MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Kategorien

23. 7. bis 29. 7. 1972

Unter der Leitung der Herren J. Gray (Urbana) und H. Schubert (Düsseldorf) fand eine Tagung über Kategorien statt.

Teilnehmer

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Vortragsauszüge

Baumgartner, K.: Categories admitting free algebras.

Let A be any small algebraic theory and D be an arbitrary category. Then we consider the full subcategory ${}_\pi[A,D]$ of product-preserving functors (i.e. of algebras) in $[A,D]$. The central question is, whether the functors I: ${}_\pi[A,D] \to [A,D]$ and V: ${}_\pi[A,D] \longrightarrow D$ are r-adjoint.

Now for a functor U: D $\to$ M consider the following diagramm:

$$\begin{tikzcd}
 & {[A,D]} \arrow[rr, "{A,U}"] & & {[A,M]} \\
{{}_\pi[A,D]} \arrow[ur, "I"] \arrow[rr, dashed, "{}_\pi U"'] \arrow[d, dashed, "V"] & & {{}_\pi[A,M]} \arrow[ur, "I'"] \arrow[d, dashed, "V'"] & \\
D \arrow[rr, "U"] & & M &
\end{tikzcd}$$

If U: D $\to$ M preserves products ${}_\pi U = [A,U] \cdot I$ is called the algebraic lifting of U. Now we consider two problems:

P 1. Under what conditions ${}_\pi U$ is (together with U) a r - adjoint?

<u>P 2.</u> Consider a factorisation:

$$\begin{tikzcd} & \bullet \arrow[dr, "U"] & \\ \bullet \arrow[ur, "V = K"] \arrow[rr, shift left, "U'"] & & \bullet \arrow[ll, shift left, "F'"] \end{tikzcd}$$

of a r - adjoint U' over a faithful functor U. Under what conditions V is a r - adjoint? (Note, that if ${}_\pi U$ and V' are r - adjoints, then P 2 applies for V' ${}_\pi U$ = U' = UV).

A detailed study of P 1 and P 2 finally yields:

<u>Theorem:</u> Let A be a small algebraic theory and let D be a category satisfying

(1) D has products

(2) D has natural ($\mathcal{M}$-Mono, $\mathcal{E}$-Epi) factorisations

(3) D is $\mathcal{E}$ - cowell prowered

(4) For any cardinal number n always $f \in \mathcal{E}$ implies $f^n \in \mathcal{E}$

Assume further that M is a $\mathcal{E}$-epi - coreflexive subcategory of D, such that V': ${}_\pi[A,M] \to M$ is already a r - adjoint. Then for any $\mathcal{E}$ - epi - reflexive subcategory $\overline{D}$ of D the functors $\overline{V}$: ${}_\pi[A,\overline{D}] \to \overline{D}$ and $\overline{I}$: ${}_\pi[A,\overline{D}] \to [A,D]$ are r - adjoints.

<u>Final remarks:</u>

(1) Let D be a coreflexive subcategory of <u>Top</u> resp. <u>Unif</u>, then for M = Ens the theorem shows the existence of free algebras over topological categories.

(2) Starting with M = Komp^o one gets the same result for coalgebras.

(3) There are $\mathcal{E}$ - epi - reflexive subcategories $\overline{D}$ of locally presentable, categories D being not locally presentable, but again by the theorem we have free algebras over $\overline{D}$.

<u>Beck,</u> J.: <u>Rational homotopytheory and coalgebras</u>

This is to suggest a new proof of Quillen's theorem (Annals 1969)

$$\mathrm{Ho}_Q\, \underline{\mathrm{Top}}_2 \simeq \mathrm{Ho}\ (\underline{\mathrm{DGC}}/Q)_2 .$$

These homotopy categories are formed by inverting all maps of 2 - reduced spaces $X \to Y$ which induce rational homology isomorphisms (n-reduced means (n-1) connected), and all maps $C \to D$ of DG cocommutative coalgebras/Q which are connected and have $C_1 = D_1 = 0$. The idea of the proof is that the deviation of the diagonal chain map from commutativity is measured by Steenrod operations, but these become trivial over the rational numbers.

Obviously $\mathrm{Ho}_Q\, \underline{\mathrm{Top}}_2 \quad \mathrm{Ho}_Q\, \underline{\mathrm{SS}}_2$, where $\underline{\mathrm{SS}}$ is the category of simplicial sets. The adjoint functors are cotripleable:

$$\begin{tikzcd} & \underline{FD} \arrow[dl, shift left, "U"] & & \\ \underline{SS} \arrow[ur, shift left, "F"] \arrow[rr, "\sim"] & & (\underline{FD})_G \arrow[ul] \arrow[r, "\sim"] & \underline{DGG} \end{tikzcd}$$

Thus $\mathrm{Ho}_Q\, \underline{\mathrm{SS}}_2 = \mathrm{Ho}_Q\, \underline{\mathrm{DGG}}_2$. Here U is the underlying simplicial set functor of simplicial Q-vector spaces, and G = UF is also written for the cotriple on DG Q - vector spaces which corresponds under the Dold-Kan equivalence $\underline{FD} \to \underline{DG}$.

There is also a cotriple W on $\underline{DG}$ whose coalgebras are unstable infinitely homotopy cocommutative DG coalgebras. The cotriple C will be the one whose coalgebras are strictly cocommutative. The first step of the proof is to show, that the cotriple maps $G \to W \leftarrow C$ induce homology isomorphisms

(1) $\quad H_*\ (AG) \longrightarrow H_*(AW) \longleftarrow H_*(AC)$

for every $A \in \underline{DG}$. This requires studying cohomology operations in these categories (which reduce to cup products).

Contensor products supply right adjoints:

$$\underline{DGG} \underset{(\ )\square_W G}{\overset{}{\rightleftarrows}} \underline{DGW} \underset{(\ )\square_W C}{\overset{}{\leftrightarrows}} \underline{DGC}$$

Any $A \in \underline{DGW}$ has a standard cosimplicial resolution $A \to AW^*$. Using the fact that these categories are enriched/$\underline{SS}$, together with simple connectivity, we map A into the corealization $|AW^*| = \varprojlim |AW^*|^{(n)}$ whose finite coskeleta are sub DG modules of $\Pi\, \mathrm{Hom}(\Delta_i, AW^{i+1})$, $o \leq i \leq n$. This is dual to the usual concept of geometric realization. The right derived functor of the cotensor product $A \square_W C$ (for example) is then defined as

$$A \overset{\mathbb{R}}{\square}_W C = |AW^*| \ \square_W C .$$

The right derived functors are well defined on the homotopy categories and give adjoint functors

$$(2) \qquad \mathrm{Ho}\ \underline{DGG}_2 \rightleftarrows \mathrm{Ho}\ \underline{DGW} \leftrightarrows \mathrm{Ho}\ \underline{DGC}$$

In $\underline{DGW}$ we have diagrams

$$(3) \qquad A \longrightarrow |AW^*| \longleftarrow |AW^*| \ {}_W C$$

The first arrow is a homology equivalence by the usual contracting homotopy argument, and the second is such by (1) and a kind of Eilenberg-Moore spectral sequence; similarly for $G \to W$. It is this part of the argument that is uncertain, as there are convergence difficulties. Probably the fact that in all of these coalgebra categories the "fundamental theorem" holds, that every coalgebra is a union of finite type ones, will give the necesasary convergence. Arrows (3) being homology equivalences implies the functors in (2) are equivalences.

These equivalences should be compatilde with the fibration, etc. structures on the homotopy categories, and are probably fairly computable. I do not know how this compares with Quillen's equivalence, which is far harder to compute. It is this incomputability which justifies a new approach to the theorem.

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<u>Benabou</u>, J.: <u>2-dimensional limits and colimits of distributors</u> (or how to glue together categories)

In this work we define two constructions permitting to glue together categories, based on a multiplicative category $\mathcal{U}$, where the attaching maps are $\mathcal{U}$-Distributors.

Precisely: Let $\mathbb{C}$ be a category and Π be a morphism of $\mathbb{C}^*$, considered as a one dimensional bicategory, in the bicategory $\mathcal{U}$- Dist of $\mathcal{U}$ -Distributors; Π being defined by the following data:

1 - for each object A of $\mathbb{C}$, a $\mathcal{U}$-Category $\mathcal{X}_A$

2 - for each α: A → B in $\mathbb{C}$, a $\mathcal{U}$ - distributor $T_\alpha : \mathcal{X}_B \to \mathcal{X}_A$

3 - for each $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathbb{C}$ a morphism of $\mathcal{U}$ - distributors

$$\mu_{\alpha,\beta} : T_\alpha \; T_\beta \longrightarrow T_{\beta\alpha}$$

4 - for each object B of $\mathbb{C}$ a morphism of $\mathcal{U}$ - distributors

$$\eta_B : \mathrm{Id}(\mathcal{X}_B) \longrightarrow T_{\mathrm{Id}(B)}$$

making commutative the following diagrams

$$\begin{tikzcd}
T_\alpha T_\beta T_\gamma \arrow[r, "\mu_{\alpha,\beta} T"] \arrow[d, "T_\alpha \mu_{\beta,\gamma}"'] & T_{\beta\alpha} T_\gamma \arrow[d, "\mu_{\beta\alpha,\gamma}"] \\
T_\alpha T_{\gamma\beta} \arrow[r, "\mu_{\alpha,\gamma\beta}"'] & T_{\gamma\beta\alpha}
\end{tikzcd}
\qquad
\begin{tikzcd}
T_\alpha \, \mathrm{Id}(\mathcal{X}_A) \arrow[d, "T_\alpha \eta_A"'] & T_\alpha \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathrm{Id}(\mathcal{X}_B) \, T_\alpha \arrow[d, "\eta_B T_\alpha"] \\
T_\alpha \, T_{\mathrm{Id}(A)} \arrow[ur, "\mu_{\alpha,\mathrm{Id}(A)}"'] & & T_{\mathrm{Id}(B)} \, T_\alpha \arrow[ul, "\mu_{\mathrm{Id}(B),\alpha}"]
\end{tikzcd}$$

<u>Theorem 1:</u> If $\mathcal{U}$ is right complete and $\otimes$ commutes with right limits, then Π has a 2 - dimensional right limit (i.e: there exists a $\mathcal{U}$ category $\mathcal{X}_\Pi$ together with $\mathcal{U}$ - distributors $J_A : \mathcal{X}_A \to \mathcal{X}_\Pi$ and maps of $\mathcal{U}$ - distributors $J_\alpha : J_A \; T_\alpha \to J_B$ making commutative:

$$\begin{tikzcd}
J_A \; T_\alpha \; T_\beta \arrow[r, "J_A \, \mu_{\alpha,\beta}"'] \arrow[d, "J_\alpha \; T_B"'] & J_A \; T_{\beta\alpha} \arrow[d, "J_{\beta\alpha}"] \\
J_B \; T_\beta \arrow[r, "J_\beta"'] & J_c
\end{tikzcd}$$

($\mathfrak{X}$, J_A, J_α) being universal with respect to this property. Moreover it turns out that the J_A are in fact functors. This construction yields as special cases:

1) The Kleisli category of a triple (take $\mathbb{C} = 1$, $\mathfrak{V} =$ Sets, T_1 a functor)

2) The generalisation of this to the $\mathfrak{V}$ - based case, or to the "pro - triples"

3) The construction given by Grothendieck in "Categories fibrées et descente" of the category associated with a pseudo functor, and its universal property (not given in this paper)

4) taking all the $\mathfrak{X}_A$ to be a fixed monoid $\mathfrak{X}$ in $\mathfrak{V}$ and the T_α, $\mu_{\alpha,\beta}$ to be identities, one gets the $\mathfrak{V}$ category $\mathfrak{X}$ ($\mathbb{C}$) (the group ring when $\mathfrak{V}$ = Ab and $\mathbb{C}$ is a group).

With the "obvious" definitions we have:

<u>Theorem 2:</u> If $\mathfrak{V}$ is closed symmetric and left complete any comorphism π from a small category $\mathbb{C}^*$ to $\mathfrak{V}$- dist has a 2 - dimensional left limit $\mathfrak{X}^{\pi}$.

We mention a few particular cases, the reader will convince himself that many well known, and apparently different, constructions are covered by this process.

1) The Eilenberg-Moore category (ordinary, $\mathfrak{V}$- base, or with respect to a pro - cotriple).

2) The functor categories $\mathfrak{X}^{\mathbb{C}}$ ($\mathfrak{X}$ a $\mathfrak{V}$- Category and $\mathbb{C}$ a category). In particular the left adjoint of the forgetful functor $\mathfrak{V}$ - Cat $\to$ Cat.

Borceux, F.: Les limites relatives

Soit F: $\underline{A} \to$ Ens un foneteur; le foncteur d'oubli $\Phi_F : \underline{R}_F \to \underline{A}$ défine sur sa catégorie de représentation $\underline{R}_F$ admet l' objet L de $\underline{A}$ comme limite a gauche ssi L est le reflet de F par le plongement de Yoneda Y : $\underline{A} \to$ Nat ($\underline{A}$, Ens); en particulier, le plongement de Yoneda admet un adjoint à gauche dès que $\underline{A}$ est complète à gauche, localement petite et possède un cogñérateur.

Ces faits nous suggèrent de relativiser comme suit la notion de limite soient $\underline{V}$ une catégorie fermée monoidale symétrique, F: $\underline{A} \to \underline{V}$ et G: $\underline{A} \to \underline{B}$ deux $\underline{V}$ - foncteurs; un object L de $\underline{B}$ est dit être la $\underline{V}$ - limite à gauche de G modulo F - (L = $\varprojlim_F$ G) - ssi L est le reflet de F par le foncteur

$$B^* \xrightarrow{Y} \underline{V} - \mathrm{Nat}\ (\underline{B}, \underline{V}) \xrightarrow{[G,1]} \underline{V} - \mathrm{Nat}\ (\underline{A}, \underline{V})$$

die Y est le $\underline{V}$ - plongement de Yoneda de $\underline{B}$.

Cette notion de limite permet d'établir les résultats suivants

(1) Un $\underline{V}$ - foncteur F: $\underline{A} \to \underline{V}$ est $\underline{V}$ - representable ssi

(a) $\varprojlim_F 1_{\underline{A}}$ existe

(b) F commute avec cette limite.

