MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 13/1969

Kontinuumproblem

11.5. bis 17.5.1969

Unter der Leitung von Prof.Dr. G.H. Müller (Heidelberg) und Prof.Dr. D. Scott (Amsterdam) fand in der Zeit vom 11. bis 17.5. 1969 im Mathematischen Forschungsinstitut Oberwolfach eine Tagung über das Kontinuumproblem statt. Von den aus allen Teilen der Welt eingeladenen 48 Teilnehmern wurden insgesamt 20 Referate gehalten.

Teilnehmer:

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Berendregt, H.P., Utrecht
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Davis, M., London
Derrick, J., Leeds
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Vortragsauszüge (in zeitlicher Reihenfolge)

P. HAJEK: Survey of recent results concerning the continuum hypothesis

1) There are recursive formulas expressing all powers by means of the unary function $\daleth(\aleph_\alpha) = \aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}$.

2) If cardinals $\aleph_\alpha, \aleph_\beta$ are defined by "good" definitions and if ($\aleph_\alpha$ regular and $\mathrm{cf}(\aleph_\beta) > \aleph_\alpha$) is provable then the following is consistent: For every γ, (a) $\mathrm{cf}(\aleph_\gamma) < \aleph_\alpha \rightarrow \daleth(\aleph_\gamma) = \aleph_{\gamma+1}$,

(b) $\aleph_\alpha \leq \mathrm{cf}(\aleph_\gamma) < \mathrm{cf}(\aleph_\beta) \rightarrow \daleth(\aleph_\gamma) = \aleph_\beta \cdot \aleph_{\gamma+1}$,

(c) $\mathrm{cf}(\aleph_\beta) \leq \mathrm{cf}(\aleph_\gamma) \rightarrow \daleth(\aleph_\gamma) = \aleph_{\beta+1} \cdot \aleph_{\gamma+1}$.

Consequences for the powers of 2:

$\aleph_\gamma < \aleph_\alpha \rightarrow 2^{\aleph_\gamma} = \aleph_{\gamma+1}$, $\aleph_\alpha \leq \aleph_\gamma < \mathrm{cf}(\aleph_\beta) \rightarrow 2^{\aleph_\gamma} = \aleph_\beta$,

$\mathrm{cf}(\aleph_\beta) \leq \aleph_\gamma \rightarrow 2^{\aleph_\gamma} = \aleph_{\beta+1} \cdot \aleph_{\gamma+1}$.

This is the consistency result on violating GCH (generalized continuum hypothesis) "at $\aleph_\alpha$". In a similar way, the GCH can be violated simultaneously at all regular cardinals.

3) Assuming something on the power of continuum, what statements remain consistent?

(a) Projective hierarchy.

(1) CH + there is a projective well-ordering of reals

(2) CH + there is a projective set of the power of continuum without perfect subsets, a projective set without the property of Baire, a projective non-Lebesgue measurable set.

(3) CH + every projective well ordering of (some) reals is at most countable.

(4) CH + every projective set is Lebesgue measurable, has the property of Baire and if uncountable contains a perfect subset.

(5) non CH + there is a projective set of power exactly $\aleph_1$

(6) non CH + as in (4)

(b) Duality of the notions "Lebesgue measure 0" and "first category".

Definition:

(K) ≡ every set of power less than continuum is of first category

(M) ≡ every set of power less than continuum has measure 0.

(L) ≡ there is a set of power of continuum which intersects every set of first category in an at most countable set.

(S) ≡ there is a set of power of continuum which intersects every set of measure 0 in an at most countable set.

Theorem. CH ≡ (K + L) ≡ (M + S)

Consistent assumptions:

(7) L + non K + M + non S

(8) S + non M + K + non L

(c) Suslin hypothesis. (SH) ≡ every complete dense linear ordering in which every system of disjoint open intervals is at most countable contains a dense countable subset.

Consistent assumptions:

(9) non SH + CH

(10) non SH + non CH

(11) SH + non CH.

Problems: consistency of SH + CH; consistency of the existence of projective sets of power $\aleph_2$ if $2^{\aleph_0} > \aleph_2$.

