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Verallgemeinerte Kac-Moody Algebren

19.07.–25.07.1998

Einleitung

The aim of this conference, organised by R. E. Borcherds (Cambridge) and P. Slodowy (Hamburg), was to bring together researchers from various areas interested in generalisations of Kac-Moody algebras. Thus, the 22 talks presented covered material from the theories of Lie algebras, super algebras, algebraic groups, quantum groups, modular forms, from algebraic geometry and theoretical physics. In addition, and conforming with the intentions of the 'Mathematisches Forschungsinstitut Oberwolfach' numerous discussion groups, from an ad hoc basis at the meal tables to separate talks, formed themselves during the conference. All participants, among them many young people from various countries visiting Oberwolfach for the first time, enjoyed the atmosphere of conference and place.

For details on the talks presented compare the abstracts below.

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Vortragszusammenfassungen

BRUCE N. ALLISON

The structure of extended affine Lie algebras

In this talk, we survey recent work on the structure of extended affine Lie algebras (EALA's) and their root systems. An EALA by definition is a Lie algebra $\mathfrak{g}$ with a finite dimensional self-centralizing abelian diagonalizable subalgebra H and a non-degenerate invariant form satisfying 3 natural axioms including the ad-nilpotency of all elements in root spaces corresponding to nonisotropic roots. EALA's were introduced in 1990 by Høeyh-Krohn and Torresani as higher nullity generalizations of affine Kac-Moody Lie algebras.

Herein an EALA $\mathfrak{g}$ with root system R, the quotient root system $\overline{R}$, modulo the isotropic roots, is a finite (possibly nonreduced) root system, whose type is called the type of $\mathfrak{g}$. If the type of $\mathfrak{g}$ is reduced and $\mathfrak{g}$ is tame, the structures of $\mathfrak{g}$ and R have been determined combining the work of several authors. For example, if $\mathfrak{g}$ has type C_l ($l \geq 4$), the core of $\mathfrak{g}$, modulo its centre, is the symplectic algebra of $2l \times 2l$-matrices over a quantum torus with involution, and the root system R is described in terms of a semilattice determined by the quantum torus. The result was proved recently by Y. Gao and the speaker.

VLADIMIR BARANOVSKY

Sheaves on surfaces and geometric action of Heisenberg algebras

Let S be a smooth projective surface from the following list: $\mathbb{P}^2$, ruled surfaces (and blowups of these), $K3$, abelian. Choose a polarization H such that $HK \leq 0$.
$M^G(r,n)$ — Gieseker moduli space of stable sheaves with $c_2 = n$.
$M^U(r,n)$ — Uhlenbeck moduli space of stable bundles.
All bundles/sheaves are with fixed c_1 such that $c_1 \cdot H$ is coprime to r.
Theorem 1: *There exists a stratification of $M^U(r,n)$ s.t. the natural map $M^G(r,n) \to M^U(r,n)$ is strictly semismall.*
Let P_t stand for the intersection homology Poincare polynomial.
Corollary 2:

$$\sum_{k=0}^{\infty} q^k P_t(M^G(r,n)) = \left(\sum_{k=0}^{\infty} q^k P_t(M^U(r,n))\right) \cdot \prod_{l=1}^{\infty} \frac{(1+t^{2rl-1}q^l)^{b_1}(1+t^{2rl+1}q^l)^{b_3}}{(1-t^{2rl-2}q^l)^{b_0}(1-t^{2rl}q^l)^{b_2}(1-t^{2rl+2}q^l)^{b_4}}$$

Following Nakajima's work for Hilbert schemes we define correspondences for any cycle α on S

$$P_\alpha[i] \subset \coprod_n M^G(r,n) \times M^G(r,n-i).$$

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These correspondences define convolution maps:

$$P_\alpha[i]: \quad \bigoplus_n H^\bullet(M^G(r,n)) \longrightarrow \bigoplus_n H^\bullet(M^G(r,n-i)).$$

Theorem 3: *One has the following relations*

$$[P_\alpha[i], P_\beta[j]] = (-1)^{i-1} i \cdot r\langle\alpha,\beta\rangle \delta_{i,-j}$$

where the commutator above is to be understood in the graded sense.

Corollary 4: $\bigoplus_n H^\bullet(M^G(r,n))$ *is a tensor product of an irreducible representation of an oscillator algebra constructed from* $H^\bullet(S)$, *with a space*

$$\bigoplus_n IH^\bullet(M^U(r,n)).$$

Georgia M. Benkart

Lie algebras graded by Finite Root Systems

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be a finite-dimensional split simple Lie algebra over a field F of characteristic 0, with reduced root system Δ. We say a Lie algebra L over F is Δ-graded if

1. $L = \bigoplus_{\alpha\in\Delta\cup\{0\}} L_\alpha$, where $[L_\alpha, L_\beta] \subseteq \begin{cases} L_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta\cup\{0\}, \\ (0) & \text{otherwise.} \end{cases}$
2. $L \supseteq \mathfrak{g}$.
3. $L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x \quad \forall\, h \in \mathfrak{h}\}$.
4. $L_0 = \sum_{\alpha\in\Delta} [L_\alpha, L_{-\alpha}]$.

The Δ-graded Lie algebras are a natural generalization of the finite-dimensional complex Lie algebras and include such important examples as the affine Kac-Moody algebras, the toroidal Lie algebras, finite-dimensional simple Lie algebras containing an ad-nilpotent element $x \neq 0$, and many more.

We survey the classification of Δ-graded Lie algebras and describe their derivations and central extensions.

For the non-reduced root system $\Delta = BC_r$, we say L is a Δ-graded Lie algebra if (1) (3) (4) and (2′) $L \supset \mathfrak{g}$ (a split simple Lie algebra of type B_r, C_r or D_r) hold. We discuss the recent classification of BC_r-graded Lie algebras.

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DAVID BEN-ZVI

K-dV and Algebraic Geometry

(joint work with Edward Frenkel)

The KdV hierarchy is an infinite collection of commuting flows, on the space of n^{th} order differential operators $L = \partial^n - q_1\partial^{n-1} - q_2\partial^{n-2} - \ldots - q_n$ in one variable. It is well known that these flows, for special operators L, become straight-line flows on Jacobians: i.e. one can construct such L from line bundles $\mathcal{L}$ on Riemann surfaces, which are n-fold branched covers of $\mathbf{P}^1$ so that the Jacobian action on line bundles translates into KdV flows on differential operators. Our aim in this talk was to explain a new algebro-geometric construction which extents the above correspondence $\mathcal{L} \to L$ to cover all n^{th} order operators, as well as all generalized Drinfeld-Sokolov hierarchies.

