problems (without DDM) to obtain an a-posteriori estimate for the discrete DDM solutions.
The extension of the proposed approach to the linearized incompressible Navier-Stokes problem is in progress.

*joint work with F. C. Otto

John A. Mackenzie:* *Uniform convergence of numerical approximations of singularly perturbed boundary value problems using grid equidistribution.*
We examine the convergence properties of finite difference and finite element approximations of model second-order, singularly perturbed linear boundary value problems using adaptive grids. The grids are based on the equidistribution of a positive monitor function which is a linear combination of a constant floor and a power of the second derivative of the

numerical solution. Analysis shows how the monitor function can be chosen to ensure that the accuracy of the numerical approximation is insensitive to the size of the singular perturbation parameter. Numerical results will be given for convection and reaction-diffusion problems in one and two dimensions.

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*joint work with George Beckett

Jens Melenk & Christoph Schwab: *Robust exponential convergence of hp-SDFEM for convection dominated problems.*
For $0 < \varepsilon \leq 1$ and $\Omega = (-1,1)$, consider the boundary value problem

$$u_\varepsilon \in H_0^1(\Omega): \quad B_\varepsilon(u_\varepsilon, v) = F(v) \quad \forall v \in H_0^1(\Omega)$$

where $B_\varepsilon(u,v) := \int_{-1}^{1}(\varepsilon u'v' + au'v + buv)dx$, $F(v) = \int_{-1}^{1} fv dx$ and $f, a(x), b(x)$ are analytic in $\overline{\Omega}$ and satisfy $a(x) \geq \underline{a} > 0$, $b(x) - \frac{1}{2}\, a' \geq \gamma > 0$. The solution u_ε has a boundary layer at $x = 1$.
For $\mathcal{T} = \{I_i : 1 \leq i \leq N\}$, $I_i = (x_{i-1}, x_i)$, $-1 = x_0 < x_1 < \ldots < x_N = 1$ a mesh in Ω, define $S_0^p(\mathcal{T}) = \{u \in H_0^1(\Omega) : u|_I \in P_p\ I \in \tau\}$. The hp-SDFEM is given by:

$$u_\varepsilon^{SD} \in S_0^p(\tau) : B_{SD}(u_\varepsilon^{SD}, v) = F_{SD}(v) \quad \forall v \in S_0^p(\mathcal{T})\,, \tag{1}$$

$$\text{where} \quad B_{SD}(u,v) = B_\varepsilon(u,v) + \sum_{i=1}^{N} \rho_i h_i\, p^{-2} \int_{I_i} av'\, L_\varepsilon u\, dx\,,$$

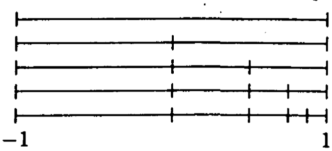
$$\text{and} \quad F_{SD}(v) = F(v) + \sum_{i}^{N} \rho_i h_i\, p^{-2} \int_{I_i} av'\, f\, dx$$

with certain positive ρ_i independent of h_i, p_i, ε and $h_i := |I_i|$.
Theorem 1 *Let $\mathcal{T} = \{(-1, 1-\varepsilon p), (1-\varepsilon p, 1)\}$. There exist C, k independent of ε, p such that*

$$\begin{aligned} \varepsilon\, |u_\varepsilon - u_\varepsilon^{SD}|_{1,2,\Omega} + \|u_\varepsilon - u_\varepsilon^{SD}\|_{0,2,\Omega} &\leq C \exp(-kp) \\ \|u_\varepsilon - u_\varepsilon^{SD}\|_{0,\infty,\Omega} &\leq C \exp(-kp) \end{aligned} \tag{2}$$

Theorem 2 *For $p = 1, 2, 3, \ldots$ consider the geometric mesh sequence $\{\mathcal{T}_p\}_p$ given by*