(2) Un $\underline{V}$ - foncteur F:$\underline{A} \to \underline{B}$ admet un $\underline{V}$ - foncteur $\underline{V}$ - adjoint à gauche ssi

(a) $\forall B \in |\underline{B}|$ $\varprojlim_{\underline{B}(B,F-)} 1_{\underline{A}}$ existe

(b) F commute à ces limites

(3) Si F: $\underline{A} \to \underline{B}$ et G: $\underline{A} \to \underline{C}$ sont deux $\underline{V}$ -foncteurs, G admet und $\underline{V}$ - extension de Kan par F dès que

(a) $\forall B \in |\underline{B}|$ $\varprojlim_{\underline{B}(B,F-)}$ G existe

(b) $\forall C \in |\underline{C}|$ $\underline{C}$ (C,-) commute avec ces limites

(ce résultat est dû à Marguerite Zandarin, Louvain)

(4) Si F: $\underline{A} \to \underline{V}$ est un $\underline{V}$ - foncteur et Y: $\underline{A}^* \to \underline{V}$ -Nat $(\underline{A},\underline{V})$ le $\underline{V}$ -plongement de Yoneda de $\underline{A}$, alors $F = \lim\limits_{\vec{F}} Y$.

Comme exemple d' application de ces théorèmes, signalons la dualité de Gel'fand qui fournit une situation d'adjonction relativisée par rapport à la catégorie des espaces topologiques, rendue fermée monoidale symétrique au moyen du bifoncteur de la convergence simple.

Cherenack, P.F.: Algebraic Homotopy Theory

Some of the usual notions of homotopy theory: loop functor, suspension, quotients exist in the category of affine schemes of a countable type over a field k. When k is the real numbers, it is possible to see that the algebraic suspension of the n sphere is the n plus 1 sphere The algebraic suspension functor is the right adjoint of the algebraic loop functor.

Cole, J.C.: Topology = Left - exact adjoints, or 2 is a category of Sets.

To a continous map f: $\langle X, O_X\rangle \to \langle Y, O_Y\rangle$ we may associate an adjoint pair of functors:

$$O_x \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} O_y \qquad f^{-1} \dashv f_*$$

with f^{-1}, the inverse - image map left - exact.

Since a map $f^{-1}: O_y \longrightarrow O_x$ between posets which is cocontinuous automatically has a coadjoint, we consider the category of cocomplete distributive lattices, and cocontinuous lattice homorphisms. (By distributive, we mean that colim's are exact), $\underline{\text{CDL}}$.

Proposition: Sober spaces form a reflective subcategory of $\underline{\text{CDL}}^{op}$.

Proof. To a cocomplete distributive lattice L associate the set of maps $L \to 2$ as the underlying set of the space, topologized in the natural way, setting as opens, $\hat{U} = \{p: L \to 2 : p(u) = 1\}$ for each $u \in L$.

Note that sober spaces form a wide category to walk in, with a sober reflection of any space.

<u>Proposition:</u> Sober spaces form a reflective subcategory of <u>Top</u>.

<u>Proof.:</u> The reflector takes a space to the open set lattice, and then to the reflecting sober space of the above Proposition. One may their define a map between sober spaces to be an adjoint pair, with left-exact left adjoint, between the open set lattices.

A map between elementary toposes is defined to be just such a pair We have for toposes a factorization of maps: Lawvere Tierney

$$\begin{tikzcd} \underline{E} \arrow[rr, shift right, "f_*"'] \arrow[dr, shift left, "a_*"] & & \underline{F} \arrow[ll, shift right, "f^*"'] \arrow[dl, shift left, "b^*"] \\ & \underline{G} \arrow[ul, shift left, "a^*"] \arrow[ur, shift left, "b_*"] & \end{tikzcd}$$

cotripleable   tripleable

i.e a reflects iso's   i.e b fully faithful

For spaces, the adjoint pair $f^{-1} \dashv f_*$ is cotripleable iff f^{-1} reflects isos, if f is surjective, and tripleable iff b_* is fully faithful if f is a subspace embedding, and hence we obtain an identical factorization theorem for maps between sober spaces.

The analogy between spaces as categories over $\underline{2}$, i.e lattices, and toposes as categories over sets, is made clear by the final.

<u>Theorem</u> Sober spaces is a coreflective subcategory of Topos/Sets.

<u>Proof</u> The inclusion is the sheaf category construction, The co-reflection is to examine the image in Sets of the subobject classifier, a cocomplete distributive lattice, which hence has a sober space associated to it. The end adjunction is an iso, whence the inclusion is fully faithful. The front adjunction is given by examining the local sections of an object or a sheaf and re-constructing from them the corresponding sheaf or object.

Deleanu, A.: Localization in homotopy theory and a construction of Adams.

A construction, due to J.F. Adams, for completing a space with respect to a homology theory by using categories of fractions is generalized to triangulated categories. It is shown that the Adams completion generalizes both the localization and the profinite completion of a space in the sense of D. Sullivan. In fact, the completion of a space with respect to a ring, in the sense of A.K. Bousfield and D.M. Kan, is shown to be a particular case of the Adams completion. It is proved that the Adams completion functor is a reflector. The relation between the Adams completion and the Kan extensions of homology theories is also discussed.

Duskin, J.: A representability-interpretation theorem for triple cohomology.

Let $\underline{B}$ be a category with finite inverse limits and $U: \underline{A} \to \underline{B}$ a tripleable functor. Under these limit assumptions, the classifying complex construction $W(K) \to \overline{W}(K)$ of MacLane is defined as well as the augmented k coskeleton triple $Cosk^k$ for augmented complexes in $\underline{A}$ and $\underline{B}$. In particular, for any abelian group object π in $\underline{A}$, the Eilenberg-MacLane complex $K(\pi,n)$ exists. Let $SS\underline{A}[K_1,K_2]$ be the set of homotopy classes of simplicial maps of K_1 into K_2 and $G^\bullet(X)$ be the standard co-triple resolution of an object X in $\underline{A}$ so that the triple cohomology groups $H^n_U(X;\pi)$ are defined.

Remark 1. (homotopy representability);

$$H^n_U(X;\pi) \cong SS\underline{A}\,[G^\bullet(X),K(\pi,n)] \,.$$

Definition: A simplicial fiber space $F \to E \to K(\pi,n)$ in $\underline{A}$ will be called a $K(\pi,n)$-torsor above X (relative to U) provided it satisfies the following conditions:

(a) E is augmented above X and is U-contractible

(b) $U(E) \xrightarrow{\sim} [Cosk^{n-2}(U(E))] \times W(K(U(\pi),n-1))$. If $TORS^n_U(X;\pi)$ is the group of connected component classes of $K(\pi,n)$-torsors, then we have the following theorem (interpretation):
$H^n_U(X;\pi) \xrightarrow{\sim} TORS^n_U(X;\pi)$. The proof is based on the construction of a "standard" $K(\pi,n)$-torsor in each class defined by a given n-cocycle ξ using a theorem of Beck and the following Lemma: (applied for $k = n-1$). There exists a canonical homomorphism leave in OK, the theory outlined hence links with $Ext^n(X;\pi)$ in the following fashion. If $\underline{A}$ has kernels, then the Moore complex of the fiber of a $K(\pi,n)$-torsor gives rise to an "n-fold extension of X by π" $o \to \pi \to N \to \ldots \to X_o \to X \to 1$ which will satisfy certain compatibility conditions (e.g. Whitehead crossed modules in the case of groups).

It links with obstruction theory as follows: let $F \to E \xrightarrow{p} K(\pi,n)$ be a $K(\pi,n)$-torsor and Tr^n be the "truncation at level n" functor. Given a map
$f: TR^{n-1}(G^\bullet(Y) \to TR^{n-1}(F)(\simeq TR^{n-1}(E))$. The V-contractibility of E defines a map $f: G^\bullet(Y) \to E$ whose composition with p is o iff f extends to the entire fiber F. The class $|p\tilde{f}| \in H^n(y,\pi)$ is the desired obstruction. For the example, the fibers of $K(\pi,2)$-torsors are groupoids, cohomology with groupoid coefficients is then applicable and $\underline{SSA}(G^\bullet(X),F) \to TORS_{\underline{V}}(\underline{Y};F)$,

Ehrig, H.: Automata theory in Monoidal Categories

The theory of automata in monoidal categories will be sketched and the main theorems concerning reduction and minimization will be stated.

1.Def. A (Mealy-)automaton is a diagram of the form
$O \xleftarrow{1} S \otimes I \xrightarrow{d} S$ in a monoidal category $(\underline{K},\otimes)$ with unitobject E not I (=Inputobject).

2.Examp. In (Sets,x), (Rel,x), ($\underline{PD}$,x), ($\underline{ND}$,x), ($\underline{Stoch}$,x), $\underline{Mod}_R$,$\otimes$) ($\underline{Top}$,x), ($\underline{GG-Haus}$,x) one will get the classical definitions of determ., relational, part. def. nondeterm., stochastic, linear and topological automata. (x=cart. pr.)

3. Def. Morphisms in the cat. Aut of automata are 3-tup. (f_I, f_O, f_S) satisfi.

$$\begin{tikzcd} O \arrow[d, "f_O"'] & S \otimes I \arrow[l, "l"'] \arrow[r, "d"] \arrow[d] \arrow[l, phantom, "=" near end, shift left=0] & S \arrow[d, "f_S"] \\ O' & S' \otimes I' \arrow[l, "l'"] \arrow[r, "d'"'] & S' \end{tikzcd}$$

$$= f_S \otimes f \qquad =$$

Serialcomposition $A' \circ A$ can be defined for $I' = O$ in categories with diagonal-morphisms. This makes Aut a hypercategory and allowes DECOMPOSITION - THEORY (cf. Budach-Hoehnke, Eilenberg).

4. Prop. Def. $d!: S \otimes TI \to S$ (TI free monoid over I) by $d_o = r_S$, $d_{n+1} := S \otimes I^{n+1} \xrightarrow{d_n \otimes I} S \otimes I \xrightarrow{d} S$, then $(S, d!)$ $\underline{Ract}_{TI}$ (=right actions).

5. Theor. I fixed, $\Sigma := \underline{Ract}_{TI} \xrightarrow{V} \underline{K} \xrightarrow{- \otimes I} \underline{K}$ implies:
$\underline{Aut}(I) \cong (\Sigma \downarrow \underline{K})$. Objects $l: \Sigma S \to O$ ($l \in \underline{K}$) are called apparats. If $(\underline{K}, \otimes)$ is closed monoidal abelian, then $\underline{Aut}(I)$ is abelian: COHOMOLOGY of automata is treated by Budach-Hoehnke.

Reduction and minimization theory in closed monoidal cat. with $(\mathcal{E}, \mathfrak{M})$-factor. Let I be fixed (or restrict f_I to retractions) and $\mathcal{E} \otimes \mathrm{id} \subset \mathcal{E}$, $\underline{Aut} := \underline{Aut(I)}$

$S \otimes TI \otimes I \xrightarrow{d! \otimes I} S \otimes I \xrightarrow{l} O$ implies $M(A): S \to [TI \otimes I, O]$ $\underline{Ract}_{TI}$, this implies:

6. Theor. There is an isomorphism $M: \underline{Aut}(I) \xrightarrow{\sim} (\underline{Ract}_{TI} \downarrow [TI \otimes I, -])$
Idea of proof: $- \otimes TI \dashv V \dashv [TI, -] : K \to \underline{Ract}_{TI}$ has to be applied.

7. Def. Let the behaviourcat. $\underline{B}$ def. by objects $m: B \to [TI \otimes I, O]$ with $m \in \mathfrak{M}$ and $m \in \underline{Ract}_{TI}$ and morphisms $i \in K$ with

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "i"'] \arrow[dr, phantom, "=" near start] & {[TI \otimes I, O]} \\ B' \arrow[ur, "m'"'] & \end{tikzcd}$$

behaviourfunctor $B: \underline{Aut} \to \underline{B}$ with $A \longmapsto m$

$$\begin{tikzcd}
S \arrow[rr, "M(A)"] \arrow[dr, "e\in\mathcal{E}"'] & \arrow[d, phantom, "="] & {[TI\otimes I, O]} \\
& B \arrow[ur, "m\in\mathfrak{M}"'] &
\end{tikzcd}$$

<u>8. Theor.</u> Let $A \in |\underline{Aut}^m| \iff M(A)\in\mathfrak{M}$, $\underline{Aut}^m \subseteq \underline{Aut}$ full then

a) there exists $\Phi : \underline{B} \to \underline{Aut}$ (EQUIVALENCE PROBLEM)

b) there exists a reflector $R: \underline{Aut} \to \underline{Aut}^m$ (MINIMIZATION PROBLEM)

c) The corresponding special problem to b) is solvable (REDUCTION PROBLEM)

d) $B \dashv J\Phi : \underline{B} \xrightarrow{\simeq} \underline{Aut}^m \xrightarrow{J} \underline{Aut}$, $BJ\Phi = 1_{\underline{B}}$

(MINIMAL REALIZATION) (Goguen)

<u>9. Def.</u> Let $\underline{Aut}_*$ be the category of automata with initial state (initial state def. by $a: E \to S$, E unitobject) behaviourcategory

$\underline{B}_*$:
$$\begin{tikzcd}
TI\otimes I \arrow[r, "b"] \arrow[dr, "b'"'] & O \arrow[d, "f_O"] \arrow[dl, phantom, "="] \\
& O'
\end{tikzcd}$$
,

objects $b \in \underline{K}$; behaviourfunctor

$B_* : \underline{Aut}_* \to \underline{B}_*$, $A \mapsto b := TI\otimes I \xrightarrow{a\otimes id} S\otimes TI\otimes I \xrightarrow{d!\otimes I} S\otimes I \xrightarrow{1} O$

$A \in \underline{Aut}_*$ is called $\mathcal{E}$-connected, if $(E\otimes TI \xrightarrow{a\otimes TI} S\otimes TI \xrightarrow{d!} S)\in\mathcal{E}$

<u>10. Theorem:</u> Let $A \in |\underline{Aut}_*^m| \iff M(A)\in\mathfrak{M}$, $A \in |\underline{Aut}_*| \iff A$ $\mathcal{E}$- connected, $|\underline{Aut}_*^{\mathcal{E},m}| = |\underline{Aut}_*^{\mathcal{E}}| \cap |\underline{Aut}_*^m|$, $\underline{Aut}_*^{\mathcal{E}}$, $\underline{Aut}_*^m$ $\underline{Aut}_*^{\mathcal{E},m}$ full subcategories of $\underline{Aut}$:

a) there exists $E: \underline{Aut}_* \to \underline{Aut}_*^{\mathcal{E}}$ coreflector with $B_*E = B_*$ (up to restr.)

b) there exists $R_*: \underline{Aut}_* \to \underline{Aut}_*^m$ reflector with $B_*R_* = B_*$ (up to restr.)

c) there exists up to isomorphism a unique functor $M_*: \underline{Aut}_* \to \underline{Aut}_*^{\mathcal{E},m}$ with $B_*M_* = B_*$. Such a functor can be given by $M_* := R_*E_* \cong E\,R_*$ (MINIMIZATION PROBLEM in $\underline{Aut}_*$)

d) there exists $F \dashv B_*: \underline{Aut}_* \to \underline{B}_*$ with $B_* F = 1_{\underline{B}_*}$ (FREE REALIZATION

e) $\underline{B}_* \underset{B_* J_*}{\overset{R_* F}{\rightleftarrows}} \underline{Aut}_*{}^{,m}$ is an equivalence of cat., with

$J_* : \underline{Aut}_*^{\ell,m} \to \underline{Aut}_*^{\ell}$ inclus. (EQUIVALENCE PROBLEM)

f) $M_R := J_* R_* F : \underline{B}_* \to \underline{Aut}_*^{\ell}$ is a minimal realization in the sense of Goguen, that is : $B_* \dashv M_R$ with $B_* M_R = 1_{\underline{B}_*}$
(MINIMAL REALIZATION PROBLEM)

Remark: f) is a corollary of b) and e).