The construction is based on the following general picture: Given algebraic groups $K \subset G \supset H$, the double coset space $K\backslash G/H$ has two natural structures:

1. A tautological G-bundle with reductions to K, H.
2. An action of any subgroup $L \subset N(H)/H$ from the right.

Combining these structures, one finds the L-action lifts to the K- and G-bundles but not to H. Pulling the structures back to L under orbit maps, one obtains an equivalence between the double coset space $K\backslash G/H$ and a moduli space of G-bundles on L, equipped with: 1. reductions to H and K, 2. a flat connection having "tautological" interactions with the reductions.

The interest in this construction lies in its application to double coset spaces of loop groups, which are known to have interpretations as moduli spaces of (principal G-)bundles on Riemann surfaces. In particular we are led to consider the double coset space $LG_-\backslash LG/A_+$. Here LG is the formal loop group around a point ∞ on a Riemann surface X, LG_- are loops extending holomorphically to all of $X \setminus \infty$, and $A_+ = A \cap LG_+$ is the positive half (the part extending to ∞) of a Heisenberg subgroup of the loop group. One now interprets this space in two ways. First it can be identified with the moduli space of G-bundles on X equipped with a Cartan reduction ("principal Higgs field") near ∞. It thus consists of bundles having a spectral cover description, but only near infinity. This is a generalization of the Krichever data, where we have thrown away most of the spectral curve. That this is the relevant generalization follows from the second interpretation of the double coset space, provided by the above general construction. This identifies $LG_-\backslash LG/A_+$ with a space of flat connections on a formal subgroup of A/A_+. These connections are then easily identified with the Drinfeld-Sokolov connections. The zero-curvature formulation of the KdV flows becomes transparent, and they are identified with the simple action of A/A_+ on the double coset space. This provides the desired algebro-geometric construction of the entire phase space of KdV and its Drinfeld-Sokolov extensions, as well as suggesting many new integrable systems.

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Yuly Billig

Representations of Toroidal Lie algebras and an extension of the KdV hierarchy

The proper framework for the representation theory of the Kac-Moody algebras, affine in particular, is given by the highest weight modules. However in the toroidal case these modules have infinite-dimensional weight spaces. In this talk we explain how to construct a large family of irreducible representations for the toroidal algebras with the weight spaces of finite dimension. A special role is played by the "rank n Virasoro algebra" that is made up of the center of the toroidal algebra and the algebra of outer derivations. In fact, we have a functor from the direct product of category $\mathcal{O}$ of the representations of affine algebra and a certain category of representations of "rank n Virasoro" to the category of modules for toroidal algebras. We give explicit realizations for our modules via the vertex operator approach.

Finally we obtain a hierarchy of PDE's associated to the "basic" representation of the 2-toroidal algebra $\mathrm{sl}_2(\mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}]) \oplus \mathcal{K} \oplus \mathcal{D}$ and construct their soliton solutions. This hierarchy is an extension of the KdV hierarchy. The equations of the small degree that appear here and not in the KdV hierarchy are:

(1) $[D_1^3 D_0 + 2D_0 D_3 - 6D_1 D_2](\tau \circ \tau) = 0$.
After $f = \frac{\partial}{\partial x_1} \ln \tau$ becomes: $\frac{\partial}{\partial x}(6f_t - f_{xxy} - 6f_x f_y) - f_{yz} = 0$
and
(2) $[D_1^3 D_2 - D_1^2 D_0 D_3](\tau \circ \tau) = 0$. After $g = \ln \tau$ becomes: $g_{xxxt} + 6g_{xx}g_{xt} - g_{xxyz} - 4g_{xy}g_{xz} - 2g_{xx}g_{yz} = 0$.

Here $x = x_1$, $y = x_0$, $z = x_3$, $t = x_2$.

Alex Feingold

Minimal Model Fusion Rules from Elementary 2-groups

The usual approach to fusion rules comes from using any ordered triple of irreducible modules, (U, V, W), to label each space of intertwining operators, $I(U, V, W)$. The fact that $\dim(I(U, V, W))$ can be greater than one could be interpreted to mean that more parameters should be used to label the intertwiners. Evidence for this can be given in the case of the Virasoro algebra minimal models, with labels given by certain elementary 2-groups. I will discuss this, and how it may correspond to a specific construction of the minimal models by vertex operators.

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YUN GAO

Vertex operators arising from the Fock space for $\widehat{gl}_N$ and applications to EALAs

The homogeneous vertex operator representation for affine Lie algebras of the simply-laced type was developed by Frenkel-Kac, independently by Segal.

In my talk, I will use the underlying Fock space for the homogeneous vertex operator representation of the affine Lie algebra $\widehat{gl}_N$ to construct a family of vertex operators. More precisely, given a pair (Λ, q), where q is a nonzero complex number, Λ is a subset of $\mathbf{R}$ containing zero and closed under addition, we define vertex operators $X_{ij}(r,z)$ depending on the parameter q, for $1 \leq i,j \leq N$ and $r \in \Lambda$. These vertex operators together with the Heisenberg algebra form a Lie algebra $\mathcal{V}(\Lambda, \mathrm{II})$, which is proven to represent an affinization of the matrix algebra with entries in a skew-polynomial ring.

The case $\Lambda = \{0\}$ is trivial as $\mathcal{V}(\Lambda, \mathrm{II})$ represents the affine Lie algebra $\widehat{gl}_N$ for $N \geq 2$. However, if (Λ, q) is generic, by enlarging the Fock space, the case $\Lambda = \mathbb{Z}$ will provide an irreducible vertex operator representation for an extended affine Lie algebra of type A_{N-1} coordinatized by a quantum torus $\mathbb{C}_q$ of 2 variables. Consequently, a character formula for the basic module of the extended affine Lie algebra is obtained. Also, if $N = 1$, we get a representation for a q-analog of two dimensional Virasoro algebra studied by Kirkman-Procesi-Small.

VICTOR GINZBURG

Principal Nilpotent pairs and Representation theory

Let g be a complex semisimple Lie algebra, and G the corresponding adjoint group, i.e., the identity component of $Aut(g)$.

An element $x \in g$ is called *regular* if $Z_g(x)$, the centralizer of x in g, has the minimal possible dimension, i.e., $dim Z_g(x) = \mathrm{rk}\, g$. The most interesting, in a sense, among regular elements of g are regular nilpotent elements, called "principal nilpotents". These elements form a single $Ad\,G$-orbit in g.

We would like to "double" the above setup and replace the Lie algebra g by $Z \subset g \oplus g$, the set of all pairs $(x_1, x_2) \in g \oplus g$, such that $[x_1, x_2] = 0$, called the commuting variety of g. It turns out that there is a remarkable "doubled" counterpart of the notion of a principal nilpotent, that we call a principal nilpotent pair. The set of principal nilpotent pairs does not form a single orbit under $Ad\,G$-diagonal action on Z, but it consists of only finitely many such orbits.