I. JUHASZ: How to Generalize the Suslin Problem

It is well known that many problems of topology involving cardinals depend essentially on the (generalized) continuum hypothesis or other assumptions on cardinals. It is more surprising, however, that several such problems seem to be closely connected to the Suslin problem.

1. Does there exist any non-separable regular space which does not contain an uncountable discrete subspace?

2. Does there exist a hereditarily Lindelöf regular space which is not separable?

3. Does there exist a first countable T_1-space with the Suslin property which is not separable?

4. (Ponomariov) Does there exist a non-separable, perfectly normal compact Hausdorff space? (Perf. normal means that every closed set is a G_δ). We can observe that this problem is "contained" in any of the above three.

5. Is every compact Hausdorff space with the Suslin property the

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continuous image of a product of separable compact spaces? Although all these problems are of a much more general character than the Suslin problem itself, the only counter-examplr known to the author is a Suslin continuum. Therefore, the following question seems to be justified: Is the Suslin problem just a particular case of a more or less general "principle "?

A. MOSTOWSKI: Models of the second order arithmetic

Consider the relational systems $M = \langle N, S, +, \times, \varepsilon \rangle$ where the elements of N are the "integers of M", elements of S are the "sets of M", + and × are the "arithmetical operations of M" and ε is the "elementhood relation of M". If all the axioms of the second order arithmetic are valid in M, then M is called a weak model. If M is elementarily equivalent with the standard model $M_0 = \langle \omega, P(\omega), +, \times, \in \rangle$, then M is called a strong model. In the lecture several examples were given of weak and strong models with singular properties. The main result was the existence of a strong model M whose standard part M^+ is not even a weak model (the integers of M^+ are the ordinary integers and its sets have the form $\{n : n \varepsilon x\} \cap \omega$ where x is a set of M). Situations were described where the problem of existence of a model satisfying given condition depends on assumptions made in the meta-system. It was printed out that the question whether the set W of sentences true in M_0 is constructible and if so what is its place in the constructive hierarchy depends on meta-mathematical assumptions. Led by some analogies between this example and the continuum problem, the author expressed the opinion that future mathematics will perhaps reject the full axiom of choice and that the continuum problem might then lose its importance.

J. DRAKE: The Status of the Continuum Hypothesis in Some Generic Extensions

Let M be an arbitrary well-founded model of ZF, in which the C.H. is false. It is known that if a is a generic ultrafilter on the Boolean algebra $RO(2^{\omega_1}, < \omega_1 \text{ top.S.})$ (for adding a new subset of

ω_1 but no new subset of ω) then in the extension M[a] the continuum hypothesis holds: i.e. 2^{ω} collapses to ω_1 in the extension (whatever value it had in M). We discuss the problem of finding similar cases where the status of the continuum in an extension is decided. Boolean algebras satisfying 2^{ω}-chain condition are provided by Sacks forcing (with perfect closed sets), by Mathias forcing and by Silver forcing. In all these cases ω_1 can be shown preserved, but the question is open whether the C.H. holds in the extension. Another problem of the same kind is to find a method to add new subsets to ω_1, but no new subset to ω_0, in such a way that 2^{ω} is not collapsed although $> \omega_1$ in M.

J.E. FENSTAD: On the Axiom of Determinateness

The aim of the lecture was to present a survey of results connected with the axiom of determinateness.

We first gave an introduction to the work of Addison and Moschovakis (1967) concentrating on the prewellordering theorem and some of the consequences thereof (in particular reduction principles).

In the second part of the lecture we gave the known positive results on determined games, the best result being due to M. Davis (1964), stating that every $F_{\sigma\delta} \cup G_{\delta\sigma}$ set is determined.

We next presented a simple example of a non-determined game, assuming the existence of a non-principal ultrafilter on N (due to S. Aanderaa): Let D be such a filter, then the set

$X_D = \{\alpha \in N^N \mid \{i \mid \mu j[\alpha(j) \geq i] \text{ is even } \} \in D\}$ is non-determined.