H.-G. Ertel and H. Schubert: Universal topological algebra.

The algebraic theories considered here are skeletons of Kleisli categories for arbitrary triples over $\underline{Ens}$ (the category of sets of some universe). Let $\underline{X}$ be a top category over $\underline{Ens}$ in the sense of Wyler (= Initialkategorie of Wischnewsky), e.g., topological spaces, and let $\underline{A}$ be an algebraic theory. Let $\langle \underline{A},\underline{X} \rangle$ be the category of those algebras over $\underline{Ens}$ whose carrier is given a "topology" and whose operations satisfy a given class of partial continuity conditions; morphisms are homomorphisms which map the carriers continuously. Then the forgetful functor
$\langle \underline{A},\underline{X} \rangle \to \underline{X}$ is tripleable and $\langle \underline{A},\underline{X} \rangle$ has nice properties (e.g., is complete, cocomplete, well powered and co-well-powered). The same holds if continuous actions of a fixed $\underline{X}$-object on algebras are taken into account, e.g., modules over a topological ring or continuous action of a topological group on spaces.

In the above, $\underline{X}$ can be replaced by an epireflective subcategory $\underline{Y}$.

If $\underline{Y}$ is co-well-powered and products of epis are epis, then corresponding results hold for continuous algebras over a epireflective subcategory $\underline{Z}$ of $\underline{Y}$, e,g., $\underline{X}$= topological spaces, $\underline{Y}$ completely regular spaces, $\underline{Z}$ = compact spaces.

Functors which forget a part of the continuity conditions (in the case of $\underline{X}$ or $\underline{Y}$) are tripleable, and so are functors which are induced by top functors, by the inclusion $\underline{Z} \subset \underline{Y} \subset \underline{X}$, or by theory - morphisms (algebraic functors).

The proofs use a pullback of forgetful functors, Manes's result on Birkhoff subcategories (slightly generalized), and Dubuc's adjoint triangles.

<u>Fakir, Sabah:</u> <u>α-injective objects in locally α-presentable categories</u>

Let α be an infinite regular cardinal and $\underline{A}$ be a locally α - presentable category (Gabriel-Ulmer, Lecture Notes N: 221). Let Mono ($\underline{A}$) be the full subcategory of Mor($\underline{A}$) whose objects are the monos of $\underline{A}$. <u>Prop.</u> Mono($\underline{A}$) is a locally α-presentable category and its α-presentable objects are the monos A >—> B such that A is α-generated and B is α-presented.

<u>Def.:</u> An object E is called α-injective if it is injective relatively to these monos.

<u>Theorem:</u> 1) If $\underline{A}$ has enough α-injectives then monos are couniversal.
2) If $\alpha = \aleph_o$ and monos are couniversal then $\underline{A}$ has enough $\aleph_o$-injectives.

<u>Examples:</u> Mod_A , C^*- comm.algebras, Bool (category of Boolean algebras).
<u>Counter examples:</u> Groups, Monoids, Cat, Comm.Rings, due to the existence of <u>simple objects</u>.

<u>Prop</u> (due to Sabbagh): if in a category monos are couniversal, then every subobject of a simple object is a simple object.

<u>Prop:</u> If $\alpha = \aleph_o$, then $\aleph_o$-inj => absolutely pure in the sense of P.M. Cohn. The converse is true iff monos are couniversal.

<u>Definition:</u> a category is called locally α-coherent if it is locally α-presented and if every α-generated subobject of an α-presented object is α-presented.

<u>Theorem:</u> Let A be locally α-presented and with enough α-injectives. The following conditions are equivalent

1) A is locally α-coherent.
2) Every α-cofiltered colimit of α-injective objects is α-injective
3) Every α-reduced product (and in case $\alpha = \aleph_0$ every ultraproduct) of α-injectives is α-injective.

<u>Examples:</u> Mod_A where A is a coherent ring, $Mod_A \backslash N$ where N is a coherent object in Mod_A . Finally we characterize small α-cocomplete category such that $Cont_\alpha\ (U^o, Ens)$ is α-coherent.

<u>Fletcher, R.W.: Casimir elements for functors</u>

The natural transformations from the identity functor to an endofunctor are called Casimir elements and those from an endofunctor to the identity functor Co-Casimir elements. A Casimir element u and a Co-Casimir element ε act on map $f: FX \to FY$, to give a map $\varepsilon fu: X \to Y$. The natural transformation εu is called the value of the Casimir and Co-Casimir elements.

If (H,ε,Δ) is a cotriple we consider the action of a Casimir element of H and the Co-Casimir element ε. If $S \to R$ is a ring extension and $H = R\otimes_S -$ (on left R-modules) then Casimir elements are multiplication on the left by elements α of $R\otimes_S R$ satisfying $r\alpha = \alpha r$ for all r in R (the classical Casimir elements). If (K,u,M) is a triple we consider Co-Casimir elements and the Casimir element u. In the case, C is a coalgebra and $K = -\otimes C$ (on right C-comodules) Co-Casimir elements are given by elements α in the linear dual $(C\otimes C)^*$ such that

$$\begin{tikzcd}
C\otimes C \arrow[r, "1\otimes\Delta"] \arrow[d, "\Delta\otimes 1"'] & C\otimes C\otimes C \arrow[d, "\alpha\otimes 1"] \\
C\otimes C\otimes C \arrow[r, "1\otimes\alpha"'] & C
\end{tikzcd}$$

commutes where Δ is the comultiplication of the coalgebra C.

If a cotriple H has a Casimir element with invertible value (with the Co-Casimir element ε) all objects are H-projective. Dually for triples. As the space of integrals of a Hopf algebra is a direct summand of the Casimir elements and the space of integrals of the dual of a Hopf algebra is a direct summand of the Co-Casimir elements we can deduce Sweedler's results on the semi-simplicity of Hopf algebras.

If the endofunctor F is both a triple and a cotriple an object A is F-projective if and only if there exists $f: FA \to FA$ such that $\varepsilon f u$ is the identity on A. (For example $F = R\otimes_S -$, when $S \to R$ is a Frobenius extension, which gives the Gaschütz - Ikeda - Kasch theorem).

If (R,S) and (S,T) are adjoint pairs of functors the Casimir elements of the cotriple RS are isomorphic to the Co - Casimir elements of the triple TS. Further, if $f: SA \to SB$, the action of a Casimir element on Rf gives the same map $A \to B$ as the action of the corresponding Co-Casimir element on Tf. An example of such a system of functors is given by a ring extension $S \to R$ where the functor R is $R\otimes_S -$, T is $\mathrm{Hom}_S(R,-)$ and the functor S is the forgetful functor from left R-modules to left S-modules.

<u>Freyd, Peter:</u> <u>Aspects of Aspects of Topoi</u>, Section 5.6 of <u>Aspects of Topoi</u> (Bul. Austral. Math. Soc. Vol 7(1972), 1-76) is all wrong.

The error occurs earlier, namely the stupid assumption that a colimit of functors, each of which preserves epimorphic families, does the same. The lines about preserving epimorphic families in 3.21, through 3.24 should be struck. Fortunately we didn't use those lines untill 5.6 where the results though false are the answers to the right questions.

- 19 -

There exist bicomplete topoi, even Boolean and 2-valued, with no exact functors into the category of sets. Nothing like a stalk functor. On the other hand, every countable exact subtopos of any bicomplete topos is exactly embeddable in a power of the category of sets. Perhaps nothing better demostrates the utility of the elementary version of topoi. The way in which one would likely use the existence of enough exact set-valued functors is to verify elementary assertions. Knowing that the countable subtopoi allow enough exact set-valued functors is, of course, sufficient for this use.

We say that a topos is N-STANDARD if it has a natural numbers object N and the maps $1 \xrightarrow{n} N$, through the standard natural numbers, form an epimorphic family.

Theorem: Every countable N-standard (Boolean) topos is exactly (logically) embeddable in a product of N-standard wellpointed topoi.

Theorem: There exists a Boolean 2-valued N-standard topos of the power of the continuum with no exact functors to the category of sets.

There exists a Boolean 2-valued bicomplete topos with no exact functors to the category of sets.

Gray, John: 2-Categories and Broich group.

In the category of 2-categories, the N-dimensional cube Q_N in the 2-category described by the following generators and relations: the objects are sequences $I = (1_1, \ldots, 1_N)$ where $1_k = 0, 1$. $\underline{0}$ denotes the sequenes with all 0's and $\underline{1}$ the one with all 1's. If I has a 0 in the m'th place, then I(m) is the sequence agreeing with I except in the m'th place, where it has a 1. In this case, there is a basic 1-cell $t_{m,I}: I \to I(m)$ and the underlying category of Q_N is freely generated by these basic 1-cells. If I has zeroes in the m'th and n'th places, $m < n$, there is a basic 2-cell as indicated in the diagram

$$\begin{tikzcd} I \arrow[r, "t_{m,I}"] \arrow[d, "t_{n,I}"'] & I(m) \arrow[d, "t_{n,I(m)}"] \\ I(n) \arrow[r, "t_{m,I(n)}"'] \arrow[ur, phantom, "\Nearrow t_{m,n,I}"] & I(m,n) = I(m)(n) \end{tikzcd}$$

The 2-cells of Q_N are the required compositions of the 2-cells with each other and with 1-cells, subject to the axioms of a 2-category together with the relations for all $m < n < p$,

$$(t_{n,p,I(m)}\ t_{m,I}) \cdot (t_{n,I(m,p)}\ t_{m,p,I})\ (t_{m,n,I(p)}\ t_{p,I})$$

$$= (t_{p,I(m,n)}\ t_{m,n,I})\ (t_{m,p,I(n)}\ t_{n,I})\ (t_{m,I(n,p)}\ t_{n,p,I}).$$

(This says that a 3-dimensional cube commutes.)

<u>Theorem:</u> Q_N is locally partially ordered. (I.e,,its hom categories are partially ordered.)

<u>Proof.:</u> (Sketch). By induction, it is sufficient to treat the category $\underline{C} = Q_N\ (\underline{0},\underline{1})$. The objects in $\underline{C}$ can be represented by permutions $A = a_1 \ldots\ a_N$ of $1,\ldots,N$ where a_m represents the basic 1-cell

$$t_{m,\underline{0}(a_{m+1},\ldots,a_N)}$$

There is a morphism in $\underline{C}$ formed by composing a single basic 2-cell with 1-cell from A to B iff A and B agree exept at two succesive places, say m and m+1, and $a_m = b_{m+1}$, $a_{m+1} = b_m$, $a_m < a_{m+1}$. Call this morphism σ_{mAB}. The morphisms of $\underline{C}$ consist of all composible words in there subject to two types of relations:

R1. If $\sigma_{m\,C\,D}\ \sigma_{m+1\,B\,C}\ \sigma_{mAB}$ is defined, then there exist unique B' and C' with $\sigma_{m+1C'D}\ \sigma_{mB'C'}\ \sigma_{m+1AB'}$ defined and these are equal.

R2. If $\sigma_{mBC}\ \sigma_{nAB}$ is defined with $|m-n| \geq 2$, then there is a unique B' with $\sigma_{nB'C}\ \sigma_{mAB'}$ defined and these are equal.

Lemma: There is a faithful representation $\underline{C} \xrightarrow{P} B_N$ where B_N is the braid group on N-strings.

Proof: Take σ_{mAB} to the generator σ_m of B_N.

These generators satisfy

$$\sigma_m\ \sigma_{m+1}\ \sigma_m = \sigma_{m+1}\ \sigma_m\ \sigma_{m+1}$$
$$\sigma_m\ \sigma_n = \sigma_n\ \sigma_m \qquad |m-n| \geq 2 .$$

The hard part is to show that equality of words in the braid group implies commutativity of the corresponding diagrams in $\underline{C}$. This is accomplished by two steps.

1. Words in the image of P can be brought into canonical form remaining entirely within the positive semi - group of B_N.

2. Regard the symmetric group S_N as the quotient of B_N by adding the relations $\sigma_m^2 = 1$ for all m. Then there is a section of $B_N \to S_N$ whose values consist precisely of the canonical forms of words in the image of P.

Applications.1. The structure of the 2-theory of moniods (as well as of other 2-theories) can be deduced from this. This yields the usual cokernel theorem for monoidal categories (which are models of this theory) as well as all cokernel properties for morphisms between such categories.

2. The coherence for the ⊠-product of 2-categories corresponding to the internal hom given by F un (-,-) requires the details of the structure of Q_4. Once this is established, one gets $Q_n \cong \underline{2} \boxtimes \underbrace{\cdots\cdots\cdots}_{n} \boxtimes \underline{2}$

where $\underline{2}$ is the usual arrow category regarded as a locally discrete 2-category.