In the case $g = gl_n$, for instance, the conjugacy classes of principal nilpotent pairs are parametrized essentially (up to transposing matrices) by Young diagrams with n-boxes. This comes about as follows. Given a Young diagram, we enumerate its boxes in some order, and label the standard base vectors in $\mathbb{C}^n$ by the box with the

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corresponding number. We now define an endomorphism $e_1 \in gl_n$ by letting it act "along the rows of the diagram", i.e., by sending the base vector labelled by a box to the base vector labelled by the next right box, if this box belongs to λ, and to 0 otherwise. Similarly, we define $e_2 \in gl_n$ by letting it act "along the columns", from bottom to top. It is easy to see that the operators e_1, e_2 thus defined commute, and form a principal nilpotent pair. Note that if λ consists of either a single row or a single column, then the corresponding principal nilpotent pair is either of the form $(e, 0)$ or $(0, e)$, where e is a principal nilpotent in gl in the ordinary sense. Moreover, any principal nilpotent pair for $g = gl_n$ is associated to a Young diagram in the above way (up to conjugation and transposing matrices).

To each principal nilpotent pair in an arbitrary semisimple Lie algebra g, and each simple finite-dimensional g-module, we associate a certain two-variable analogue of Kostant's partition function, and prove the corresponding (s,t)-weight multiplicity formula. Furthermore, we wish to express our two-variable weight multiplicity in terms of intersection cohomology of a double-loop Grassmannian, similar to the way, the q-analogue of weight multiplicity introduced by Lusztig is related to perverse sheaves on the loop Grassmannian.

We also associate to a principal nilpotent pair in g a bi-harmonic polynomial on the Cartesian square of a Cartan subalgebra of g. This polynomial has very interesting properties, in particular transforms under an irreducible representation of the Weyl group. In the special case $g = gl_n$ the irreducible representation of the symmetric group generated by the polynomial turns out to be parametrized by the Young diagram labelling the principal nilpotent pair. Our construction is closely related to the theory of Springer representations.

VALERY GRITSENKO

Some coincidences in the theory of generalized Kac-Moody Lie algebras

I) Automorphic Lorentzian Kac-Moody algebras.

Let $\mathbf{g}$ be a generalized Kac-Moody (GKM) superalgebra (without real odd roots) with the root lattice M of signature $(1, n-1)$. We consider the Weyl-Kac-Borcherds denominator function of $\mathbf{g}$ as a function of the complex homogeneous domain (of type IV) associated to M,

$$\begin{aligned}\phi_{\mathbf{g}}(z) &= \sum_{w \in W} det(w) \left(e^{\langle w(\rho), z\rangle} - \sum_{\alpha \in_s \Delta^{im}} m(\alpha) e^{\langle w(\rho+\alpha), z\rangle} \right) \\ &= e^{\langle \rho, z\rangle} \prod_{\alpha \in \Delta_+} \left(1 - e^{\langle \alpha, z\rangle}\right)^{mult(\alpha)}\end{aligned}$$

(see Borcherds, J. of Alg. '88; Gritsenko-Nikulin, Am. J. Math. '97).

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Definition: $\mathbf{g}$ is called automorphic KMA if $\phi_{\mathbf{g}}(z)$ is an automorphic form with respect to a group of finite index in an integral orthogonal group $O_{\mathbb{Z}}(\tilde{M}_{2,n})$ of signature $(2, n)$.
Theorem: *If ${}_s\Delta^{im} = \emptyset$ (or ${}_s\Delta^{im}$ is finite) then $\phi_{\mathbf{g}}(z)$ is not automorphic.*

The first example of an AKMA was found by R. Borcherds. This is the famous fake monster Lie algebra ($n = 26$, $\rho^2 = 0$). Some examples of AKM algebras are given by

II) Automorphic discriminants of moduli spaces of K3 surfaces with a condition on the transcendental lattice. Such AKM algebras are related to the arithmetic mirror symmetry for K3 (see Gritsenko-Nikulin, Math. Res. Let. '96).

III) Classification.
A theorem of Nikulin says that if $\mathrm{rank}(M) \geq 3$, then there are essentially a finite number of AKM algebras (Nikulin, Jzv. Math. '96). The two papers of Gritsenko-Nikulin in Int. J. Math. '98 are devoted to the classification for $n = 3$. We proved that there are only 12 elliptic (i.e. $\rho^2 > 0$) Lorentzian Cartan matrices of rank 3: $A_{a,b}$ where $a = 1$, 2, 3 and $b = 0$, I, II, III. The corresponding automorphic forms are 3-dimensional generalizations of Dedekind η-functions.

IV) Canonical differential form on a Calabi-Yau 3-fold.
The AKM algebra with the Cartan matrix (of the real simple roots) $A_{3,II}$ (this is a 6×6-matrix) is given by a Siegel modular form of weight 1, which we denote by Δ_1.
Theorem (Gritsenko-Hulek): *$\Delta_1(Z)^3 dZ$ generates the space $H^{3,0}(\tilde{\mathcal{A}}^{(2)}_{1,3})$, where $\tilde{\mathcal{A}}^{(2)}_{1,3}$ is a smooth compact model of the moduli space $\mathcal{A}^{(2)}_{1,3}$ of (1,3)-polarized abelian surfaces with a 2-level structure. $\tilde{\mathcal{A}}^{(2)}_{1,3}$ is a Calabi-Yau 3-fold with Euler number $= 80$.*
Conjecture: The L-function $L_{\Delta_1^3}(s)$ of the form Δ_1^3 is related to the Hasse-Weil L-function of the 3-fold $\tilde{\mathcal{A}}^{(2)}_{1,3}$.

V) The second quantized elliptic genus (SQEG) of Calabi-Yau manifolds.
The SQEG was introduced in the papers of R. Dijkgraaf, G. Moore, E. Verlinde, H. Verlinde (see hep-th). We proved:
Theorem (Gritsenko-Nikulin): *Let M_d be a Calabi-Yau manifold of complex dimension $d = 2$, 4, 6 or 8. Then $SQEG(M_d, Z)$ is given by a product of $d/2$ modular forms which are powers of denominator functions of AKM algebras. For $d = 2$, 4, 6 these algebras have elliptic Cartan matrices (of real simple roots) $A_{a,b}$ mentioned above. For $d = 8$ AKMA with parabolic root systems (i.e. $\rho^2 = 0$) appear in the*

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formula. In particular

$$
\begin{aligned}
SQEG(\text{Enriques surface}; Z) &= \Delta_5^{-1}(Z) \quad \textit{related to} \quad A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \\
SQEG(\text{K3 surface}; Z) &= \Delta_5(Z)^{-2}, \\
SQEG(CY_4; Z) &= \Delta_2(Z)^{4\chi_0 - e/6}\Delta_{11}^{-\chi_0},
\end{aligned}
$$

where χ_0 = arith. genus(CY_4), e = Euler number(CY_4), $\Delta_5(Z)$ is the product of all even Siegel theta-constants, $\Delta_2(Z)^4$ is the first cusp form for the paramodular group $\Gamma_2 \subset Sp_4(\mathbb{Q})$ and $\Delta_{11}(Z)$ is the first cusp form for Γ_2 of odd weight (the index denotes the weight of the modular form).