$\{\mathcal{T}_p\}$: $p = 1$, $p = 2$, $p = 3$, $p = 4$, $p = 5$, $\vdots$



Then:

i) if $p > c_0\,|\log \varepsilon|$, the estimate (2) hold with $C(c_0)$,

ii) for any fixed $\Omega_0 \subset\subset \Omega$ exist C, $\tilde{k} > 0$ independent of ε s.t. for all p

$$\|u_\varepsilon - u_\varepsilon^{SD}\|_{1,2,\Omega_0} \leq C \exp(-\tilde{k}p)\,.$$

Stefano Micheletti:* *A new Galerkin Framework for the Drift-Diffusion Equations in Semiconductors.*
In this lecture we propose a new Galerkin formulation for dealing with the drift-diffusion equations in semiconductor device modelling. The approach is based on the choice of a suitable weighted inner product that allows to symmetrize the convection-diffusion operator. The issues of the scaling and of the dependence of the induced norm on the size of the electric potential are also addressed. Several exponentially fitted finite element methods are then considered and analyzed. In the one-dimensional case, two new second-order accurate schemes are devised and successfully validated and compared with a Petrov-Galerkin first-order method. A family of monotone schemes is then proposed for the discretization of the two dimensional problem. Three methods, differing in the choice of the average of the exponentially varying diffusion coefficient, are average of the exponentially varying diffusion coefficient, are examined. A linear convergence theorem in the new energy norm is established for two of the schemes, while the remaining one is shown to be equivalent with the Scharfetter-Gummel Box method. Its remarkable stability and accuracy properties are finally demonstrated in the study of several highly convection-dominated test problems.

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*joint work with Emilio Gatti and Riccardo Sacco

Mariana Nikolova:* *Adaptive Refinement Procedure for Singularly Perturbed Convection-Diffusion Problems in 2D.*
A finite difference method is presented for singularly perturbed convection-diffusion problems with discretization error estimates of nearly second order, which hold uniformly in the singular perturbation parameter ε. A theorem proving this fact is given. The method is based on a defect-correction technique and special adaptively graded and patched meshes. In a standard adaptive refinement method certain slave nodes appear where the approximation is done by interpolating the values of the approximate solution at adjacent nodes. This deteriorates the accuracy of truncation error. In order to avoid the slave points we change the stencil at the interface points from a cross to a skew one. The efficiency of this technique is illustrated by numerical experiments in 2D.

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*joint work with Owe Axelsson

Robert E. O'Malley, Jr.: *Naive Singular Perturbations Theory.*
The talk demonstrates, via extremely simple examples, the shocks, spikes, and initial layers that arise in solving certain singularly perturbed initial value problems. As examples from stability theory, they are basic to many asymptotic solution techniques for differential equations. First, we note that limiting solutions of linear equations $\epsilon\dot{x} = -a(t)x$ on $t \geq 0$ are specified by the zeros of $A(t) = \int_0^t a(s)ds$, rather than the turning points where $a(t)$ becomes zero. Further, solutions to the solvable equations $\epsilon\dot{x} = -a(t)x - b(t)x^k$ for $k = 1, 2,$ or 3 can feature *canards*, where the trivial limit continues to apply after it becomes linearly

unstable. Solutions of the separable equation $\epsilon\dot{x} = a(t)c(x)$ likewise involve switchings between the zeros of $c(x)$ above and below $x(0)$, if they exist, at zeros of $A(t)$. Finally, we note that limiting solutions of many other problems follow by using special functions and their asymptotic expansions. For example, solutions of $\epsilon\dot{x} = t^2(t^2 - x^2)$ can be given in terms of the Bessel functions $K_j(t^4/4\epsilon)$ and $I_j(t^4/4\epsilon)$ for $j = 3/8$ and $-5/8$.