Hoffmann, R.-E.: <u>The Categorial Idea of Initial and Final Topology.</u>

The starting point of the investigation was the definition of initial and final topology in the work of BOURBAKI: these concepts (being non-categorial until 1965) are used to define products and coproducts in Top. - The come - out of my investigation are several types of functors (↪ denoting inclusion of classes)

		amb - idt. triangles		
		↗		
not introduced here	{ (Σ-)semi - idt. triangles	↪ idt.triangles	↪	topological functors
				↑
		ps-idt.triangles		ps.-topological functors

(ps = pseudo, idt = identifying);

the most interesting ones being "idt. triangles" and "topological functors". The concept of (Σ-) idt. triangle is related to those of

Colimit	functor having a full and faithful left adjoint	fibration in the sense of Grothendieck 61 and Gray 65	and to the

varions definitions of something like "topological functor" occuring in the literature: c.e.s. complête of ANTOINE (Bull. Math. Soc. Belg. 66), initial functor of ROBERTS (J. of Algebra 68), initial completeness of BRÜMMER (Thesis, Cape Town 71) - Husek's s-categories (Comment.Math.Univ.Carol. 1964 sqq.) and the "projective generating" of the Praque school: all these definitions have turned out to be (nearly) equivalent to the concept of Top-category of Wyler (1. Archiv d.Math. 71, z.General Top 71), which has been earlier defined by Kennison (65) (parts of this result ore due to SHUKLA, thesis KANPUR/INDIA 71 and WISCHNEWSKY, Diss. München 72; the whole result is due to the author).

Examples of topological functors are e.g. the forgetful functors Top-, proximity apaces, measurable spaces, Dynkin-systems. preordered sets, sets with a relation inscribed $\to$ Ens; also topological groups, rings etc. $\to$ groups, rings. Idt. triangles and (the dual concept) co-idt. triangles (= amb-idt. triangles) at the same time are e.g.: Mod (=category of all modules, see GRAY) $\to$ unitary, ass. Rings, Object: cat $\to$ Ens and directed graphs $\to$ Ens, all of them not being topological functors.

<u>Definition:</u> V: $\underline{C} \to D$ a functor, (λ, A) a cone in $\underline{C}$ with domain T: $\Sigma \to \underline{C}$: (λ, A) V-idt. (V-identifying : $\Leftrightarrow$

for every cone (η, B) in $\underline{C}$ with domain T and every morphism

u : VA $\to$ VB in D,

so that $u_\Sigma\ V{*}\lambda = V{*}\eta$, there is just one morphism f: A $\to$ B, so that Vf = u and $f_\Sigma\lambda = \eta$ ("test-situation").

If the test-situation can be satisfied (at least) for isomorphisms u : VA $\xrightarrow{\cong}$ VB, then (λ, A) is called V-pseudo-identifying (V-ps-idt).

<u>Examples:</u> 1. V = forget: Top $\to$ Ens, then (λ, A) is V-idt, iff A has the "final" or "identfying" topology

2. $\underline{D}$ = 1: (λ, A) V-idt. $\Leftrightarrow$ (λ, A) colimit

3. $\Sigma = 1$: (λ,A) is just a morphism in $\underline{C}$, let ns say 1!

4. $\Sigma = \emptyset$: (λ,A) can be replaced by A:

(λ,A) V-idt $\Leftrightarrow$: A V-discrete $\Leftrightarrow$ (V): $\mathrm{Hom}(A,-) \to \mathrm{Hom}(VA,V-)$ is an isomorphism

if V = forget: Top $\to$ Ens, the "discrete spaces" are so characterized; if V = Object: cat $\to$ Ens: "discrete categories".

A V-date $(T;\gamma,D)$ consists of a diagram T: $\Sigma \to \underline{C}$, and a cone γ: $VT \to D_\Sigma$; $(\lambda,A;\ i\colon D \xrightarrow{\cong} VA)$ is called a V-idt. lift of the V-date (T; ,D) iff.

1. λ has domain T, 2. $i_\Sigma \gamma = V * \lambda$, 3. (λ,A) V-idt.

V is called an idt. triangle, iff every V-date of type Σ, Σ being $\mathfrak{U}$-small, has a V-idt. lift ($\underline{C}$ co-complete, then $\underline{C} \to 1$ is an idt. triangle, other examples you find above). If $V \cong W$ V idt. triangle, then W idt. triangle (therefore i in the lift is assumed to be not necessarily an identity). An idt. triangle has a full and faithful left adjoint and it respects colimits (not necessarily being itself left adjoint); its domain is co-complete iff its range is.

Theorem: D co-complete, V: $\underline{C} \to \underline{D}$ a functor:

V idt. triangle $\Leftrightarrow$ 1. $\underline{C}$ co-complete
2. V respects colimits
3. V has a full and faithful left adjoint

By this criterion e.g. Object: cat $\to$ Ens is recognized to be an amb-idt. triangle (= idt. + co-idt.).

<u>Definition:</u> V: $\underline{C} \to$ D is a "topological functor" iff

1. every V-date of type Σ, Σ discrete and $\mathfrak{U}$-small, has a V-idt lift

2. for every $D \in \mathcal{O}b\, D$ $\{C \in \mathcal{O}b\,\underline{C} \mid VC \cong D\}/_{\cong_{\underline{C}}}$ has $\mathfrak{U}$-small cardinality

if V lifts isomorphisms (not necessarily unique), 2. can be replaced by 2'. the V-fibre of D has a $\mathfrak{U}$-small skeleton

<u>Theorem:</u> V: $\underline{C} \to D$ topological, then

a) V is faithful, b) every V-date (Σ map be not discrete and card Σ >, =, < card $\mathfrak{U}$) has a V-idt. lift (this suggests a nother definition of "topological"), c) V^{op}: $\underline{C}^{op} \to \underline{D}^{op}$ is also topological ("<u>duality theorem</u>") (c) is still known under the assumtion that V lifts isomorphisms).

a) generalizes Freyd's result on $\mathfrak{U}$ small complete categories (consider $\underline{C} \to 1$).

<u>Remark:</u>if every V-date has at least a V-ps.-co-idt. lift and "2." is satisfied, V is called ps.-topological: V has a full and faithful left adjoint and is faithful: forget: T_1-spaces $\to$ Ens is ps.-topological, but not topological.

Let V: $\underline{C} \longrightarrow \underline{D}$ be topological: V has a full and faithful left adjoint and a full and faithful right adjoint; $\underline{C}$ is (co-)complete, iff D is; $\underline{C}$ is (co-)wellpowered, iff $\underline{D}$ is

<u>Theorem:</u> let D be co-well powered and co-complete, V: $\underline{C} \to \underline{D}$ a functor: V topological $\Leftrightarrow$

1. $\underline{C}$ is co-wellpowered and co-complete
2. V has a full and faithful left adjoint and a full and faithful right adjoint
3. V is faithful.

<u>Remark:</u> for any category $\underline{C}$ there is at most one topological functor (up to isomorphisms) $\underline{C} \to$ Ens.

For the following result remember the construction of "pseudo-functor" in GRAY'S paper (La Jolla Conference 65):

<u>Theorem:</u> Let V: $\underline{C} \to$ D lift isomorphism (!)
V <u>co</u>-idt. triangle $\Leftrightarrow$ V is a fibration and
(i.e. V^{op} idt. triangle)
1. the fibres are complete categories
2. the functor between these fibres, which are induced by the morphisms of $\underline{D}$, preserve limits

V is topological, iff furthermore the fibres are preordered $\mathcal{U}$-classes with small skeleta; if V lifts isomorphisms uniquely, then the inducing "pseudo-functor" is just a "topological theory" in the sense of WYLER.

Herefrom "forget": Mod → Rings is seen to be an amb-idt. triangle, and especially Mod(= all modules) (see e.g. GRAY) is seen to be complete and co-complete.

Jensen, C.U.: A survey of the latest results about $\varprojlim^{(i)}$

The lecture gives a survey of some results about $\varprojlim^{(i)}$ found jointly with L. Gruson.

Definition: For a left R-module M define L- dim M as the length of a minimal pure injective resolution of M.

Theorem 1: If L-dim M = n < ∞, then $\varprojlim^{(i)}(A_\alpha \otimes_R M) = o$ for any projective system of finitely presented right R-modules A_α and all $i > n$.

Theorem 2: Let R be right coherent and P a flat left R-module. For an integer n the following are equivalent:

1) L-dim P $\leq$ n

2) $\varprojlim^{(i)}(A_\alpha \otimes P) = o$ for any projective system of finitely presented right R-modules A_α and all $i > n$

3) $\varprojlim^{(i)}(F_\alpha \otimes P) = o$ for any projective system of finitely generated free right R-modules A_α and all $i > n$

4) $\mathrm{Ext}^i_R(Q,P) = o$ for all flat left R-modules and all $i > n$.

Remark: 1 ⟹ 4) holds without assuming the coherence condition on R.

Remark: If L-dim P $\leq$ n for any flat R-module P, then the projective dimension of any flat left R-module $\leq$ n.

<u>Theorem 3:</u> For any (not necessarily coherent) ring R of cardinalety $\aleph_t$ one has L-dim M $\leq$ t + 1 for all R-modules M.

<u>Corallary:</u> For R as above any flat module has projective dimension $\leq$ t + 1 and (l. gl. dim R- v.gl. dim R) $\leq$ t + 1.

<u>Theorem 4:</u> Let R be right Noetherian of finite Krull-dimension (in the sense of Gabriel) d. Then the L-dimension of any flat left R-module is $\leq$ d.

<u>Corollary:</u> For R as above any flet left R-module has projective dimension $\leq$ d.

<u>Theorem 5:</u> Assume R and R $[[x]]$ coherent. Then L-dim R = L-dim R $[[x]]$

If A is a flat R-module the modules of the form $\mathrm{Ext}^1_R(A,R)$ are the same as those of the form $\varprojlim^{(1)} F_\alpha$, F_α finitely generated free.

Results about $\varprojlim^{(i)}$ give information of "Whitehead-like" probleme for various classes of rings. As typical results we mention.

<u>Theorem 6:</u> Let R be a countable Dedekind domain. Then the following conditions are equivalent:

1) For any countable torsion-free R-module A $\mathrm{Ext}^1_R(A,R)$ is compact (in a suitable topology)

2) For any maximal ideal m of R, R/m is finite.

In the uncountable case the following result is useful.

<u>Theorem 7:</u> Let R be Dedekind ring with quotient field Q and write

$$\mathrm{Ext}^1_R(Q,R) \cong Q^{(d)}$$

Then the d's which can occur are

1) any infinite cardinal number

2) among the finite cardinal numbers exactly those of the form $p^n - 1$, p being a prime number.

<u>Corallary:</u> There exists a principal ideal domain R and torsion-free A such that $\mathrm{Ext}^1_R(A,R) \cong Q/R$.

<u>Kock, A. and Mikkelsen, C.J.: A factorization Theorem for first-order preserving functors between toposes.</u>

<u>Theorem:</u> Given a functor between elementary toposes, $\varphi : \underline{E} \to \underline{E}_o$ which "preserves 1. order logic", then φ can be factored

$$\begin{tikzcd} \underline{E} \arrow[rr, "\varphi"] \arrow[dr, "\bar{\varphi}"'] & & \underline{E}_o \\ & \underline{E}^* \arrow[ur, "\psi"'] & \end{tikzcd}$$

when $\underline{E}^*$ is a topos and $\bar{\varphi}$ and ψ are 1. order logic preserving, and when further

$\bar{\varphi}$ <u>preserves higher order logic</u> (exponentiation)

ψ <u>preserves elements</u> (i.e. $\hom_{\underline{E}^*}(1,Y) \to \hom_{\underline{E}_o}(1, Y)$ is bijective).

The factorization is motivated by non-standard analysis, when higher order properties of extensions <u>in some sense</u> (namely $\bar{\varphi}$ are preserved, and in some sense (namely φ) are <u>not</u> preserved. The key word in this contradiction is the word "internal", as used in higher order non-standard analysis. <u>There</u> one changes the logic by inserting the word "internal" on the quantifiers; "change of logic" in topos theory should be replaced by "change of topos". This is what our shift from $\underline{E}_o$ to $\underline{E}^*$ does.

There are some different ways to interpret the phrase "first order logic preserving "; we shall take it to mean: "preserving finite inverse limits, epies, and Ω". The theorem is also true if to this we add "preserves universal quantification".

Construction: Since φ preserves products, it has a natural structure as closed functor:

$$(1) \qquad \varphi(B^A) \xrightarrow{\hat{\varphi}_{A,B}} \varphi B^{\varphi A} \quad ;$$

maps $\varphi B \to \varphi A$, whose name $1 \to \varphi B^{\varphi A}$ factor through $\hat{\varphi}$ will be called internal maps. A subobject $A_o \rightarrowtail \varphi A$ is called an internal subobject, provided its characteristic map $\varphi A \to \Omega \cong \varphi \Omega$ is an internal map. The objects of $\underline{E}^*$ are now taken to be triples (A_o, a, A) where $A_o \overset{a}{\rightarrowtail} \varphi A$ is an internal subobject. Maps from (A_o, a, A) to (B_o, b, B) are maps $f: A_o \to B_o$ such that "the graph"

$$A_o \overset{<1,f>}{\rightarrowtail} A_o \times B_o \longrightarrow \varphi A \times \varphi B \cong \varphi(A \times B)$$

is an internal subobject.

The only hard thing in the proof is that $\underline{E}$ has exponentiation. We get help from the

Lemma: A category with finite inverse limits and a subobject classifier Ω has exponentiation if it has exponentiation of form Ω^X.

The form of the theorem involving universal quantification depends on the following "extensionality" statement:

Proposition: For a left exact, Ω-preserving functor $\varphi: \underline{E} \to \underline{E}_o$, the following are equivalent

(i) φ preserves universal quantification

(ii) $\hat{\varphi}_{A,B}$ (as in (1)) is monic for all A,B.

Further details may be found in "Non-standard extensions in the theory of toposes", Aarhus Universitet, Preprint-seried no 25 1971/72.

the category of presheaves over a small category (and an analogous situation for elementary topos), and various categorie of pre-ordered sets, the latter by B. BALLINGER.