Corollary: *Let $M_4^{(\chi_0,e)}$ be a Calabi-Yau 4-fold with fixed arithmitic genus χ_0 and Euler number e. Then*

$$
\begin{aligned}
SQEG(M_4^{(2,48)}) &= \Delta_{11}(Z)^{-2} \quad \textit{related to} \quad A_{2,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}, \\
SQEG(M_4^{(2,36)}) &= (\Delta_{11}/\Delta_2)^{-2} \quad \textit{related to} \quad A_{2,I} = \begin{pmatrix} 2 & -2 & -4 & 0 \\ -2 & 2 & 0 & -4 \\ -4 & 0 & 2 & -2 \\ 0 & -4 & -2 & 2 \end{pmatrix}, \\
SQEG(M_4^{(0,\pm 12)}) &= \Delta_2^{\mp 2} \quad \textit{related to} \quad A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}.
\end{aligned}
$$

Theorem (Gritsenko-Nikulin): *SQEG(CY_3) is a power of a Siegel automorphic form of weight 0, which is the denominator function of an AKM superalgebra with a real root system, which has a fundamental polygon of hyperbolic type. (This is the third possible type of root systems after elliptic, $\rho^2 > 0$, and parabolic, $\rho^2 = 0$, ones.)*

GERALD HÖHN

Elliptic genera of symmetric products and generalized Kac-Moody Lie algebras

In the first part of my talk, I gave a survey of the different types of elliptic genera. They are defined formally as the S^1-equivariant index $\varphi(q, \mathcal{L}X)$ of hypothetical elliptic operators on the loop space $\mathcal{L}X$ of a manifold X (de Rham, Dirac, Dolbeault, signature complex), generalizing the "classical" genera. They map the bordism ring to some ring of modular functions.

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For a finite group G acting on a manifold X, the orbifold elliptic genus of the quotient space X/G is defined by

$$\varphi(q,\mathcal{L}(X/G)) := \frac{1}{|G|}\sum_{\substack{g,h\in G\\ [g,h]=1}} \varphi(h;q,\mathcal{L}_gX).$$

Here, $\varphi(h;q,\mathcal{L}_gX)$ is given by formally applying the equivariant Atiyah-Singer index theorem to $T\mathcal{L}_gX|_{X^{\langle g,h\rangle}}$, where $\mathcal{L}_gX = \{\alpha : \mathbb{R}\to X \mid \alpha(t+1) = g\alpha(t)\}$ is the g-twisted loop space. Let $S^nX = (X\times X\times\ldots\times X)/S_n$ be the n-fold symmetric product.

Theorem:

$$(a)\quad \sum_{m\geq 0} e(q,\mathcal{L}(S^mX))p^m = \prod_{m=1}^{\infty}(1-p^m)^{-e(X)}. \qquad \textit{(Hirzebruch/Höfer)}$$

$$(b)\quad \sum_{m\geq 0}\chi_y(q,\mathcal{L}(S^mX))p^m = \prod_{m>0}\prod_{n\geq 0,l}(1-yp^mq^n)^{-c(nm,l)},$$

where $\chi_y(q,\mathcal{L}X) = \sum_{n\geq 0,l} c(n,l)y^lq^n$.

$$(c)\quad \sum_{m\geq 0}\hat{A}(q,\mathcal{L}(S^mX))p^m = \prod_{m>0}\prod_{n\in\mathbb{Z}}(1-p^mq^n)^{-a(nm)},$$

where $\hat{A}(q,\mathcal{L}X) = q^{-\dim X/24}\hat{A}(X,\bigotimes_{n=1}^{\infty}S_{q^n}T_{\mathbb{C}}X) = \sum_{n\in\mathbb{Z}}a(n)q^n;\quad 24\mid\dim X.$

Part (b) was obtained by Dijkgraaf, Moore, Verlinde and Verlinde in a more physical formulation.

Product expansions of this kind have been studied by Borcherds in connection with character formulas of generalized Kac-Moody algebras. The above products together with an extra correction factor are automorphic forms on $SO(s+2,2)/(SO(s)\times SO(2))$ for some arithmetic subgroup of $SO(s+2,2)$. They can be obtained by some singular theta correspondence from $\varphi(q,\mathcal{L}X)$.

For X a $K3$ surface, it is a conjecture of the physicists mentioned above, that $\chi_y(q,\mathcal{L}X^{[m]}) = \chi_y(q,\mathcal{L}(S^mX))$. Here, $X^{[m]}$ is the Hilbert scheme of dimension zero subschemes of length m, which is a smooth resolution of S^mX. More generally, B. Totaro made the conjecture that $\chi_y(q,\mathcal{L}X)$ can be defined for arbitrary singular varieties over $\mathbb{C}$.

It is the idea of Hirzebruch, that there may exist a 24-dimensional manifold X_{24} on which the Monster group acts, such that for g in the Monster the equivariant elliptic genus $\hat{A}(g;q,\mathcal{L}X)$ equals the Thompson series $\sum_n \mathrm{tr}(g|V_n^\natural)q^n$ of the Moonshine module $V^\natural$. Assuming the existence of such a manifold, we obtain an induced action on S^mX and an equivariant version of Theorem (c) gives one side of the equivariant

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denominator identity of the Monster Lie algebra — suggesting the possibility that this generalized Kac-Moody Lie algebra can be obtained in a new geometrical way.

VICTOR KAC

Highest weight modules over Kac-Moody superalgebras

Let I be an index set, $\tau \subset I$ a subset, and $A = (a_{ij})_{i,j \in I}$ a matrix over $\mathbb{C}$. To this data one associates a Lie superalgebra $\mathfrak{g}(A, \tau)$ as follows. Denote by $\tilde{\mathfrak{g}}(A, \tau)$ the Lie superalgebra on generators e_i, f_i, h_i ($i \in I$), the e_i and f_i with $i \in \tau$ being odd, all other generators even, subject to the usual relations: $[h_i, h_j] = 0$, $[e_i, f_j] = \delta_{ij} h_i$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$. Let Q be a free abelian group on generators α_i ($i \in I$), and introduce a Q-gradation of $\tilde{\mathfrak{g}}(A, \tau)$ letting $\deg e_i = -\deg f_i = \alpha_i$, $\deg h_i = 0$. Let $J(A, \tau)$ be the maximal Q-graded ideal of $\tilde{\mathfrak{g}}(A, \tau)$ intersecting trivially the Cartan subalgebra $\sum_i \mathbb{C} h_i$. Then

$$\mathfrak{g}(A, \tau) = \tilde{\mathfrak{g}}(A, \tau) / J(A, \tau).$$

The Lie superalgebra $\mathfrak{g}(A, \tau)$ is called a generalized Kac-Moody superalgebra if its even part is a generalized Kac-Moody algebra whose representation on odd part is integrable. The talk was a survey of results on the denominator and character formulas for finite-dimensional and affine Kac-Moody superalgebras. At the end a free field construction of all integrable highest weight $\mathrm{sl}(m, n)^\wedge$-modules of level 1 was briefly discussed (joint work in progress with Wakimoto).