Using the axiom of constructibility we showed (following Mycielski, 1964) that there is a $\underset{\sim}{\Pi}^1_1$ non-determined set. The main open problem in this area seems to be whether every Borelgame is determined.

Concerning consistency results we gave the result of Solovay (1967) that Cons (ZF + Ax.Det.) implies that Cons (ZF + AC + "there exists a measurable cardinal").

In general it seems that the continuum must be very large if the full Ax.Det. is adopted, there is e.g. the following recent result of Moschovakis = Let $\underset{\sim}{\delta}^1_n$ be the least ordinal not the ordertype of a $\underset{\sim}{\Delta}^1_n$ prewellordering of the reals. If we assume full Ax.Det., then

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$\underset{\sim}{\delta}^1_n$ is a cardinal $\geq \aleph_n$.
On the other hand it seems that certain versions of "definable determinateness" is provable on the assumption that large cardinals exist, e.g. D. Martin has shown that $\underset{\sim}{\Sigma}^1_1$- determinateness follows from the existence of a measurable cardinal.

J. PARIS: L[D] and G.C.H. and Large Cardinals

For A a κ-additive ultrafilter on κ in V ($\kappa > \omega_0$) we can iterate the ultrapower construction with respect to A to obtain a sequence of well founded classes $V_A^{(\alpha)}$ and embeddings $E_{\alpha\beta}: V_A^{(\alpha)} \to V_A^{(\beta)}$ for $\alpha \leq \beta$. Using this process Kunen has shown:
i) If D_1, D_2 are normal ultrafilters on κ_1, κ_2 in $L[D_1]$, $L[D_2]$ respectively and $\kappa_1 \leq \kappa_2$ then $\exists \alpha$ s.t. $D_2 = E_{0\alpha}(D_1)$ (taken in $L[D_1]$).
ii) If κ is measurable in V and $|2^\kappa| > \kappa^+$ then V contains a (set) model for ZFC + $\exists$ m.c.. In fact a theory $0^\dagger$ exists which corresponds to L[D] (D a normal ultrafilter) just as $0^\#$ corresponds to L.
On the "positive" side however for any subset a of κ of regular cardinals which is measure zero w.r.t. some normal measure on κ in V $\exists$ a Boolean extension of V in which κ is still measurable and G.C.H. fails at all $\alpha \in a$.

R.B. MANSFIELD: The measurable cardinal and Σ^1_3 sets

We use the somewhat surprising fact that if κ is a measurable cardinal, sequences of ordinals with length κ can be Gödel numbered by a single ordinal to define trees for Π^1_2 sets. To each Π^1_2 formula φ we associate an ordinal definable tree T^*_φ such that for any real number α $\exists\beta\ \varphi(\alpha,\beta)$ relativizes to $L(T^*,\alpha)$. We then can go on to prove the analogs of the Kondo-Addison theorem and the perfect set theorem for Π^1_2 sets. Also, if $\omega_1^{(L(T^*))}$ is countable, every Σ^1_3 set is Lebesgue measurable.

A. LEVY: On the decomposition of sets of reals to Borel sets

Let us say that a set A of real numbers has the decomposition property if it is the union of at most $\aleph_1$ Borel sets. The basic facts about this property in the set theory ZFC (Zermelo-Fraenkel set theory with the axiom of choice) are as follows.
(1) Every Σ^1_2-set has the decomposition property (a classical result). (2) It is not provable that any set other than a Σ^1_2-set has the decomposition property (Martin-Solovay). (3) If $2^{\aleph_0} > \aleph_1$, then there is a set which does not have the decomposition property (easily seen). (4) If ZFC is consistent with the existence of an inaccessible cardinal then ZFC is consistent with $2^{\aleph_0} = \aleph_\Theta$, where Θ is any "reasonable" fixed ordinal and with "every real-ordinal-definable set or reals has the decomposition property".