If we look at the category $\underline{B} = \underline{A}^{opp}$ and choose a projective with respect to regular epis in $\underline{B}$, let $U = (P,-)$ and F be its left adjoint Ens $\to$ B, and put $\mathcal{T} = (UF, \eta, U\varepsilon F)$ for the associated triple. Then one considers the EILENBERG - MOORE category $\mathrm{Ens}^{\mathcal{T}}$, the comparsion functor $K\colon \underline{B} \to \mathrm{Ens}^{\mathcal{T}}$, and the left adjoint M of K. One calculates $MK = Q$, thus obtaining another interpretation of localization on $\underline{A}$ (colocalization on $\underline{B}$). As a corollary one obtains the following variant of LINTON's Theorem: B is equational with respect to $(P,-)$ if and only if $\underline{B}$ is cocomplete and has kernel pairs, Q is the identity, and every equivalence relation is a kernel pair. It follows, for example, that the opposite of every GROTHENDIECK category is equational, in view of the GABRIEL - POPESCU theorem.

Laudal, O.A.: Obstructions for the existence of sections of functors.

Let $\Pi\colon R \to S$ he a surjective homomorphism of commutative rings and assume $(\ker\Pi)^2 = 0$. We may then consider the functor $\Pi\colon \underline{C}_R \to \underline{C}_S$ defined by $\Pi(A^1) = A^1 \otimes_R S$ with $\underline{C}_R$ (resp $\underline{C}_s$) denoting one of the categories R-(resp S-) modules, R-(resp S -) algebras.

Let $\underline{c}$ be any subcategory of $\underline{C}_s$ and let $\underline{C}$ be the full subcategory of $\Pi^{-1}(\underline{c})$ given by the objects A^1 for which $\mathrm{Tor}_1^R(A^1,S) = 0$. Consider the obvious restricted functor $\Pi\colon \underline{C} \to \underline{c}$. Given any motphism $\psi\colon A \to B$ of $\underline{C}$ let $H^i(\psi)$ denote either $\mathrm{Ext}^i_s(A, B \otimes_s \ker\Pi)$ or the Andri cohomology group $H^i(S,A;\, B \otimes_s \ker\Pi)$ according to the choice of category $\underline{C}_s$.

Then H^i is a contravariant functor on the category $\underline{\mathrm{Mor}}\ \underline{c}$ defined.

Lambek, J. and Rattray, B.: Localization at injectives in complete categories

Consider an object I in a complete category $\underline{A}$. Following FAKIR, we define a functor Q: $\underline{A} \to \underline{A}$ and a natural transformation $k: Q \to S = I^{(-,I)}$ as the equalizer of the two canonical natural transformations $S \to S^2$. We call I k-injective if, for any object A of $\underline{A}$, every map $Q(A) \to I$ can be extended to a map $S(A) \to I$. Let Fix Q be the full subcategory of $\underline{A}$ consisting of all objects A for which the canonical map $A \to Q(A)$ is an isomorphism. Then Fix Q is the limit closure of I in $\underline{A}$ if and only if I is k-injective. This result depends on Fakir's Theorem, which says that Q is idempotent if and only if S(k(A)) is mono, and on a lemma which asserts that k(A) is the joint equalizer of all pairs of maps $S(A) \rightrightarrows I$ which are equalized by the canonical map $A \to S(A)$. If I is injective with respect to all regular monos, the reflector $\underline{A} \to$ Fix Q preserves all regular monos.

Example 1: TIETZE's Theorem assures that the interval $|0,1|$ is k-injective in the category of topological spaces. Q(A) is the STONE - CECH compactifications of A, and this is essentially Čech's original construction. In the category of uniform spaces, $[0,1]$ is even injective with respect to all regular monos and Q(A) is the SAMUEL compactification.

Example 2: If $\underline{A}$ = Mod R, R an associative ring, and I is any injective R-module, Q is the usual localization functor associated with I. This agrees with the localization of GABRIEL - BOURBAKI, if one takes the filter $\underline{D}$ of all right ideals D for which $\mathrm{Hom}_R(R/D,I) = 0$ and defines $Q(A) = \varinjlim \mathrm{Hom}(D, A/T(A))$, where $T(A) = \{a \in A \mid a^{-1}0 \in \underline{D}\}$. Conversely, any Gabriel filter of right ideals $\underline{D}$ gives rise to an injective which is the product of all injective hulls of modules R/K, K rangig over those right ideals K for which $\forall_{r \in K}\, r^{-1}K \notin \underline{D}$.

Other examples that have been studied are:

the opposite of Mod R, the category of bimodules à la DELALE, by: the objects are the morphisms of $\underline{c}$ and the morphisms $\varphi_1 \to \varphi_2$ are the commutative diagrams of the form

$$
\begin{tikzcd}
\cdot \arrow[r, "\psi_1"] \arrow[d, "\varphi_1"'] & \cdot \arrow[d, "\varphi_2"] \\
\cdot & \cdot \arrow[l, "\psi_2"]
\end{tikzcd}
$$

The main result is then the following,

<u>Theorem:</u> There exists an obstruction $\underline{\sigma}_o$ $\varprojlim_{\underline{Mor}\ \underline{c}} H^2$ s.t. if $\underline{O}_o = o$ there exists an obstruction $\underline{O}_1$ $\varprojlim^{(1)}_{\underline{Mor}\ \underline{c}} H^1$ s.t. if $\underline{O}_1 = o$ there exists a set of obstructions $O_2 \subseteq \varprojlim^{(2)}_{\underline{Mor}\ \underline{c}} H^o$ s.t. $\underline{O}_o = o$, $\underline{O}_1 = o$, $6 \in \underline{O}_2$ is nessecary and sufficicient for the existence of a section σ of π.

<u>Corollary:</u> If X is an S-<u>scheme</u> of finite type then there exists an obstruction $O_o \in H^o(X, \mathfrak{X}^2)$ s.t. if $\underline{O}_o = o$ then these exists an obstruction $\underline{O}_1 \in H^1(X, \mathfrak{X}^1)$ s.t. if $\underline{O}_1 = o$ there exists an obstruction $\underline{O}_2$ $H^2(X, \mathfrak{X}^o)$ s.t. $\underline{O}_o = o$, $\underline{O}_1 = o$, $\underline{O}_2 = o$ is nessecary and sufficient for X to be liftable from S to R.

Several other applications were mentioned, as was the relationship with the work of Lichtenbaum - Schlesinger, Grothendieck and Illusie.

The proofs are found in the Preprint Series - Mathematics, No. 12 (may 1971), Department of Mathematics, University of Oslo, Oslo, Norway.

Linton, F.E.J.: Algebren über Tripel bzw. über Theorien im Rahmen relativer kategorieller Algebra

We deal in the framework of $\underline{V}$-categories, where $\underline{V}$ is closed, or monoidal, or a full subcategory of any such. Where $\underline{M}$ in the strctly associative monoidal category describing the multilinear-structure on $\underline{V}$, we often need to expand our framework to $\hat{\underline{V}}$-categories, where $\hat{\underline{V}} = S^{\underline{M}^{op}}$, a complete, cocomplete, closed monoidal category containing $\underline{V}$ as a full subcategory.

Given a $\underline{V}$tuple $\mathcal{T}$ on a $\underline{V}$category $\underline{A}$, we may form the category $A^{\mathcal{T}}$ of $\mathcal{T}$-algebras in $\underline{A}$; while $\underline{A}^{\mathcal{T}}$ is not in general a $\underline{V}$-category unless $\underline{V}$ has certain equalizers, it is canonically a $\hat{\underline{V}}$-category, as are also all $\underline{V}$-valued contravariant functor categories $\underline{V}^{\underline{X}^{op}}$, where $\underline{X}$ is an arbitrary $\underline{V}$-category. The two instances of $\underline{X}$ we need are $\underline{X} = \underline{A}$ and $\underline{X} = Kl(\mathcal{T})$, the Kleisli category built out of the $\underline{V}$-tuple $\mathcal{T}$, a $\underline{V}$category in a natural fashion.

Operational-style algebras over the tuple $\mathcal{T}$ are described in terms of functors on $Kl(\mathcal{T})$ precisely as the following pullback:

$$\begin{tikzcd} \text{pullback} \arrow[r] \arrow[d] & \underline{V}^{(Kl(\mathcal{T}))^{op}} \arrow[d] \\ A \arrow[r, "\text{Yoneda}"'] & \underline{V}^{A^{op}} \end{tikzcd} \qquad (\text{among } \hat{\underline{V}}\text{-categories})$$

The fundamental identification theorem asserts that $\underline{A}^{\mathcal{T}}$ is this pullback, on the nose.

The proof uses as an intermediary stage the category of coalgebras, in $\underline{V}^{A^{op}}$, over the cotuple on $\underline{V}^{A^{op}}$ arising from composition- in - advance with the structure of $\mathcal{T}$.

More precisely, referring to the diagram

$$\begin{tikzcd}
A^{\mathcal{T}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\underline{V}^{\mathcal{T}} \quad PB"] & (\underline{V}^{A^{op}})_{\mathcal{T}_0} \arrow[r, "(*)"] \arrow[d] & \underline{V}^{Kl(\mathcal{T})^{op}} \arrow[d] \\
A \arrow[r, "Yoneda"'] & \underline{V}^{A^{op}} \arrow[r, equal] & \underline{V}^{A^{op}}
\end{tikzcd}$$,

one sees, by use merely of the $\underline{V}$-Yoneda Lemma, that square PB is a pullback diagram, and, by examination of the nature of the two kinds of data, that the arrow (*) is an isomotphism of categories. Further details will be omitted here.

If is worth pointing out the curiosity that the trileable situation $A^{\mathcal{T}}{}_{op} \to A$ on the left is the pullback of the cotripleable situation $(\underline{V}^{A^{op}})_{-o\mathcal{T}} \to \underline{V}^{A^{op}}$ (or $\underline{V}^{Kl(\mathcal{T})^{op}} \to \underline{V}^{A^{op}}$) on the right. This seems paradoxical ... Oder?

MacLane, S.: A Survey of Recent Results an Coherence

A coherence theorem specifies conditions when two parallel canonical arrows must be equal. A typical case is that of monoidal categories which are categories with a $\otimes$-product , and structure arrows

$$\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

$$\beta : I \otimes A \to A, \qquad \gamma : A \otimes B \to B \otimes A$$

which made certain basic diagrams commute. The canonical arrows are then instances of α, β and γ, closed under $\otimes$ and composition. The basic coherence theorem is that for a closed category (monoidal plus $-\otimes A$ has a right adjoint), proved by Kelly-MacLane (Coherence in closed categories, J.of Pure and Applied Algebra 1 (1971), 91-140). The essential method used is that of cut-elimination. We report here on several extensions of such results.

First, G. Lewis has found a similar coherence theorem for a closed functor Φ: $V \to V'$ between two closed categories. However, canonical arrows are not allways equal; for example there are two different arrows $\Phi I \longrightarrow \Phi I \otimes \Phi I$.

Next, Voreadou has in her thesis (Chicago) extended the results of Kelly-MacLane. They proved that canonical arrows between <u>proper</u> shapes are equal if they have the same graph. She treats certain improper shapes as well by using an <u>extended</u> graph, which links not only variables but the constants I.

Also, Kelly-MacLane extend then a coherence theorem from a closed category $\underline{V}$ to the case of a natural transformation $\Theta : F \to G$, where $F,G: \underline{A} \longrightarrow \underline{B}$ are functors between two $\underline{V}$-categories $\underline{A}$ and $\underline{B}$.

Finally LaPlaza treats the coherence problem for distributivity: Functors $\oplus$ and $\otimes$, with $\otimes$ distributive on both sides over $\oplus$. He obtains the following complete result: Two canonical arrows are equal if they have the same distortion, where the distortion is obtained by mapping the whole situation to a certain standard category with $\oplus$ and $\otimes$. This problem had previously been solved in unpublished work of Benabou.

A general setting for any coherence problem has been developed by Kelly. He introduces a non-symmetric product in <u>Cat</u>/$\underline{P}$ where $\underline{P}$ is the category of permutations (objects, natural numbers, arrows $n \to n$, permutations of n). A <u>club</u> is a $\circ$-monoid in <u>Cat</u>/$\underline{P}$. Kelly shows that each suitable coherence problem has a characteristic club (where the objects over n are all the functors of n variables). He extends the original Kelly-MacLane cut-elimination theorem to this case.

All the results cited (except those of Voreadou and Benabou) will appear in the Springer Lecture Notes Volume 281: Coherence in Categories.

Malraison, P.J., Jr.: Ho-equivalences of topological categories

Notations:

Top = compactly generated spaces

Top/A = spaces over A. $U:\underline{Top/A} \to \underline{Top}$ sends an object to its domain as a map in Top.

Ho(Top/A) = category of fractions with respect to maps $f: X \to Y$ such that U(f) is a weak homotopy equivalence, i.e. it induces an isomorphism on all homotopy groups.

$\underline{Top}^G$ = G-spaces = triple algebras for the triple $- \times G$, where G is a topological monoid. $V:\underline{Top}^G \to \underline{Top}$ is the canonical forgetful functor.

$Ho(\underline{Top}^G)$ = category of fractionss with respect to maps $f:X \to Y$ such that V(f) is a weak homotopy equivalence.

If G is a topological monoid such that $\pi_o(G)$ is a group under the induced multiplication, BG is its Dold-Lashof classifying space. If A is a space, OA is the Moore loops on A and thus a topological monoid. (Moore loops = maps $w: R^+ \to A$ together with a length parameter r, such that w(t) = the basepoint for $t > r$. Multiplication is juxtaposition and adding length parameters.)

Results:

1: $Ho(\underline{Top/A}) \simeq Ho\ (\underline{Top}^{OA})$ for A connected, with a given basepoint.

2: If $f: A \to B$ is a weak homotopy equivalence for $A,B \in |\underline{Top}|$, basepoint preserving, and A or B (and hence both) connected, then

$Ho(\underline{Top/A}) \simeq Ho(\underline{Top/B})$

3: $Ho(\underline{Top/BG}) \simeq Ho(\underline{Top}^G)$ for G a topological monoid, with $\pi_o(G)$ a group.

Remarks:

1. 2: follows from 1: via the following special case of a theorem of Beck: If $h: G \to G'$ is a homomorphism, and as a map

of topological spaces a w.h.e., then

$Ho(\underline{Top}^G) \simeq Ho(\underline{Top}^{G'})$.

2. An intermediate category in 1: is the category of regular, transitive fibrations over A with a fixed path lifting.

That category is denoted Fib/B and is tripleable over Top/B as well Ho-equivalent. (Weak homotopy equivalences being maps which are so when forgotten to Top/B).