Literature:

1. V.G. Kac, Lie Superalgebras, Adv. Math. **26** (1977), 8–26.
2. V.G. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, Progress in Math. **123** (1994), 415–456 (and references there).

SEOK-JIN KANG

Crystal bases for Kac-Moody superalgebras

Let $A = (a_{ij})_{i,j \in I}$ be a Cartan matrix satisfying the following conditions:
$I = I^{even} \cup I^{odd}$, $\tau = I^{odd}$

1. $a_{ii} = 2$ for all $i \in I^{even}$,
2. $a_{ii} = 0$ or 2 for $i \in I^{odd}$,
3. if $a_{ii} = 2$ and $i \neq j$, then $a_{ij} \leq 0$,
4. $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

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Consider the Kac-Moody superalgebra $\mathfrak{g} = \mathfrak{g}(A, \tau)$ associated with (A, τ), and let $U_q(\mathfrak{g})$ be the quantized superalgebra which is a deformation of the universal enveloping algebra of $\mathfrak{g}$. Let $\mathcal{O}_{\text{int}}$ denote the category of $U_q(\mathfrak{g})$-modules satisfying the conditions: If $M \in Ob\,\mathcal{O}_{\text{int}}$, then:

1. $M = \bigoplus_{\lambda \in P} M_\lambda$, $M_\lambda = \{u \in M \mid q^h u = q^{\lambda(h)} u \quad \forall h \in P^\vee\}$.
2. $\dim M_\lambda < \infty \quad \forall \lambda \in P$.
3. $\forall i \in I^{even}$, M is locally $U_q(\mathfrak{g}_i)$-finite.
4. $\forall i \in I^{odd}$, $a_{ii} = 2$, M is locally $U_q(\mathfrak{g}_i)$-finite.
5. If $M_\mu \neq 0$, then $\mu(h_i) \geq 0$ for $i \in I^{odd}$, $a_{ii} = 0$.
6. If $M_\mu \neq 0$, $a_{ii} = 0$, $\mu(h_i) = 0 \Rightarrow e_i M_\mu = f_i M_\mu = 0$.

Then we develop the crystal base theory for the $U_q(\mathfrak{g})$-modules in $\mathcal{O}_{\text{int}}$, generalizing the corresponding theory for Kac-Moody algebras. In particular the tensor product rule was generalized to this case, and for the general linear Lie superalgebra $gl(m,n)$, the crystal bases for the irreducible highest weight modules in $\mathcal{O}_{\text{int}}$ were realized as the set of semistandard (m,n)-hook Young tableaux, which are given the crystal structure by admissible readings of tableaux.

OLIVIER MATHIEU

On modular representations of affine Lie algebras at the critical level

For an affine Kac-Moody Lie algebra, we set

$$\begin{aligned} M(-\rho) &= \text{the Verma module with highest weight } -\rho, \\ L((p-1)\rho) &= \text{the simple module with highest weight } (p-1)\rho, \\ \Delta((p-1)\rho) &= \Gamma(G/B, \mathcal{L}(1-p)\rho)^*, \text{where } p = \text{characteristic of the field.} \end{aligned}$$

We investigate these modules.

Theorem 1:

$$\mathrm{ch} L((p-1)\rho) = e^{(p-1)\rho} \prod_{\alpha \in \Delta^+_{re}} \frac{1}{1-e^{-\alpha}}.$$

In particular $L((p-1)\rho) \neq \Delta((p-1)\rho)$, which is different from the finite dimensional case (and Steinberg theorem).

Theorem 2: *Set* $A = \mathrm{End}_G L((p-1)\rho)$. *Then* A *is commutative, and* $\mathrm{Spec} A$ *is naturally identified with the space of* $\underline{h}^\vee$*-valued forms* α, $\alpha = (\frac{\rho}{z} + \sum_{i \geq 0} h_i z^i) dz$ *such that* $C\alpha = \alpha$, *where* C *is the Cartier operator.*

Now we investigate $\tilde{A} = \mathrm{End}_G(M(-\rho))$. Accordingly to Feigin-Frenkel philosophy, $\tilde{A}$ should be a space of functions on a space of connections or $\mathcal{D}$-modules. In characteristic p, the p-curvature is in some sense a map on the space of connection. So we expect the existence of an endomorphism of $\tilde{A}$. Indeed we prove:

Theorem 3: $\tilde{A}$ *is commutative, and isomorphic to* $A \otimes A \otimes \cdots$. *The tensor product decomposition depends on the choice of coordinates, but the shift* $\tilde{A} \to \tilde{A}$, $a_1 \otimes a_2 \otimes \ldots \mapsto 1 \otimes a_1 \otimes a_2 \otimes \ldots$ *is canonical, and corresponds with* C.

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GREGORY MOORE

Some Remarks on BPS Algebras associated to Calabi-Yau Manifolds

String theory in a black box: For this talk, a string theory T is a map from a collection of manifolds X; s.t. $\dim X \le 9$, X compact, Riemannian, admits covariantly constant spinors, to a set of data; including:

- "moduli of groundstates" $\mathcal{M}_T(X)$
- "Hilbert space of 1-particle states" $\mathcal{H}_T(X,\mu)$, $\mu \in \mathcal{M}_T(X)$
- "S-matrix" $\mathcal{S}_T : \bigoplus_{n\ge 0} S^n\mathcal{H}_T \to \bigoplus_{n\ge 0} S^n\mathcal{H}_T$
- "supergravity theory"

Here $\mathcal{H}_T$ is a (highly reducible) unitary rep^n of a superpoincaré algebra $\mathcal{SP}^{(\mathcal{N})}(1,s)$ with $\mathcal{N}$ real supercharges, $s = 9 - \dim X$. There are 5 known examples : IIA, IIB, I, HE, HO. If X admits "enough" covariantly constant spinors then $\mathcal{SP}^{(\mathcal{N})}(1,s)$ admits "BPS" or "short" representations which enjoy special rigidity properties. Under these conditions there is a distinguished subspace $\mathfrak{A}_T(X,\mu) \subset \mathcal{H}_T(X\mu)$ defined by the BPS states. Using analytic continuation of S-matrix elements we define the algebra structure $\mathfrak{A}_T(X) \otimes \mathfrak{A}_T(X) \to \mathfrak{A}_T(X)$ by the residue of the BPS pole.

Two examples: Let $p, q \in \mathbb{Z}_+$, $p \ge q$, $p - q \equiv 0\,(8)$.
Notation: $\mathcal{B}^{p,q} := \{\text{projections } II^{p,q} \otimes \mathbb{R} \to \mathbb{R}^{p,0} \perp \mathbb{R}^{0,q}\}$, $II^{p,q}$:= even unimodular lattice of signature $(-1)^p(+1)^q$, $\mathcal{N}^{p,q} := O(II^{p,q})\backslash\mathcal{B}^{p,q}$. If $P \in II^{p,q}$ the projection gives $P = P_L + P_R$.