Thomas F. Russell: *Eulerian-Lagrangian Localized Adjoint Methods (ELLAM) for Transient Convection-Diffusion Problems.*
A prototypical transient convection-diffusion problem is $u_t + vu_x - Du_{xx} = 0$, with appropriate initial and boundary conditions. The convection-dominated case can be viewed as a singularly perturbed problem with respect to the parameter $\epsilon = Pe^{-1} = D/vL$, where L is a characteristic length of the system and Pe is the Péclet number. The solution exhibits traveling fronts of width $O(\sqrt{\epsilon})$. Eulerian methods, such as centered finite differences, require a mesh of size $O(\epsilon)$ to avoid oscillations, and they have large time-truncation errors when a front passes by. Eulerian-Lagrangian methods can avoid oscillations with mesh $O(\sqrt{\epsilon})$, as one would wish, and they reduce time-truncation errors by following the flow. Earlier Eulerian-Lagrangian methods did not conserve mass and had difficulties in formulating boundary conditions. Eulerian-Lagrangian localized adjoint methods (ELLAM) overcome these drawbacks with a space-time finite element framework that represents conservation and boundary conditions systematically with integrals. The test functions are oriented along Lagrangian streamlines. The presentation summarized this background and some further developments, including theoretical analysis, extensions to complex problems, and a finite-volume ELLAM that uses piecewise-constant test functions to conserve mass locally on finite volumes that move with the flow.

Alessandro Russo: *Stabilization of Finite Element Methods via Residual-Free Bubbles.*
We study a finite element approximation of the solution of a convection-diffusion equation with a dominating convection term through a decomposition of the finite element space in low and high frequences, corresponding to continuous piecewise linear functions and residual-free bubble functions respectively. It is shown that this approach can reproduce the well-known Streamline-Upwind Petrov/Galerkin stabilization method (a.k.a. Streamline Diffusion Method).

Friedhelm Schieweck: *Nonconforming Finite Elements of Higher Order for Solving the Navier-Stokes Equations.*
We construct nonconforming finite elements of higher order with the property that each degree of freedom belongs either to the interior of an element or to the interior of an element face, i.e. it belongs to at most two elements. This property is advantageous for the parallelization of the corresponding method since the amount of local communication will be (especially in 3D) esentially smaller compared to the case of conforming elements. The constructed element pairs for velocity and pressure, respectively, satisfy the discrete Babuška-Brezzi condition. Numerical tests with the Stokes and Navier-Stokes problem indicate that in the case of a smooth solution the higher order elements are much more

efficient than the lower order ones.

V. V. Shaidurov: *Second-order monotone scheme for convection-dominated equations with adaptive triangulations.*
We deal with the boundary-value problem

$$-\varepsilon\Delta u + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} = f \quad \text{in} \quad \Omega,$$

$$u = g \quad \text{on} \quad \Gamma.$$

Two-dimentional bounded domain Ω has piecewise-smooth boundary Γ; ε is a positive small parameter; b_1, b_2, f, g are smooth given functions.
First, we construct the finite-element scheme with second-order of approximation which has M-matrix and satisfies discrete maximum principle. To construct some equations, one need an orientation of triangulation. Therefore we additionally suggest a technique for re-orientation of grid along characteristics. Second, we use Gauss-Seidel iterative process with special ordering of equations and unknowns as smoother in multigrid.
Third, to improve the rate of convergence, we use adaptive refinement of triangulation. For this purpose, new local estimator of approximate solution is used in order to divide or not divide each edge of triangulation in two non-equal parts. This algorithm results in new nested triangulation with unisotropic behaviour.
Several numerical examples confirm theoretical results.