F.W. LAWVERE: Categorical Logic and Models of Generalized Set Theories

It is suggested that the categories corresponding to Boolean models in their own right (i.e. without dividing by ultrafilter) and in fact considerably more general "models" are mathematically interesting (somewhat as arbitrary commutative rings, not only fields, are mathematically interesting). Specifically it is pointed out that for any small cat $\mathbb{B}$ equipped with Grothendieck topology, the category of sheaves sh($\mathbb{B}$) satisfies not only the "topos" axioms of Giraud (e.g. cartesian closed, etc.) but also has a truth-value object $2_{\mathbb{B}}$ in the sense that for all X, the subobjects of X $\overset{\leftrightarrow}{1\text{-}1}$ morphisms $X \to 2_{\mathbb{B}}$. For example $\mathbb{B}$ can be a complete Boolean algebra with the "canonical" topology. The "generalized set theory" is formalized by uniformly expressing λ-conversion, recursion, logic axioms etc. in terms of adjoint-functors.

R.O. GANDY: Subsystems of 2nd-order Arithmetic

The hyperarithmetic sets may be characterised (a) As the sets ($\subseteq \omega$) which are strongly representable in 2nd-order arithmetic (Z_2); (b) as the sets which are recursive in the "jump" operator: $J\alpha = \{\mathfrak{z}: \{\mathfrak{z}\}(\alpha)$ is defined$\}$; (c) as the minimum β-model for the Δ_1^1 comprehension axiom. Enderton has considered the $\mathfrak{A}$ rule: If $(\exists\alpha)(\forall n) \vdash \not\!\phi\, (\underset{\sim}{\bar{\alpha}}(n))(\underset{\sim}{m} = \text{numeral for } m)$ then $\vdash (\exists\alpha)(\forall n)\, \not\!\phi\, (\bar{\alpha}(n))$. We have defined the superjump $\mathcal{E}$ by $\mathcal{E}^3(\underset{\sim}{F}^2) = \{\mathfrak{z}: \{\mathfrak{z}\}(\underset{\sim}{F}^2)$ is defined $\}$. Theorem: The sets strongly represented in Z_2 + $\mathfrak{A}$-rule are just those recursive in $\mathcal{E}$; they are also those sets X such that X and ω-X are many-one reducible to sets inductively defined by a Σ_1^1 clause. We do not know any analogue to (c) above. Δ_2^1 comprehension axiom is too weak, and Σ_2^1 comprehension axiom is too strong.

P. HINMAN and P. ACZEL: Representability in Extensions of Arithmetic

Our results concern the sets weakly and strongly representable in the system $\mathfrak{A}$ of Enderton (cf. Gandy's note) and other systems. Let $E_1^{\#}\ (\varphi) \simeq$ (0, if $\forall\alpha\exists x[\varphi(\bar{\alpha}(x)) = 0]$; 1, if $\exists\alpha\forall x[\varphi(\bar{\alpha}(x))>0]$ undefined, otherwise). Theorem 1: For any $A \subseteq N$, equivalently (1) A is weakly $\mathfrak{A}$-representable (2) A is 1-1 reducible to a set given by a monotonic Σ_1^1 inductive definition (3) A is semi-recursive in $E_1^{\#}$. Theorem 2: For any $A \subseteq N$ equivalently, (1) A is strongly $\mathfrak{A}$-representable (2) A and N-A are both weakly $\mathfrak{A}$-representable (3) A is recursive in $E_1^{\#}$: The equivalence of (2) and (3) uses the prewellordering theorem for sets semi-recursive in $E_1^{\#}$. $E_1^{\#}$ is shown to be much stronger than $E_1 = E_1^{\#} \upharpoonright N^N$ in fact to be at least as strong as Gandy's superjump (one version). However, application of a stronger superjump to $E_1^{\#}$ and iteration leads to much more extensive subclasses of Δ_2^1. The sets recursive in E_1 are those strongly representable in a weakened $\mathfrak{A}$-system which allows $\mathfrak{A}$-inferences applied only to Φ such that $\forall n[\ \vdash_{\mathfrak{A}} \Phi(\bar{n}) \vee \vdash_{\mathfrak{A}} \neg\, \Phi(\bar{n})]$. Theorem 1 extends to systems J with