Example 1. ("Trivial case" c. 1985) $T = HE, HO$; $X = (S^1)^d$.

- superalgebra: $\mathcal{SP}^{(16)}(1, 9-d)$
- $\mathcal{M}_{HE}(X) = \mathcal{N}^{d+16,d} \times \mathbb{R}_+$
- $\mathfrak{A}_{HE}(X,\mu) = \mathbb{R}^{d+16,d} \otimes \pi_\nu(0) \oplus \bigoplus_{P\in II^{d+16,d}} \mathfrak{A}^P_{HE} \otimes \pi_\nu(m(P,\mu))$
  where $m^2(P,\mu) = \frac{1}{2}P_R^2$;
  $\pi_\nu(m)$ = BPS vectormultiplet representation of $\mathcal{SP}$ of mass m.

$\mathfrak{A}^P_{HE}$ is expressed in terms of vertex operators and, in particular, $\dim \mathfrak{A}^P_{HE} = p_{24}(\frac{1}{2}P^2 + 1)$.

The BPS algebra structure on $\mathfrak{A}_{HE}$ is a generalization of a Lie algebra. Along enhanced symmetry varieties $V_\mathfrak{g} \subset \mathcal{N}^{d+16,d}$ associated to a Lie algebra $\mathfrak{g}$, $\mathfrak{A}_{HE}(X,\mu) = \mathfrak{g} \otimes \pi_\nu(0) \oplus [\text{ massive rep}^{n'}\text{-s}]$ and the BPS algebra structure is computed to be of the form $(x \otimes \mathcal{S}_1) \cdot (y \otimes \mathcal{S}_2) = [x,y] \otimes \mathcal{S}_{12}$, $[x,y]$ = Lie bracket on $\mathfrak{g}$.

Example 2. $T = IIA$, $X = K3$ surface.

- superalgebra $= \mathcal{SP}^{16}(1,5)$
- $\mathcal{M}_{IIA} = \mathcal{N}^{20,4} \times \mathbb{R}_+$
- $\mathfrak{A}_{IIA}(X,\mu) = \mathbb{R}^{20,4} \otimes \pi_\nu(0) \oplus \bigoplus_{P\in H^\bullet(X,\mathbb{Z})} H^\bullet_{L^2}(\mathcal{M}_P) \otimes \pi_\nu(m(P,\mu))$.

$\mathcal{M}_P$:= moduli of simple coherent sheaves $\mathcal{E} \to X$ with $\mathrm{ch}\mathcal{E}\sqrt{\mathrm{Td}\,X} = P$; this description follows from the theory of "D-branes".

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String Duality: Is the idea that $IIA(K3) \cong HE\big((S^1)^4\big)$.
Some checks are nontrivial and involve automorphic forms on domains of type IV. Also, $\mathfrak{A}_{IIA}(X,\mu) \cong \mathfrak{A}_{HE}\big((S^1)^4,\mu\big)$.

Generalizations to Calabi-Yau 3-folds (Not given, for lack of time):
The spaces $\mathfrak{A}_{IIA}(X)$, $\mathfrak{A}_{IIB}(X)$ for X a smooth Calabi-Yau 3-fold have the structure:

$$\mathfrak{A}_{IIB}(X,\mu) = \mathfrak{g}^{ab} \otimes \pi_\nu(0) \oplus \bigoplus_{\gamma \in H^{odd}(X,\mathbb{Z})} \mathfrak{A}^\gamma_{IIB} \otimes \pi_h(m_\gamma)$$

$$\mu \in \mathcal{M}_{\substack{\text{cplx.}\\ \text{str.}}}(X), \qquad \mathfrak{g}^{ab} \oplus (\mathfrak{g}^{ab})^* \cong H^{odd}(X;\mathbb{R})$$

$$\mathfrak{A}_{IIA}(X,\mu) = \mathfrak{g}^{ab} \otimes \pi_\nu(0) \oplus \bigoplus_{\gamma \in H^{even}(X,\mathbb{Z})} \mathfrak{A}^\gamma_{IIA} \otimes \pi_h(m_\gamma)$$

$$\mu \in \mathcal{M}_{\substack{\text{cplxfied}\\ \text{Kähler}}}(X), \qquad \mathfrak{g}^{ab} \oplus (\mathfrak{g}^{ab})^* \cong H^{\text{even}}(X;\mathbb{R})$$

$\mathfrak{A}^\gamma_{IIA}$, $\mathfrak{A}^\gamma_{IIB}$ are cohomologies of certain moduli spaces. Some results on the dimensions of these spaces ("root multiplicities") may be derived using supergravity. In particular

1. Let $\gamma \in H_3(X,\mathbb{Z})$. Suppose X admits a complex structure so that

$$\hat\gamma = \hat\gamma^{3,0} + \hat\gamma^{0,3} \qquad (\text{“attractor equation”})$$

where $\hat\gamma$ is the Poincaré dual. If $|Z^*|^2 \gg 1$, where

$$|Z_*|^2 \equiv \frac{|\int_\gamma \Omega|^2}{|\int_X \Omega \wedge \overline{\Omega}|}, \qquad \Omega = \text{a nowhere zero holomorphic } (3,0)\text{-form on } X,$$

then $\mathfrak{A}^\gamma_B \neq \{0\}$ and $\log(\dim \mathfrak{A}^\gamma_B) \sim \pi |Z_*|^2$.

2. If $\int_\gamma \Omega = 0$ defines a nontrivial divisor $\mathcal{D}_\gamma \subset \mathcal{M}_{\text{cplx}}(X)$ not contained in the discriminant locus, then $\mathfrak{A}^\gamma_B = \{0\}$.

Literature:

1. J. Harvey and G. Moore, "On the algebra of BPS states," hep-th/9609017.
2. G. Moore, "Arithmetic and Attractors," hep-th/9807087.