3.3: follows from 1: by replacing the s.h.m. map from $G \to \underline{O}BG$ by a homomorphism which is still a w.h.e, from a new monoid $UG \to \underline{O}BG$. UG is a homotopy associative cotriple, and also has a natural $UG \to G$ a homomorphism and a homotopy equivalence. So applying Remark 1 twice yields the desired result.

Mikkelsen, C.J.: Characterisation of an Elementary Topos.

Lawvere and Tierney defined an elemtary topos to be a category $\underline{E}$ satisfying the following axioms.

(i) $\underline{E}$ has finite inverse limits.

(ii) $\underline{E}$ has finite direct limits.

(iii) $\underline{E}$ has a subobject classifier $1 \xrightarrow{\text{true}} \Omega$.

(iv) $\underline{E}$ has exponentiation.

Theorem: An elementary topos is a category $\underline{E}$ satisfying the axioms (i), (iii) and (iv) above (i.e. the finite direct limits can be construeted).

The proof is based an universal quantification, internal intersection and the universal property of the subobject classifies.

1) The initial object is the domain of $\forall_{\Omega\to 1}(\text{true})$

2) The image of $f: A \to B$ is $Im(f) = \bigcap_{D\in\mathcal{B}} D$ (internal intersection) where $\mathcal{B} = \{D \to B \mid f \in D\}$.

3) The coequalizer of an equivalence relation is the image of the exponential adjoint of the characteristic morphism of the equivalence relation.

4) The coequalizer of a pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is the coequalizer of the equivalence relation

$\equiv R$ generated on A by the relation $R = \mathrm{Im}\langle f,g\rangle$. $\equiv R$ is constructed as the internal intersection of all equivalence relations an A containing R.

5) The union operator $v: \Omega \times \Omega \to \Omega$ is defined by means of the equation $(\alpha v \beta) \Rightarrow \gamma = (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)$ where $\alpha, \beta, \gamma \in \Omega$, using the universal property of the subobject classifiers.

6) The coproduct of two objects A, B in $\underline{E}$ can now be constructed as the union of A and B imbedded in the product $\Omega^A \times \Omega^B$ by

$$\begin{tikzcd}
 & & \Omega^A \times \Omega^B & & \\
A \arrow[urr, dashed, "{\langle \{\ \}_A, \ulcorner false \urcorner \rangle}"] \arrow[rr, "e_A"'] & & A \vee B \arrow[u] \arrow[ul, phantom, "="] \arrow[ur, phantom, "="] & & B \arrow[ull, "{\langle \ulcorner false \urcorner, \{\ \}_B \rangle}"'] \arrow[ll, dashed, "e_B"]
\end{tikzcd}$$

The theorem is a joint work with F.W. Lawvere.

Mitchell, B.: The Mapping Theorem.

One can generalize a great part of noncommutative homological ring theory by replacing rings R by ringoids $\underline{C}$ (small preadditive categories), where the category Mod $\underline{C}$ of "left $\underline{C}$-modules" is interpreted as the category of covariant functors $M: \underline{C} \to Ab$.
One has the usual bifunctors.

$$\mathrm{Hom}_{\underline{C}} : (\mathrm{Mod}\ \underline{C})^* \times \mathrm{Mod}\ \underline{C} \to Ab$$

$$\otimes_{\underline{C}} : \mathrm{Mod}\ \underline{C}^* \times \mathrm{Mod}\ \underline{C} \to Ab$$

and their derived functors $\mathrm{Tor}^{\underline{C}}$ and $\mathrm{Ext}_{\underline{C}}$.

Now suppose given a map of ringoids (additive functor) $U: \underline{C} \to \underline{D}$, fixed right $\underline{C}$ and D-modules $Q_{\underline{C}}$ and $Q_{\underline{D}}$ respectively, and a map (natural transformation) $\psi: Q_{\underline{C}} \to Q_{\underline{D}} \circ U$. Such a situation induces a map

$$g: Q_{\underline{C}} \otimes_{\underline{C}} D(\ ,U(\ )) \to Q_D$$

which in turn induces maps

$$F^U: \mathrm{Tor}^{\underline{C}}(Q_{\underline{C}}, MU) \to \mathrm{Tor}^{\underline{D}}(Q_{\underline{D}}, M),\ M \in \mathrm{Mod}\ \underline{D}$$

$$F_U: \mathrm{Ext}_{\underline{D}}(Q_{\underline{D}}, N) \to \mathrm{Ext}_{\underline{C}}(Q_{\underline{C}}, NU),\ N \in \mathrm{Mod}\ \underline{D}\ .$$

Mapping Theorem: In order that F^U be an isomorphism for all M, it is necessary and sufficient that

(i) g is an isomorphism

(ii) $\mathrm{Tor}_n^{\underline{C}}(Q_{\underline{C}}, \underline{D}(\ ,U(\ )) = o$ for $n > o$.

In this case F_U is also an isomorphism for all N.

The construction of the map g, F^U, F_U, and the proof of the theorem is exactly as in Cartan-Eilenberg, page, 149. However the statement of the theorem is more general, not only because rings have been replaced by ringoids, but more important because the notion of an "augmented ring" has been eliminated from the picture. With this generality, one can deduce immediately the following corollary on the derived functors of the inverse functor.

Corollary: Let R be a ring, and let $U: \underline{C} \to \underline{D}$ be a cofinal functor between small categories where a cofinal functor between small categories where $\underline{C}$ (and hence $\underline{D}$) is filtered. Then for any $N \in (\mathrm{Mod}\ R)^{\underline{C}^*}$, we have

$$\varprojlim_{\underline{D}^*}{}^k N \simeq \varprojlim_{\underline{C}^*}{}^k NU.$$

Now define the R-cohomological dimension of a category C

$$cd_R\ \underline{C} = \sup \{k \mid \varprojlim_{\underline{C}^*}{}^k \neq o\}.$$

Then the above corollary enables one to reduce the following theorem to the case where $\underline{C}$ is a totally ordered set, in which case the proof can be carried out using results of Barbara Osofsky on the homological dimension of a direct module.

<u>Theorem:</u> Let $\underline{C}$ be a directed set and let $\aleph_n$ be the smallest cardinal number of a cofinal subset. Then

$$cd_R \underline{C} = n + 1$$

for all nonzero rings R.

<u>Müller, H.: Über Epimorphismen in der Kategorie der kleinen Kategorien</u>.

Sind $\underline{A}$, $\underline{B}$ kleine Kategorien und ist $\varphi : \underline{A} \to \underline{B}$ ein Funktor, so bezeichne

$V_\varphi : [\underline{B}, \text{Meng}] \to [\underline{A}, \text{Meng}]$ den Funktor mit

$$\begin{array}{ccc} F & \longmapsto & F\circ\varphi \\ \alpha \Big\downarrow & \longrightarrow & \Big\downarrow \alpha(\varphi) \\ G & \longmapsto & G\circ\varphi \end{array}$$

wobei $[\underline{A}, \text{Meng}]$ die Kategorie aller Funktoren von $\underline{A}$ in die Kategorie Meng der Mengen und der Abbildungen.

Eine Unterkategorie $\underline{C}$ einer Kategorie $\underline{K}$ nennen wir strikt voll, wenn sie eine volle Unterkategorie von $\underline{K}$ ist und gegenüber Isomorphismen in $\underline{K}$ abgeschlossen ist.

Eine Unterkategorie $\underline{C}$ einer Kategorie $\underline{K}$ heißt reflexive (koreflexive) Unterkategorie von $\underline{K}$, wenn der Inklusionsfunktor $\underline{C} \hookrightarrow \underline{K}$ einen Linksadjungierten (Rechtsadjungierten) besitzt.

<u>Theorem 1:</u> $\underline{A}$ sei eine kleine Kategorie; es gilt dann:

(1) Ist $\underline{C}$ eine strikt volle sowohl reflexive als koreflexive Unterkategorie von $[\underline{A},\mathrm{Meng}]$, so gibt es eine kleine Kategorie und einen Epimorphismus $\varphi : \underline{A} \to \underline{B}$ in der Kategorie der kleinen Kategorien, so daß $V_\varphi : [\underline{B},\mathrm{Meng}] \to \underline{C}$ ein Isomorphismus ist und φ auf Objekten bijektiv ist.

(2) Ist $\underline{A} \xrightarrow{\varphi} \underline{B}$ ein Epimorphismus in der Kategorie der kleinen Kategorien, so ist $V_\varphi : [\underline{B},\mathrm{Meng}] \to [\underline{A},\mathrm{Meng}]$ eine volle Einbettung, die einen Links- und Rechtsadjungierten besitzt.

<u>Theorem 2:</u> $\underline{A}$ sei eine kleine additive Kategorie; es gilt dann:

(1) Ist $\underline{C}$ eine strikt volle sowohl reflexive als auch koreflexive Unterkategorie von $(\underline{A},\mathrm{Ab})$, der Kategorie aller additiven Funktoren von $\underline{A}$ in die Kategorie Ab der abelschen Gruppen, so gibt es eine kleine additive Kategorie $\underline{B}$ und einen Epimorphismus $\varphi : \underline{A} \to \underline{B}$ in der Kategorie der kleinen additiven Kategorien, so daß $V_\varphi : (\underline{B},\mathrm{Ab}) \to \underline{C}$ ein Isomorphismus ist und φ auf Objekten bijektiv ist

(2) Ist $\underline{A} \xrightarrow{\varphi} \underline{B}$ ein Epimorphismus in der Kategorie der kleinen additiven Kategorien, so ist $V_\varphi : (\underline{B},\mathrm{Ab}) \to (\underline{A},\mathrm{Ab})$ eine volle Einbettung, die einen Links- und Rechtsadjungierten besitzt.

Theorem 2 beweist man analog wie Theorem 1.

<u>Mulvey, C.: Rings in a Topos.</u>

In a topos $\underline{E}$, as in any category with finite limits, the concepts of ring and of module may be defined. However, the internal logic of the topos also allows the consideration of those concepts which ivolve logical predicates. Thus a <u>field</u> in $\underline{E}$ man be defined by the sentence.

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$$\forall x \in R\ (\neg (x \in U(R)) \Rightarrow x = o)$$

for a ring R, where U(R) is the subobject of R defined by

$$\{\, x \in R \mid \exists\, y \in R\ (xy = 1 \wedge yx = 1) \,\}$$

Similarly a *local* ring is defined in *E* by

$$\forall x \in R\ \forall y \in R\ (x+y \in U(R) \Rightarrow (x \in U(R) \vee y \in U(R))).$$

Although the logic of *E* is intuitionstic, it may be that a proof in standard ring theory may be valid in *E* provided that the definitions involved are made appropiately in the topos and that any non-standard conditions which are necessary for the proof to be valid in *E* are added. The Theorem in the topos *E* may possibly then externalise to an extended theorem in the category of sets *S*, for example concerning the rings of sections of certain rings in *E*, once again provided that some conditions on the ring may be needed in order that this externalization can take place.

For the ring module theorist interested in algebra in the category of sets the programme might be described diagrammatically by

$$\boxed{\text{Ring Theory in } S} \ \underset{\text{Sections}}{\overset{\text{Representations}}{\rightleftarrows}} \ \boxed{\text{Ring Theory in } E}$$

that is represent ring in the category of sets and their theory by rings in a topos *E* and their theory. Then relate this theory in *E* to the theory in *S* by taking sections. Hopefully in this may, by suitable choice of represantion standard theorems in *S* may give use to extended theorems in *S*.

An example to illustrate the principle is the following: the theorem that over a local ring every projective module is free when interpreted in the topos Top(X) of sheaves on a topological space X yields extended theorems which when interpreted include Swan's theorem on vector bundles an a compact space and a theorem of Pierce on projective modules over commutative regular rings.

To extend the theorem one needs that the ring R considered is not only local but als a field. (In a topos a field need not be a local ring). Further in Top (X) the ring R must be compact: that is, that x be compact and that for x, y $\in$ X distinct there exist a section f of R over X with $f_x = 1$ and $f_y = o$. Then

Theorem: If R is a compact local field in Top (X), then for a finitely generated module A the following are equivalent:

i) A projective

ii) A free

the proof being essentially that obtained by internalizing that which proves that over a local ring every projective is free.

To externalize the theorem one notes that R compact implies that the functor

$$\text{Mod } R \longrightarrow \text{Mod } R(X)$$

obtained by taking sections is an equivalence of categories. Then noting that a free module in Top(X) is externally describable or a locally free module, one obtains:

Corollary: For a compact local field R in Top (X), the section functor establishes an equivalence between the categories of finitely generated locally free R-modules and of finitely generated projective modules over the ring R (X) of sections of R.

In the case that X is a compact space, the ring $\mathbb{R}$ in the topos Top (X) is a compact local field. The finitely generated locally free $\mathbb{R}$-modules are externally exactly the real vectorbundles on X. The theorem then externalizes as Swan's theorem, extablishing an equivalenve between the categories of vectorbundles on X. and of f.g. projective C(X)-modules.

In the case that X is the spectrum of a commutative regular ring, the affine scheme R in Top(X) is a compact local fields. The theorem obtained in this case, in particular describing Grothendieck ring of a commu_tative regular ring, is that due to Pierce.

Osius, G.: A characterization of the category of sets

Let us state main result first and explain the notions involved later:

Metatheorem: The CZ-topoi (i.e. models of topos-theory CZ) are up to equivalence the categories of sets in the models of the set-theory Z. This result still holds if we simultaneously the "same" axioms to CZ and Z (e.g. axioms of infinity, replacement, choice) thus getting in particular the topos-theory CZF and Zermelo-Fraenkel's set theory ZF.

The set-theory Z is a first order theory with one binary relation satisfying (i) the axioms of extensinality and regularity, (ii) the following axioms of set-existence: empty set unordered pair-set, powerset PM, union set $\bigcup M$, limited seperation - schema - i.e. for formulars $\mathfrak{A}(x)$ with bounded quantifiers the set $\{x \in M \mid \mathfrak{A}(x)\}$ exists for any M -, and (iii) the following two axioms:

(T) Any set is a subset of a transition set.

(TR) Any extensional well-founded relation $\langle A,R\rangle$ can be represented by restriction of the $\in$ -relation to a (unique) transitive set T: $\langle A,R\rangle \cong \langle T,\in\rangle$

The set theory Z plus the axiom of infinity and the replacement-schema is Zermelo-Fraenkel's theory ZF (which doesnot include the axiom of choice). We note that Z is finitely axiomatizable and that (T), (TR) follow from the replacementscheme and the axiom of infinity.