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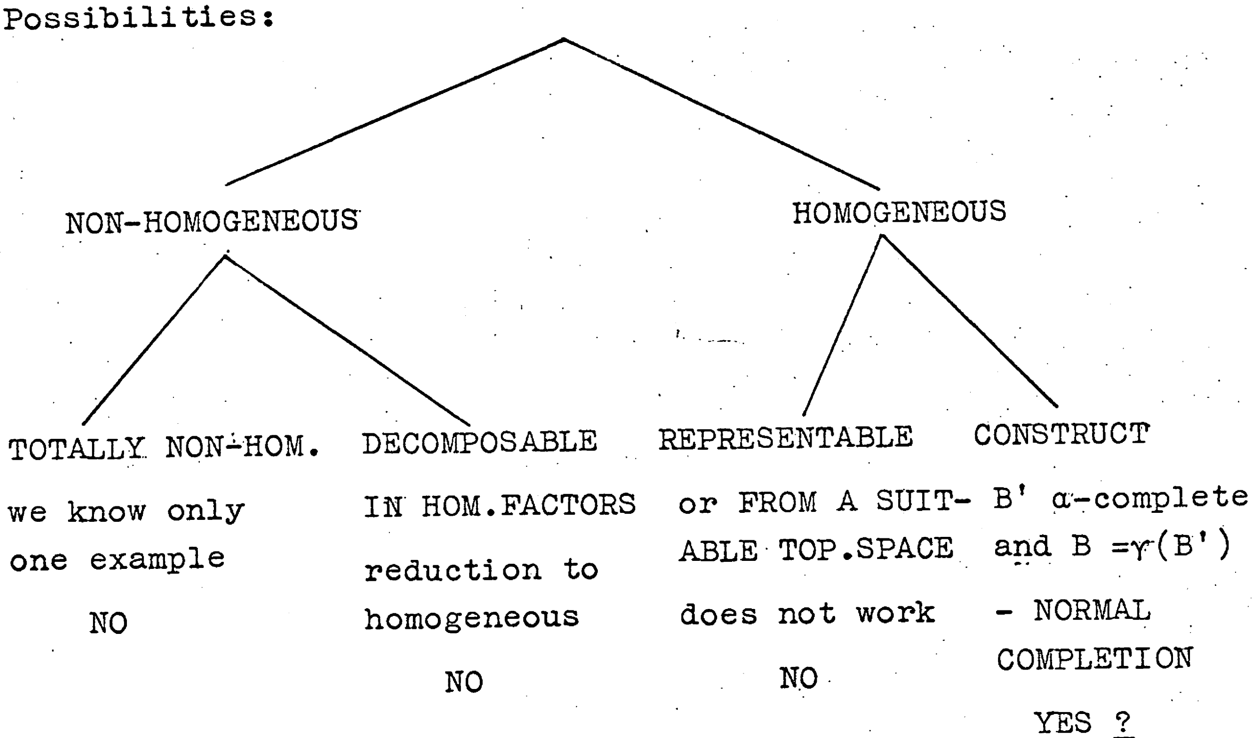
the additional quantifier symbol Q and the rules (a) if $\exists\alpha\in J\forall x \vdash_J \Phi(\alpha(x))$ then $\vdash_J Qx\Phi(x)$; (b) if $\forall\alpha\in J\exists x \vdash_J \Phi(\alpha(x))$ then $\vdash_J \neg Qx\neg \Phi(x)$, and the corresponding functional $E^{\#}_J$. For theorem 2 we must add the ω-rule and the functional ${}^2E$.

L. BUKOVSKY: <u>Constructing a Suitable Boolean Algebra</u>

Assume V=L. We want to construct a complete Boolean algebra B s.t. in the model ${}^BV$ 1) $(\exists x)(x\subseteq\omega_{\omega_0} \wedge x\notin L)$
2) $(\forall n)(\forall y)(y\subseteq\omega_n \wedge n\in\omega_0 \to y\in L)$.
Equivalent conditions for B: 1) $(\omega_{\omega_0},2)$-nondistributive
2) $(\omega_n,2)$-distributive for $n\in\omega_0$.
Possibilities:



<u>Theorem</u> (for completion): Let B be an α-complete B.alg. with α-generators. If α^+ is not collapsed in ${}^{\gamma(B)}V$, then B is (β,δ)-distr. = $\gamma(B)$ in (β,δ)-distr. for any $\delta^\beta\leq\alpha$.