SATOSHI NAITO

Character formula of Kac-Wakimoto Type for Generalized Kac-Moody Algebras

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a symmetrizable GKM algebra over $\mathbb{R}$. This means that $\mathfrak{g}$ is the real contragredient Lie algebra associated to a real square matrix $A = (a_{ij})_{i,j\in I}$ such that

$$\begin{cases} (C1) \text{ either } a_{ii} = 2 \text{ or } a_{ii} \le 0, \\ (C2)\ a_{ij} \le 0 \text{ if } i \neq j \text{ and } a_{ij} \in \mathbb{Z} \text{ if } a_{ii} = 2, \\ (C3)\ a_{ij} = 0 \Leftrightarrow a_{ji} = 0. \end{cases}$$

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Assume $\#(I) < +\infty$. Let $K^{m.g.} \equiv \{\lambda \in \mathfrak{h}^* \mid 2(\lambda+\rho|\alpha) \gneqq (\alpha|\alpha) \text{ for } \alpha \in \Delta_+^{im}\}$, $(K^{m.g.} \supset K^g)$. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{h}, \mathbf{R})$, Δ_+^{im} is the set of positive imaginary roots, $(\cdot \mid \cdot)$ a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ which is W-invariant. In addition ρ is a Weyl vector ($\rho \in \mathfrak{h}^*$ s.t. $\rho(\alpha_i^\vee) = \frac{1}{2}a_{ii}$ for $i \in I$, $\{\alpha_i^\vee\}_{i\in I}$ simple coroots). Then we have:

Theorem: *Let $\lambda \in K^{m.g.}$ be such that $(\lambda+\rho)(\alpha^\vee) > 0$ for $\alpha \in \Delta^\lambda \cap \Delta_+$. ($\Delta^\lambda = \{\alpha \in \Delta^{re} \mid \lambda(\alpha^\vee) \in \mathbf{Z}\}$. $L(\lambda)$ is the irreducible h.w. module with $\lambda \in \mathfrak{h}^*$.) Then*

$$\mathrm{ch}L(\lambda) = \frac{\sum_{w\in W^\lambda} \det(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}},$$

where $W^\lambda = \langle r_\alpha \mid \alpha \in \Delta^\lambda\rangle$, $\mathrm{mult}(\alpha) = \dim_{\mathbf{R}} \mathfrak{g}_\alpha$, $\alpha^\vee$ coroot of α.

HIRAKU NAKAJIMA

Blowup formula for moduli spaces of sheaves on surfaces

I am running the following project: "Study homology groups of moduli spaces of sheaves on surfaces by using the representation theory of affine algebras."

We do not have enough understanding to make a precise conjecture yet, but we *do* have lots of interesting examples. We would like to add one example, the blowup formula in terms of the character of the representation of $\widehat{\mathfrak{gl}_r}$ with level 1.

Let $X = \mathbb{C}^2$ and $\hat{X}$ = blowup at the origin. We consider moduli spaces of sheaves on both X and $\hat{X}$.

$M^\times(r, 0, c_2)$ = the framed moduli space of sheaves E on X with rank $E = r$, $c_1(E) = 0$, $c_2(E) = c_2$.

$M^{\hat{\times}}(r, c_1, c_2)$ = the framed moduli space of sheaves $\hat{E}$ on $\hat{X}$ with with rank $\hat{E} = r$, $c_1 = (\hat{E}) = c_1$, $c_2(\hat{E}) = c_2$.

Here "framed moduli" more precisely means the set of isomorphism classes of pairs (E, ϕ), where E is a sheaf on the compactification $X \cup l_\infty = \mathbb{C}P^2$, and ϕ is an isomorphism $\phi : E|_{l_\infty} \xrightarrow{\cong} \mathcal{O}_{l_\infty}^{\oplus r}$.

Using torus actions on moduli spaces, we find (joint work with Yoshioka)

$$\sum_{c_1,c_2} P_t(M^{\hat{\times}}(r,c_1,c_2))q^\Delta \Big/ \sum_{c_2} P_t(M^\times(r,0,c_2))q^\Delta \qquad (\Delta = c_2 - \frac{r-1}{2r}c_1^2)$$

$$= \left\{\prod_{d\geq 1} \frac{1}{1-(t^{2r}q)^d}\right\}^r \cdot \sum_{(a_1,\dots,a_r)\in(\frac{1}{r}\mathbf{Z})^{\oplus r}} t^{\sum a_i(r+1-2i)}(t^{2r}q)^{\sum a_i^2/2},$$

where P_t is the Poincaré polynomial. If we set $t = -1$ (i.e. generating function of the Euler numbers), we get the character of the basic representation of $\widehat{\mathfrak{gl}}_r$.

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Hermann Nicolai

Kac-Moody Symmetries in Gravity and Supergravity

Dimensional reduction of (super)gravity theories leads to theories of matter coupled gravity in lower dimensions with "hidden symmetries". More specifically, the scalar sectors are governed by G/H σ-models with G noncompact and H the maximal compact subgroup of G. For $d \geq 3$ (d is the dimension of the reduced space-time) we have $\dim G$, $\dim H < \infty$ whereas for $d = 2$, G is the affine extension of the corresponding group in $d = 3$ and H again the maximal compact subgroup. The coset space in this case is the moduli space of solutions of the higher dimensional field equations with certain commuting Killing vectors. The reduction to $d = 1$ suggests the emergence of hyperbolic KM symmetries (in accordance with a conjecture by B. Julia) whose Dynkin diagrams are obtained by adding an $A_2 = \mathrm{sl}_3$ diagram (where the sl_3 acts on the vierbein) to the Dynkin diagram of the $d = 3$ theory. The corresponding (huge) coset space for the dimensionally reduced maximal ($D = 11$) supergravity may be relevant for M-Theory.

See e.g. 1. B. Julia & H.N., Nucl. Phys. B482 (1996) **431**;
2. H.N.: hep-th/9801090.

Urmie Ray

A characterization of a certain class of Lie superalgebras

As the definition of GKM superalgebras via generators and relations is difficult to use in many situations, it is worth finding different ways of characterizing them. Borcherd's characterization, which roughly says that the Lie algebra is a GKM algebra if it is graded and has an almost positive definite invariant symmetric bilinear form, only holds for Lie superalgebras of finite dimension when the odd part of the superalgebra is non-trivial.

Also it makes little sense to define real roots as those conjugate to simple roots of positive norm under the action of the Weyl group, as is shown by the following example: When the superalgebra $\mathfrak{g}$ is of type $B(1,1)$ ⊗—•, it has roots of norm 0, negative norm, and positive norm and $\dim \mathfrak{g} < \infty$. We show that:

Theorem *Let $\mathfrak{g}$ be a Lie superalgebra satisfying the following conditions:*

1. *$\mathfrak{g}$ has a non-degenerate symmetric, invariant bilinear form $(\cdot,\cdot)$.*
2. *There exists an even subalgebra $\mathfrak{h} \leq \mathfrak{g}$, which is self-centralizing. The eigenspaces for $\mathfrak{h}$ are finite dimensional and $\mathfrak{g}$ is the sum of the $\mathfrak{h}$-eigenspaces. The eigenvalues $\neq 0$ are called roots.*
3. *$\exists\, h \in \mathfrak{h}$ such that $C_{\mathfrak{g}}(h) = \mathfrak{h}$ and $\forall\, r \in \mathbf{R}$, there exists only finitely many roots α s.t. $|\alpha(h)| < r$.*
4. *A root α is called of finite type if $\forall$ roots β, $n\alpha + \beta$ is a root for only finitely many integers n.*

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A root α is called of infinite type either if $|\alpha| = 0$ and α is not of finite type or if $\forall$ root β s.t. $(\alpha, \beta)(\alpha, \alpha) > 0$, $\alpha + \beta$ is a root unless $|\beta| = 0$, β is of finite type and $\alpha - \beta$ is a root.
All roots are of finite or infinite type.