The topos-theory CZ is the first order-logic-formalization of Lawvere-Tierney's theory ET of elementary topos plus the following axioms: Non-triviality ($0 \neq 1$), 1 is a separator (generator), and (RM)

(RM) For any object C there is an RM-object $A \to PA$ (this will be explained below) and a monomorphism $C \to A$.

Adding the axiom of infinity - i.e. there exists a natural number object - and the categorical version of the replacement-schema we get the topos-theory CZF. We note that "strange" axiom (RM) is a consequence of the axiom of choice (i.e. epis split) which we do not include in CZF.

Let as now just touch the proof of the metatheorem. Starting with set-theory Z we get a model of CZ, namely the topos of sets in Z and conversely, starting with topos-theory CZ we can construct within CZ a model of set-theory Z:

$$\begin{tikzcd} \text{models of set-theory } Z \arrow[r, shift left, "\text{category of sets}"] & \text{models of topos-theory } CZ \arrow[l, shift left, "\text{set-theoretic model}"] \end{tikzcd}$$

Now the performance of both constructions one after the other gives a model which is equivalent to the original one. By "models" we actually mean "inner models" given by an interpretation of one theory within the other which makes over method purely syntactical.

Finally let as define RM-objects in topos-theory ET with the aid of the (covariant) power-functor P: An $A \longrightarrow PA$ is called an RM-object iff r is monic and recursive i.e. has the property that for any $PB \xrightarrow{q} B$ there is a unique (recursively defined) $A \xrightarrow{f} B$ that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "r"] & B \arrow[d, "q"] \\ PA \arrow[r, "Pf"] & Pb \end{tikzcd}$$

commutes. The RM-objects in the topos of sets in Z are up to isomorphism precisely the inclusions $T \rightarrow PT$ for transitive sets T.

The main idea behind the construction of the set-theoretical model in CZ is that the $\in$-structure of a set M can be fully recovered only in a transitive set T containing M, so that a set is actually equipped with a "structure" T. Accordingly a set in the model in CZ will be a pair $<A \xrightarrow{r} PA,\ A \xrightarrow{M} \Omega>$ where r is an RM-object. The definition of equality and $\in$ between sets in the model will not be given here.

Pfender, M.: Monoidale Theorien und monoidale Algebren

Beschreibung von Algebren in monoidalen Kategorien durch - bewahrende Funktoren von monoidalen Theorien in die monoidale Grundkategorie. Hierarchie von Kategorien von Theorien (= monoidale Kategorien mit Zusatzstruktur).

Methoden: Theorie der monadischen Algebren. Zahlen beziehen sich auf den Preprint "Monoidale Theorien".

Prämonoidale und monoidale Kategorien

2.5 Definition: $B \neq (B,\otimes) = (B,(\otimes_n : B^n \to B)_{n \in \mathbb{N}_0})$ heißt prämonoidale Kategorie.

A: T $\to$ B verträglich mit $\otimes$ heißt prämonoidaler Funktor.

2.9 Definition: des freien $\otimes$-Magmas (I) über I Meng der "beklammerten Wörter" über I. Beispiel: für $I = \{1\}$: $((11)1) \in (\{1\}) =: \tilde{N}$.

Für $v \in (I)$ sei $|v|$ die Anzahl der Buchstaben von v.

2.11 rekursive Definition von $\otimes_n : B^{|n|} \to B$ $(n \in \tilde{N})$:

$$\otimes_{((n_i)_{i<m})_m} := \otimes_m \cdot X_m(\otimes_{n_i})_{i<m} : B \xrightarrow{\Sigma |n_i|} B.$$

Beispiel: $\otimes_{((11)1)}(a,b,c) = (a \otimes b) \otimes c$.

2.13 Definition: Kategorie Sub der formalen Substitutionen:

Sub = N (Objekte) $Sub(n,m) = \{(n,\flat,m) \mid \flat\ \ Meng(|m|,|n|)\}$ ("Substitution von $|m|$ Variablen durch $|n|$ Variable"). Unterkategorien: S = Ass $(\flat = 1_{|n|}$, Assoziativitäten), = Sym ($\flat$ bijektiv), =Diag, = Term.

<u>2.14 Def.</u> Kategorie ∇ B:

$$\begin{tikzcd}
B^m_T \arrow[rr, "F"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\eta"] & & B^n_U \arrow[dl] \\
& B &
\end{tikzcd}$$

Für B = (B,$\otimes$) gilt ∇ B $\in$ Pmkat (prämon.Kat.).

<u>2.15 Def.</u> B = (B,$\otimes$,can), can: S $\to$ (∇B,$\otimes$) prämonoidal, heißt <u>monoidal</u> für S = Ass, <u>symmetriesch</u> für S = Sym, ... , <u>substitutiv</u> für S = Sub, halb S, falls die Familie can$b$ ($b \in$ S) die Funktortransformations-Eigenschaft nicht verlangt wird.

Monoidale Theorien

<u>4.1 Def.</u> Th_I := $Pmcat_{(I)}$ = I-stellige prämonoidale Theorie (prämonoidale Kategorien mit festem Objekt-Magma (I)).

<u>4.2</u> Kategorien von mehrstelligen formalen Substitutionen:

$$\begin{tikzcd}[column sep=small]
& & \mathrm{Diag}_I \arrow[dr, hook] & & \\
\mathrm{Ass}_I \arrow[r, "\subseteq"] & \mathrm{Sym}_I \arrow[ur, hook] \arrow[r, hook] & \mathrm{Term}_I \arrow[r, hook] & \mathrm{Sub}_I &
\end{tikzcd}$$

in Th_I.

<u>45 Theorem.</u> In

$$\begin{tikzcd}[column sep=small]
LTh_I \arrow[dd] & & & & & & \\
& \mathrm{Diag}\,Th_I \arrow[rr] \arrow[dd] & & \mathrm{Sym}\,Th_I \arrow[r] \arrow[dd] & \mathrm{Ass}\,Th_I \arrow[dd] & & \\
\mathrm{Sub}\,Th_I \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Term}\,Th_I \arrow[ur] \arrow[dd] & & & & \\
& \mathrm{Diag}_I{\downarrow}Th_I \arrow[rr] & & \mathrm{Sym}_I{\downarrow}Th_I \arrow[r] & \mathrm{Ass}_I{\downarrow}Th_I \arrow[r] & Th_I \arrow[r] & \mathrm{Graph}_I \\
\mathrm{Sub}{\downarrow}Th_I \arrow[ur] \arrow[rr] & & \mathrm{Term}_I{\downarrow}Th_I \arrow[ur] & & & &
\end{tikzcd}$$

alles stark monadisch.

<u>4.8 Satz.</u> Sei $\underline{T}$ $\in$ 4.5, F —| U: $\underline{T} \to Graph_{(I)}$, $\Omega \in Graph_{(I)}$ (Operationen), G $\subseteq$ FΩ' X'FΩ $\in$ $\underline{T}$ (cartes.Quadrat der Morphismenmengen bei fester Objektmenge). Zur <u>Gattung</u> (Ω,G) gehörige Theo-

rie $D(\Omega,G) := \mathrm{dcok}(pr_1, pr_2 : G \to F\Omega)$ rel. U. Dies definiert Funktor D: Gatt-$\underline{T} \to \underline{T}$. Die Konstruktion ist möglich wegen 4.5.

Monoidale Algebren

5.2 Satz. B monoidal mit $/\!/$, $\otimes$ distributiv rel. $/\!/$ [und mit Dcok's von Kernpaaren mehrstellig vertauschbar] $\Rightarrow$ $U: \mathrm{Alg}(\Omega,B) \to B^I$ [stark] monadisch. $(A \longmapsto (A_i)_I\ ,\ f \longmapsto (f_i)_I\ )$.

5.3-5.5 Sätze. $\mathrm{Alg}(\Omega,B) \simeq \mathrm{Funkt}_{\otimes S}(F\Omega,B)$ natürlich in $\Omega \in \mathrm{Graph}_{(I)}$, $B \in [h]$ Spmkat, $F \dashv U: S_I \downarrow Th_I \to \mathrm{Graph}_{(I)}$. Dabei $A \in$ $\mathrm{Funkt}_{\otimes S}$: $\Leftrightarrow$ A respektiert can: $S \to \nabla F\,\Omega$.

5.7 Theorem. $(\Omega,G) \in \mathrm{Gatt}_{S_I \downarrow Th_I}$ algebraisch (d.h. Coarität 1), $B \in [\bar{h}]$ SPmkat mit Eigenschaften wie in 5.2

$$\begin{tikzcd}
\mathrm{Alg}((\Omega,G),B) \arrow[r, "\subseteq", "\text{stark monad}"'] \arrow[rr, bend right, "= \text{ stark monad.}"'] & \mathrm{Alg}(\Omega,B) \arrow[r, "\text{Vergißfunktor}", "\text{stark monad.}"'] & B^I
\end{tikzcd}$$

Damit Konstruktion von Limites und Colimites in $\mathrm{Alg}((\Omega,G),B)$.

5.8 Satz. B wie in 5.7. $f: (\Omega,G) \to (\Omega',G')$ Morphismus von algebraischen Gattungen $\Rightarrow$ algebraischer Funktor

$\mathrm{Funkt}_{\otimes S}(Df,B) : \mathrm{Funkt}_{\otimes S}(D(\Omega',G'),B) \to \mathrm{Funkt}_{\otimes S}(D(\Omega,G),B)$

stark monadisch.

Rattray, B.A.: Torsion Theories in Non-Additive Categories

The following is intended to describe the common features of classical torsion theories in abelian categories, sheaf reflectors

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in functor categories and the separated completion reflector in uniform spaces.

Let $\underline{A}$ be a complete, cowell powered category with finite colimits in which each map factors into an epi followed by an r-mono (i.e. equalizer). Let r-injective object, r-essential extension, r-injective hull be defined as usual but using r-monos. We call a (full, replete) subcategory of $\underline{A}$ a TFD subcategory if it is the limit closure of a class of r-injectives. Adjoining all r-subobjects to a TFD subcategory we obtain a TF subcategory.

<u>Theorem 1:</u> A limit closed subcategory $\underline{B}$ is TFD iff:

(i) it has r-injective extensions in $\underline{A}$ and (2) the inclusion $\underline{B} \to \underline{A}$ reflects r-monos.

<u>Theorem 2:</u> A limit closed subcategory $\underline{C}$ is TF iff (1) as above and (2) any r-subobject of an object of $\underline{C}$ is in $\underline{C}$.

<u>Theorem 3:</u> If C is TF then there is an epi-reflector T: $\underline{A} \to \underline{C}$ and $T(M(\underline{A})) \subset M(\underline{A})$, where $M(\underline{A})$ is the class of r-monos of $\underline{A}$.

<u>Theorem 4:</u> If $\underline{C}$ is TF then:

(1) there is a unique TFD subcategory $\underline{B}$ such that $\underline{C}$ is the category of r-subobjects of $\underline{B}$;

(2) there is a reflector D: $\underline{C} \to \underline{B}$,

(3) $D(M(\underline{A}) \cap C) \subset M(\underline{B})$.

<u>Theorem 5:</u> If $\underline{B}$ TFD then there is a reflection Q: $\underline{A} \to \underline{B}$ and Q preserves r-monos, i.e. $Q(M(\underline{A})) \subset M(\underline{B})$.

If $\underline{A}$ has r-injective hulls then the converses of Theorem 3 and 5 are true.

The following concepts play basic roles in the proofs: dense and closed r-subjects, closure of an r-subobject, complete (or divisible) object. Objects in $\underline{C}$ are called separated (or torsion free).

Semadeni, Z.: The category of logical Kits (Summary)

<u>Definition:</u> A <u>logical kit</u> shortly a <u>kit</u>, is a quintuple (X,A,V,f,p) where X,A,V are sets and

$$f\colon X \times A \to V \qquad \text{and} \qquad p\colon V \to A$$

are functions such that the diagram

$$\begin{tikzcd} X \times A \arrow[rr, "f"] \arrow[dr, "\pi"'] & & V \arrow[dl, "p"] \\ & A & \end{tikzcd}$$

is a commutative, where π is the second coordinate projection.

<u>First motivation</u>. In educational experiments in kindergarten one uses various kits to teach elements of logic and set theory. In such a kit one can distinguish:

a set X of <u>things</u>,
a set A of <u>features</u>,
if $a \in A$, a set V_a of <u>values</u> of the feature a,
if $a \in A$ and $x \in X$ then $f(x,a)$ is the value of feature a at the thing x Moreover, we set $V = \coprod_{a \in A} V_a$ (disjoint union), $p\colon V \to A$ - canonical projection. The most popular is the <u>classical kit</u> of Dienes in which:

$A = \{\text{shape, color, size, thickness}\}$,

$V_{shape} = \{\text{square, oblong, triangle, disc}\}$,

$V_{color} = \{\text{red, yellow, blue}\}$, etc.

$X = V_{shape} \times V_{color} \times V_{size} \times V_{thickness}$,

$f_a : X \to V_a$ is the a-th coordinate projection.

The notion of a kit originated from some classification problems for logical kits already used in kindergarten and kits to-be-invented.

<u>Second motivation.</u> So e mathematicians working in computer science (e.g., Z.Pawlak) claim that the notion of a logical kit should be very useful in the theory of classification (e.g., recognizing shapes by computer). Yet, no particular results are known to me.

THE CATEGORY OF KITS is defined in a natural way. It was investigated by A. Wiweger and me. Also F.W. Lawvere made some interesting observations about it; in fact, the definition of an kit written above is Lawvere's modification of the definition originally proposed.

The category is - as one may expect - both complete and cocomplete. Yet, coproducts and coequalizers - when written explicity - are somwhat strange. Left and right adjoints of some functors can also be given in an explicit form.

<u>Street, R.: Abstract. Two universal properties for the category of sets in the 2-category of categories.</u>

For simplicity in this abstract, size considerations will be ignored.

Let $\underline{K}$ be a 2-category in which 2-pullbacks and comma objects exist. The commaobject of

$$\begin{tikzcd} A \arrow[dr, "r"'] & & B \arrow[dl, "s"] \\ & D & \end{tikzcd}$$

determinesa span

$$\begin{tikzcd} & r/s \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

Those span from A to B which appear as commaobjects are called <u>distributors</u> from A to B. Let Dist (A,B) denote the category of distributors from A to B as a full subcategory of the category Spn(A,B) of spans from A to B. By pullback Dist (A,B) becomes 2-functorial in B.