T. JECH: Trees

We are considering uncountable trees whose all levels are countable. A natural question is how many long branches does such a tree have, if any. One extreme is when a tree has no branch of length ω_1. Existence of such trees is a classical result. The generalization of this property for bigger cardinals gives, in the inaccessible case, a characterization of weakly compact cardinals. Existence of a tree without long branches and without big antichains is equivalent to the famous Suslin's problem. Models are produced both for positive and negative solution. Similarly, there are models for both Kurepa's conjecture and its negation (which is the other extreme): no tree has too many branches (i.e. $\geq \aleph_2$). Moreover, in the constructible universe L, there are both Suslin and Kurepa trees.

L. HENKIN: Multi-models

L, a first-order language. If $\mathfrak{M}$ is an L-structure and R an equivalence rel'n on M, we call $\mathfrak{M}^* = \langle \mathfrak{M}, R \rangle$ a multi-L-structure. Let $\langle M_i \rangle_{i \in I}$ be the R-partition of M, and set $W = \bigcup_i {}^{\omega}M_i (\subseteq {}^{\omega}M)$. With respect to $\mathfrak{M}$, each formula φ of L determines the set $\overline{\varphi}$ of all $x \in {}^{\omega}M$ which satisfy φ; similarly, with rep. to $\mathfrak{M}^*$, we define the set $\overline{\varphi}^*$ of all those $x \in W$ which satisfy* φ, specifying, e.g., $\overline{\exists v_k \psi}^* = \{x \in W \,/\, \exists y \in \overline{\psi}^*, y_\lambda = x_\lambda \text{ for all } \lambda \neq k\}$. If Γ is a set of sents. of L, a multi-model of Γ is an $\mathfrak{M}^*$, $\overline{\varphi}^* = W$ iff $\Gamma \vdash \varphi$. Every consistent Γ has a multi-model. Theorem: Let $\mathfrak{M}^*$ be any multi-L-structure; then for each formula φ of L there is some $\psi \in L$, $\overline{\varphi} \cap W = \overline{\psi}^*$. Similar results hold for languages L_n with only n individ. vars. (where $W = \bigcup_i {}^{n}M_i \subseteq {}^{n}M$), and for related cylindric algebras. A form of "elimination of quantifiers" is used in the proof.

I. JUHÁSZ: Some more problems on topology

1) (P.S. Alexandrov) Is every first countable compact T_2-space of cardinality $\leq 2^{\aleph_0}$? 2) Is it true that for every hereditarily separable T_2-space R, $|R| \leq 2^{\aleph_0}$? 3) Does there exist a T_2-space which is (ω_n, ω)-compact for all $n<\omega$ but not (ω_ω, ω)-compact?

K. McALOON: A theorem of Krivine

Theorem: Every Boolean Algebra of power $\aleph_\alpha$ can be embedded in the algebra of regular open sets of ${\aleph_\alpha}^{\aleph_0}$.

A. OBERSCHELP: Bemerkungen zum Platonismus

Es wurde vorgeschlagen (nach Carnap), die Diskussion über die Existenz von Objekten zu ersetzen durch eine Diskussion über die Wahl eines Sprachrahmens. Die Gründe für die Wahl einer platonistischen Sprache sind dann nicht so sehr verschieden von den Gründen, die zur Annahme eines physikalischen Systems führen. Außerdem wurde darauf hingewiesen, daß die Tatsache, daß es verschiedene mengentheoretische Systeme gibt, kein Anlaß ist, von der An-sich-Auffassung abzugehen, da diese Systeme ja auch verschiedene Arten von Mengen (oder Klassen) beschreiben sollen. Schließlich wurden einige "Sowohl-als-auch"-Argumente vorgebracht, die sich nicht nur gegen die Mengenlehre richten, sondern auch gegen die (besser begründete) Zahlentheorie. Wenn diese Argumente dann gegen die "bessere" Theorie nicht ernst genommen werden, so verlieren sie auch gegen das platonische System ihr Gewicht.