Grigorii I. Shishkin:* *Singularly Perturbed Convection-Diffusion Problems with the Flow Coming to an Impermeable Wall: ε-Hypersensitivity and Grid Approximations.*
On the segment $[0, d]$ we consider the Dirichlet problem for a singularly perturbed parabolic equation with varying time-directions in $[-T, T]$. For $\varepsilon = 0$ the parabolic PDE degenerates into a first-order hyperbolic equation. If the time variable is interpreted as a space one, then the reduced equation describes the process of stationary transport. The flow in this transport equation is directed to an impermeable wall. As $\varepsilon \to 0$, a parabolic boundary layer appears in a neighbourhood of the "impermeable wall" disposed along the time-axis. Such problems arise in modelling of heat and mass transfer in moving fluids with large Peclet numbers in the stationary two-dimensional case, when we neglect heat and mass conductivity along the stream.
Unlike problems considered previously, the present problem has a new property of ε-hypersensitivity inherited by a finite difference scheme. More precisely, the solution is not stable with respect to the right-hand side, uniformly in ε. We choose the right-hand sides and their disturbances from a newly introduced class of functions for which the solution is ε-uniformly bounded in L_∞. It is shown that, under these assumptions, a monotone finite difference operator and a special *piecewise uniform* mesh yield an ε-uniformly convergent (in the L_∞-norm) difference scheme. Numerical experiments confirm this theoretical result.

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Gisbert Stoyan: *Towards discrete Velte decompositions and narrow bounds for inf-sup constants.*
The stable solvability of the Stokes problem with first kind homogeneous boundary conditions depends on the so-called inf-sup condition. For the constant β_0 appearing in this condition, narrow upper and lower bounds are given for several domains along with some values of the corresponding constant β_h of the "discrete" inf-sup condition, for specific discretizations. Such values are useful for computable error bounds and for tuning iterative methods.
Also considered is the equivalent of the well-known Helmholtz decomposition for vector functions in $\vec{H}_0^1$. This decomposition is due to W. Velte, On optimal constants in some inequalities (Lecture Notes in Maths. 1431, pp. 158–168) and contains, besides the rotation-free and divergence-free vector functions, a third orthogonal subspace. We show the close connection of the inf-sup condition and the third orthogonal subspace of the Velte decomposition.
Discrete Velte decompositions should be useful for the investigation of numerical methods. Therefore, we derive some first results in this direction for finite element methods including the Taylor-Hood family. For the staggered grid difference approximation of the Stokes problem on a unit square we are able to give all details of the discrete Velte decomposition.

Katarina Surla:* *On Global Approximation of the Solution of a Singular Perturbation Problem.*
The semilinear singularly perturbed reaction-diffusion problem

$$\begin{cases} Ly = \varepsilon^2 y'' + f(x,y) = 0 \quad x \in I = [0,1], \\ y(0) = 0, \quad y(1) = 0, \end{cases}$$

where ε is a small positive parameter and $f(x,y) \in C^2(I \times \boldsymbol{R})$, $f_y(x,y) > \beta^2 > 0$ for all $(x,y) \in I \times \boldsymbol{R}$ is considered. The approximate solution is given in the form of the quadratic polynomial spline. The collocation method on a slightly modified Shishkin mesh is applied and the approximation of the almost second order global uniform accuracy in small parameter is obtained. The middles of intervals are used as the collocation points and the corresponding values of unknown function are replaced by the averages of its values at the ends of intervals.
The global uniform accuracy of the approximations of the normalised diffusive flux and the function $\varepsilon^2 y''(x)$ are also proved. Numerical results, which verify these rates of convergence, are presented.

*joint work with Zorica Uzelac

Song Wang: *Conforming Exponentially Fitted Triangular and Tetrahedral Finite Elements for a Singularly Perturbed Convection-Diffusion Equation.*
In this talk we discuss construction of some new piecewise exponential basis functions on triangles and tetrahedra for Galerkin finite element approximations of convection-diffusion equations of the form

$$-\nabla \cdot (\varepsilon \nabla u - \mathbf{a}u) + bu = f$$

in a polygonal or polyhedral region Ω with a homogeneous Dirichlet boundary condition on $\partial\Omega$. Both the interpolation error and the finite element approximation error were discussed. We also discussed the ε-uniform convergence of the method on meshes of Shishkin type. It is shown that, if a mesh of Shishkin type is used, the convergence rate of the method is $Ch^{1/2}$ for problems with elliptic boundary layers, where C is an arbitrary constant, independent of ε, h and u.

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