5. *Let α, β be positive roots of infinite type or of norm 0. Suppose that $\forall$ roots γ s.t. $0 < \gamma(h) < \alpha(h)$, $[x, \mathfrak{g}_{-\gamma}] = 0$ for $x \in \mathfrak{g}_\alpha$. Then $(\alpha, \beta) = 0$ implies $[x, \mathfrak{g}_\beta] = 0$.*

Then $\mathfrak{g}$ is a direct sum of GKM superalgebras, finite dimensional simple classical Lie superalgebras and affine Lie superalgebras.

KYOJI SAITO

Non-negativity of Dirichlet coefficients of elliptic L-functions

Let $c_{(R,G)}$ be the Coxeter element for a marked elliptic root system (R, G) and let us decompose the characteristic polynomial $\det(\lambda I - c_{(R,G)})$ in the form $\prod(\lambda^i - 1)^{e(i)}$. The elliptic eta-product $\eta_{(R,G)}(\tau)$ attached to (R, G) is $\prod \eta(i\tau)^{e(i)}$, where $\eta(\tau)$ is the Dedekind eta-function. Using $m = 24/(24, \sum ie(i))$, the eta-product gets the Fourier expansion $\eta_{(R,G)}(m\tau) = \sum_{n\in\mathbf{Z}} c(n)q^n$. So we define the elliptic L-function $L_{(R,G)}(s)$ attached to (R, G) by $L_{(R,G)}(s) := \sum_{n=1}^{\infty} c(n)n^{-s}$. The main result of the talk is:

Theorem: *There exists a Kummer extension $E_{(R,G)} := \mathbb{Q}(\zeta_m, \chi^{Y_{m^*}})$ for the cyclotomic field $\mathbb{Q}(\zeta_m)$ ($\zeta_m = exp(2\pi\sqrt{-1}/m)$) and representation(s) ρ (or $\rho^{(\pm)}$) : $\mathrm{Gal}(E_{(R,G)}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbf{Z}[\sqrt{-1}])$ such that the elliptic L-function is one of the following forms:*

$$L_{(R,G)}(s) = \begin{cases} L(\rho, s), \\ \frac{1}{4}(L(\rho^{(+)}, s) - L(\rho^{(-)}, s)), \text{ or} \\ \frac{\sqrt{-1}}{4}(L(\rho^{(+)}, s) - L(\rho^{(-)}, s)) \end{cases}$$

where $L(\rho, s)$, etc. are Artin L-functions attached to the representations ρ, etc., respectively.

As a consequence, one gets explicite formulae for the Dirichlet coefficients $c(n)$. Then, one verifies the equivalence : $c(n) \geq 0 \quad \forall n \in \mathbf{Z}_+ \Leftrightarrow \eta_{(R,G)}$ is not cuspidal $\Leftrightarrow$ (R, G) is of type either $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ or $E_8^{(1,1)}$. After the talk V. Kac pointed out strange coincidence of the eta product for the above 4 cases with some finite algebras of rank 2.

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CHRISTOPH SCHWEIGERT

Twining characters and orbit Lie algebras
(based on joint work with J. Fuchs, U. Ray and A.N. Schellekens)

We consider some algebraic structures associated to a class of outer automorphisms of generalized Kac-Moody (GKM) algebras. The twining character is defined as a character-valued index, i.e. the trace of the action of the outer automorphism on certain irreducible highest weight modules.

The main theorem states that the twining character is essentially the ordinary character of some other GKM, the orbit Lie algebra.

Applications to moduli spaces of flat bundles, to two-dimensional rational conformal field theory have been presented. A conjecture for the Verlinde formula for non-simply connected structure groups has been proposed.

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ERIC VASSEROT

Hall Algebras and quantum groups at roots of 1

Notation:

$$\begin{aligned} \epsilon &= exp(2i\pi/n) \\ U_\epsilon(sl(d)) &= \text{restricted specialization} \\ V_\epsilon(\lambda) &= \text{simple module} \\ W_\epsilon(\mu) &= \text{Weyl module} \end{aligned}$$

Theorem (Kazhdan, Lusztig, Kashiwara, Tannisaki):
$[W(\mu) : V(\lambda)]$ *is given by the value at 1 of some Kazhdan-Lusztig polynomials of type* $A^{(1)}_{d-1}$.

Goal: We explain an alternative approach via global basis of the Hall algebra of the cyclic quiver acting on the Fock space.

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WEIQIANG WANG

Dual pairs and infinite dimensional Lie algebras

Let $\Psi_m^{\pm,k}$, $m \in \frac{1}{2} + \mathbf{Z}$, $k = 1, \ldots, l$ be the generators of a Clifford algebra Cl: $\{\Psi_m^{\pm,p}, \Psi_n^{\pm,q}\} = 0$, $\{\Psi_m^{+,p}, \Psi_n^{-,q}\} = \delta_{p,q}\delta_{m,-n}$. We denote by $\mathcal{F}^{Dl}$ the Fock space of Cl with a highest weight vector annihilated by $\Psi_m^{\pm,k}$, $m > 0$, $k = 1, \ldots, l$.

On $\mathcal{F}^{Dl}$ one has the natural action of an infinite dimensional Lie algebra $\widehat{\mathfrak{gl}}_\infty$. $\widehat{\mathfrak{gl}}_\infty$ admits Lie subalgebras of B, C, D types, denoted by b_∞, c_∞, d_∞ respectively. There is also a natural action of the affine Lie algebra $\widehat{so(2l)}$ (and $\widehat{\mathfrak{gl}(l)}$) on $\mathcal{F}^{Dl}$. The action of the horizontal subalgebra $so(2l)$ (resp. $gl(l)$) of $\widehat{so(2l)}$ (resp. $\widehat{\mathfrak{gl}(l)}$) can be lifted to the orthogonal group $O(2l)$ (resp. $\mathrm{GL}(l)$). The actions of $\mathrm{GL}(l)$ and $\widehat{\mathfrak{gl}}_\infty$ (maximally) commute with each other, and thus form a dual pair in the sense of R. Howe. We also show that $O(2l)$ and d_∞ form a dual pair, and present an explicit decomposition of $\mathcal{F}^{Dl}$ into the isotopic subspaces of $O(2l)$ and d_∞, which admits many favourable properties. We also indicate that there are several variations which produce many other dual pairs between classical Lie groups and infinite dimensional Lie algebras.

Furthermore, one has a natural Lie algebra homomorphism from $W_{1+\infty}$ to $\widehat{\mathfrak{gl}}_\infty$, where $W_{1+\infty}$ is the central extension of the Lie algebra of differential operators on the circle. We show that one can replace $\widehat{\mathfrak{gl}}_\infty$ (resp. d_∞) by $W_{1+\infty}$ (resp. some natural Lie subalgebras of $W_{1+\infty}$) in the above discussion. The $W_{1+\infty}$ and its subalgebras are most natural from the viewpoint of vertex algebras.

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