J. REZNIKOFF: Remarks on the evolution of set theory

Starting from the feeling that the Axiom of Determinateness is certainly consistent with ZF (without AC.) one wonders what the situation is. New axioms are found and everything is put up and down, usual notions are distroyed or loose their meaning (e.g. every set is Lebesgue measurable). And then either one desires some notions to have a basic meaning either ...? But the situation is not new, when Axiom of Zermelo appeared there came a

"trouble". Recalling of the attitude of Russel (who thought it is contradictory) and in France that of Borel (perhaps the most impressed and looking to narrow constructivism), Lebesgue (calling himself a "Kroneckerian"), Baire (denying even the existence of the power set of ℕ), and, opposite, that of Hadamard (admitting Zermelo's axiom on the same level as others and denying even interest to Hilbert's attempts in proving consistency) whose attitude prevailed for many years in France (see e.g. Bourbaki), one sees that not only the Axiom of Choice played a role in the evolution of Mathematics but also in some mathematical careers... Of course the present situation is different, but is it really so different? Looking to the past experience one could suggest 1) To try to accelerate the evolution by finding new axioms of non constructible existential character 2) Return to Proof theory (trying to settle the axioms by sharper deduction considerations e.g. infinite) 3) For teaching mathematics: one has not necessarily to choose between set theoretical doubtful frame or intuitionistic one (or Markov's) some alternative can certainly be found (see for instance Bishop's <u>Foundations of Constructive Analysis</u>, 1967)

D. SCOTT: <u>On the Future of Set Theory</u>

We discussed at this conference many independence proofs and technical results but did so without much regard for their foundational significance. One simple point in connection with the continuum hypothesis (CH) that should be kept in mind is this: There are many properties (P(m) of cardinals for which we can proof <u>without</u> any hypothesis that $\aleph_1$ is the <u>unique</u> cardinal having this property. Thus P($2^{\aleph_0}$) is a "cheap" form of (CH), e.g. P(M) could be: every set of cardinality m is the union of a chain of its countable subsets. For more "essential" applications of (CH) one should consider such propositions as (K) & (L) (cf. the lecture of Hajek) or problems as: the <u>existence of a $2^{2^{\aleph_0}}$ chain of sets of reals</u> or <u>the existence of a $2^{2^{\aleph_0}}$ family of "almost" disjoint sets of reals</u> ("almost" disjoint means

having a countable intersection).
Aside from simply giving up set theory in the face of the independence proofs, there seem to be two attitudes both of which might be called "realist" but one is <u>absolute</u> and the other <u>relative</u>. Both hold that the notion of <u>set</u> (better <u>set of</u>as in the usual cumulative hierarchy) is definite and that the questions (say, of cardinal arithmetic) are precise. The absolute position claims that the set of <u>all</u> subsets is an "absolute" totality but agrees that the current axioms have not determined all its properties. The models for independence results do not distroy this faith in the "complete" powerset since the meaning of set in the new model is clearly "unintended". What is needed is the discovery of "new" and "correct" axioms. On the other hand the relative position questions the idea of a "final" powerset because the models show how easy it is to adjoin "new" subsets which, of course, appear unintended from the old model.
What is needed now for the sake of the relative position is a good theory of the variety of (well-founded!) models so we can appreciate the sence and order of the <u>various</u> possible cardinal arithmetics - the notion of cardinal being precise but <u>relative</u> to the model. If a reasonable theory is forthcoming we might then be satisfied with a "potential" concept of powerset. In view of the really remarkable number of "mathematical" consequences of various hypotheses (such as V = L, Martin's Axiom, Measurable Cardinals, Axiom of Determinateness) the proper theory of models for set theories should be very respectable. Whether it is a good <u>foundation</u> will have to be answered in the light of consideration of the properly formulated theory.

A. Prestel (Bonn)

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