The first universal property of $\ulcorner A$ is that it should 2-represent Dist (A,-) up to equivalence. That is, there should be a 2-natural equivalence of categories

$\underline{K}(B,\Gamma A) \cong \mathrm{Dist}\ (A,B)$.

<u>The representation arrow</u> is defined to be the arrow $A \xrightarrow{Y_A} \Gamma A$ which corresponds to the distributor

$$\begin{tikzcd} & A/A \arrow[dl, "d_o"'] \arrow[dr, "d_1"] & \\ A & & A\ . \end{tikzcd}$$

<u>Theorem</u>: The functor $K(B,\Gamma A) \to \mathrm{Dist}\ (A,B)$ which takes

$B \xrightarrow{h} \Gamma A$ <u>to</u> $\begin{tikzcd}[column sep=small] & Y_A/h \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$ <u>is an equivalence of categories</u>.

A second universal property for ΓA was developed in joint work with R.F.C. Walters for the case $\underline{K} = \underline{V}$ - Cat and $\Gamma A = [A^{op},\underline{V}]$. This property is that there should be an arrow $Y_A: A \to \Gamma A$ satisfying the following condition.

SW Given any arrow $f: A \to B$, there exists an arrow $c_f: B \to \Gamma A$ (the characteristic arrow of f) and a 2-cell

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "Y_A"'] & \arrow[d, phantom, "\overset{\psi_f}{\Rightarrow}"] & B \arrow[dl, "c_f"] \\ & \Gamma A & \end{tikzcd}$$

unique up to isomorphism of such with the property that, for any arrow $X \xrightarrow{u} A$, the 2-cell.

$$\begin{tikzcd} & X \arrow[dl, "fu"'] \arrow[dr, "Y_A u"] \arrow[d, phantom, "\Downarrow \psi_f u"] & \\ B \arrow[rr, "c_f"'] & \ & \Gamma A \end{tikzcd}$$

exhibits f u as a left lifting of Y_A u through c_f. Furthermore, ψ_f exhibits c_f as a left extension of Y_A along f.

<u>Theorem:</u> The representation arrow satisfies SW

In this setting a great deal of theory can be developed which results are familiar for K = V-Cat. For example, the hom-set version of adjointness, <u>pointwise</u> left extensions, cocomplete objects, the relations between Kleisli and Eilenberg-Moore algebra constructions, an embedding in $\underline{K}$ (K,ΓK)-Cat,....

<u>Thiébaud, M.: Algebras associated to an arbitrary functor</u>

Every functor U: $\underline{B} \to \underline{A}$ considered as a $\underline{B}$-$\underline{A}$-bimodule U_* has a canonical coadjoint U^* (U_* is $\underline{A}$ considered as a $\underline{B}$-$\underline{A}$-bimodule via U, U^* is $\underline{A}$ considered as an $\underline{A}$-$\underline{B}$-bimodule.) The composite $U^* o U_*$ (= $\underline{A} \otimes_{\underline{B}} \underline{A}$) is an $\underline{A}$-$\underline{A}$-bimodule canonically equipped with a comonoid structure. Sending U to $U^* o U_*$ defines a functor, which we call the structure functor, from the category (<u>Cat</u>, $\underline{A}$) of categories over $\underline{A}$ to the category <u>Com</u> ($\underline{A}$) whose objects are $\underline{A}$-$\underline{A}$-bimodules equipped with a comonoid structure. A functor in the other direction is defined by associating to an object G in <u>Com</u> ($\underline{A}$) a category $\underline{A}$(G) over $\underline{A}$, extending the Eilenberg-Moore construction. (The objects of $\underline{A}$(G) are the set-like, or group-like, elements of the coalgebra G.) We call this functor the semantics functor. Structure is adjoint to semantics and by composition they define on (<u>Cat</u>,$\underline{A}$) a monad $Alg_{\underline{A}}$ whose value at a category U: $\underline{B} \to \underline{A}$ over $\underline{A}$ we denote by $\bar{U}$: $\underline{A}(U) \to \underline{A}$. $\underline{A}$(U) is the category of U-algebras. If U has an adjoint F (or a coadjoint R) there $\underline{A}(U) = \underline{A}^T$, the category of algebras associated to the induced monad T = FU on $\underline{A}$ (resp. $\underline{A}(U) = \underline{A}_G$, the category of coalgebras associated to the induced comonad G = RU on $\underline{A}$). If U is the inclusion of a full subcategory of $\underline{A}$ then $\underline{A}$(U) is the smallest full subcategory of $\underline{A}$ containing $\underline{B}$ and closed under retracts. If U: $\underline{B} \to \underline{1}$ is the unique functor to a terminal object in <u>Cat</u> (in those examples we are considering categories based on sets) then $\underline{1}$(U) is $\pi_o(\underline{B})$, the discrete category of connected components of $\underline{B}$.

If U is a fibration (or an opfibration) then $\overline{U}$ is the fibration (resp. the opfibration) with discrete fibres associated to it.

We call an algebra in the sense of Eilenberg-Moore over the monad $\mathrm{Alg}_{\underline{A}}$ a category algebraic over $\underline{A}$. In all of the above examples, but not in general, a category $U: \underline{B} \to \underline{A}$ is algebraic over $\underline{A}$ if and only if $\underline{B} \cong \underline{A}(U)$. Thus, in particular, in the presence of an adjoint (or a coadjoint) algebraic and monadic (resp. comonadic) mean the same. Exponentiation and pulling back of algebraic categories are algebraic i.e. given $U: \underline{B} \to \underline{A}$ algebraic over $\underline{A}$ then, for any $\underline{D}$, $U^{\underline{D}} : \underline{B}^{\underline{D}} \to \underline{A}^{\underline{D}}$ is algebraic over $\underline{A}^{\underline{D}}$ and, for any functor $g: \underline{A}' \to \underline{A}$, the pullback of U along g is algebraic over $\underline{A}'$.

Tierney, M.: Foundations of Analysis in Topos

Let $\underline{E}$ be a topos with natural numbers object $\mathbb{N}$. Mimicing the ordinary constructions in the category of sets $\underline{S}$ one obtains the objects $\mathbb{Z}$ and $\mathbb{Q}$ of integers rational numbers respectively. The internal theory of order on $\mathbb{Q}$ is uncomplicated, since the intuitionistic and classical theory of the rationals coincide. To define the reals $\mathbb{R}$, we have a symmetric definition of Dedekind cut: a cut in $\mathbb{Q}$ is a pair $<c,c'>$ with $c' \rightarrowtail \mathbb{Q}$, $c \rightarrowtail \mathbb{Q}$, c' a lower cut c an upper cut, $c' \wedge c = o$ and $c - c'$ = the upper cut determined by o. Expressing these conditions internally yields $\mathbb{R} \rightarrowtail \Omega^{\mathbb{Q}} \times \Omega^{\mathbb{Q}}$. If $\underline{E}$ = sheaves (T) then $\mathbb{R}$ in the sheaf of germs of continuous real valued functions on T. If $\mathcal{T}$ is an intuitionistic first order theory, then one can consider the classical notion of a model of $\mathcal{T}$ whose truth values are open subsets of T - call these topological models of . Then the adjoint pair sheaves $(T) \rightleftarrows \underline{S}$, where $\Gamma(F)$ = global section of F, and $\hat{x}$ = constant sheaf of x, yields a 1 - 1 correspondance between constant models of $\mathcal{T}$ in sheaf (T) and topological models of $\mathcal{T}$ in $\underline{S}$. Consideration of work of Dona Scott's on topological models of intuitionistic analysis together with the previous computation of $\mathbb{R}$ in sheaf (T) showe that topological models cannot capture intuitionistic notions of developement.

In an arbitrary topos $\underline{E}$ one can show that $\mathbb{R}$ is an intuitonistic field ond one can consider the category of internal finite dimensional vector over $\mathbb{R}$. In sheaves (T) this category is equivalent to the category of finite dimensional vector bundles on T.

Volger, H.: Abstract, Logical categories, polyadic categories and topoi

In 1965 Lawvere suggested a definition of an elementary theory as an application of functorial semantics to model theory. As suggested by Lawvere a completeness theorem has been proven for elemtary theories and the more general logical categories. In particular, one obtains a completeness theorem for higher order logic, if one considers logical categories with exponentiation.

Polyadic categories are certain regular categories (in the sense of Barr), where quantification can be defined means of direct image. A polyadic category with exponentiation is a boolean topos. Now the free polyadic category resp. free boolean topes over a logical category resp. a logical category with exponentiation can be constructed using the functional relations. It should be remarked that the same construction works also in the non-boolean case i.e. if Ω is a Heyting-algebra object rather than a Boolean-algebra object.

As an application one can obtain the factorization of a first-order functor between two toposes of Kock and Mikkelsen. Moreover this result should be useful for the construction of the free topos over an arbitrary category.

Wick-Negrepoints, J.: Duality of Functors in BAN

Let BAN denote the category of all Banach spaces over $\mathbb{R}$ and linear contractions between them. BAN is a closed monoidal category, its internal hom $[X,Y]$ being given by all linear continous morphisms from X to Y, for X, Y $\in$ BAN. We denote by

Ω_X the functor $[X,-]$ and by Σ_X its left adjoint. Explicitly, $\Sigma_X(Y) = X\hat{\otimes} Y$, where $X\hat{\otimes} Y$ is the completion of the algebraic tensor product $X \otimes Y$ with respect to the greatest crossnorm. We let $[F,G]$ denote the Banach space whose unit ball is the set of natural transformations from F to G, where F,G: BAN $\rightarrow$ BAN are BAN-functors, whenever this forms a set. We remark that $[F,\Sigma_X]$ is a set for every $X \in$ BAN. The dual of an (BAN-) endofunctor on BAN, in the sense of Fuks-Svarc-Mityagin, is defined as follows: there is a functor D from $(\mathrm{BAN}^{\mathrm{BAN}})^{\mathrm{op}}$ to $\mathrm{BAN}^{\mathrm{BAN}}$ which associates to each functor F the functor DF (called the dual functor of F) defined for $X \in$ BAN, $f \in$ BAN(X,Y) by

$$DFX = [F,\Sigma_X], \quad DF(f)(\gamma) = \Sigma(f) \circ \gamma,$$

for $\gamma \in [F,\Sigma_X]$. If α: F $\rightarrow$ G, then $D\alpha$: DG $\rightarrow$ DF is the natural transformation $D\alpha_X(\gamma) = \gamma\,\alpha$, for $\gamma \in |G,\Sigma_X|$. It can easily be seen that D is self-adjoint on the right.

<u>Definition:</u> F: BAN $\rightarrow$ BAN is said to be reflexive if $F \cong D^2F$, under the morphism which corresponds to 1_{DF} under the above adjunction.

<u>Definition:</u> F is said to be finite dimensional if $F \sim \Sigma_A$ (isomorphic but not isometric), where A is a finite dimensional Banach space.

Let <u>A</u> be the full subcategory of BAN consisting of all Banach spaces X which satisfy the metric approximation property (1_X is the limit of a directed set of finite dimensional operators).

<u>Definition:</u> A functor F is said to be computable if for every $X \in$ BAN, $FX = \varinjlim FY$, where Y runs through the finite dimensional subspaces of Y.

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The following lemma is crucial the proof of the main theorem.

LEMMA. Let F: $\underline{A} \to$ BAN be finite dimensional.

Then F is computable.

Theorem: Let F: $\underline{A} \to$ BAN be finite dimensional. Then F is reflèxive.

Proposition: Let F: $\underline{A} \to$ BAN be any functor. Then DF is reflexive if and only if $D\,F\,\mathbb{R} \cong D^3 F\,\mathbb{R}$ under the canonical morphism.

For computable functors we obtain the following representation theorem.

Theorem: $DF(X^*) \cong F(X)^*$.

Using the above theorem, we are able to compute the duals of certair concrete functors. As an example of this, we derive a categorical definition of the integral operators from one spaces to another as the dual functor of a certain computable functor.

Wischnewsky, M.: Universal Algebra in Initial-Categories

Initial functors F: $\underline{K} \to \underline{L}$, the categorial generalization of Bourbaki's notion of an "intial object", reflect a lot of properties (as e.g. (co)-completeness, (co)-wellpoweredness, the existence of projective or injective objects, (co)-generators, or bicategory-structures) from the base cat. $\underline{L}$ to the initial-cat. $\underline{K}$. Especially all important theorems of universal algebra are valid in algebraic categories over $\underline{K}$ (and even in (epi)-reflective or coreflective subcategories) iff they are valid in the corresponding algebraic cat. over the base cat. $\underline{L}$. Examples of intial-cat. are(with obvious intial functors) : the cat. of topological, measurable, locally convex, limit, compactly generated, or zero dimensional spaces.

An algebraic theory ($\underline{C}$,C) consists of a small cat. $\underline{C}$ together with a set $\mathbb{C}$ of diagrams $S: \underline{D}_S \to \underline{C}$, having limits in $\underline{C}$ (generalization of an equationally defined theory in the sense of Lawvere). A theorymorphism $f: (\underline{C},\mathbb{C}) \to (\underline{D},\mathbb{D})$ is a functor f: $f: \underline{C} \to \underline{D}$, which preserves all $\mathbb{C}$-limits and which is compatible with $\mathbb{C}$ and $\mathbb{D}$ i.e. if $S \in \mathbb{C}$, then $fS: \underline{D}_S \to \underline{D} \in \mathbb{D}$. Let $\underline{K}$ be a complete cat.. The full subcat. $Alg(\underline{C},\underline{K})$ of all $\mathbb{C}$-limitpreserving functors from the functor cat. $(\underline{C},\underline{K})$ is called a $\underline{K}$-algebraic cat.. If $f: (\underline{C},\mathbb{D}) \to (\underline{D},\mathbb{D})$ is a theory-morphism then the functor $Alg(f,K)$: $Alg(\underline{D},\underline{K}) \to Alg(\underline{C},\underline{K})$, induced by f, is a $\underline{K}$-algebraic functor. The complete category $\underline{K}$ is called universal-algebraic iff for all theories $(\underline{C}.\mathbb{C})$, the inclusion $I: Alg(\underline{C},\underline{K}) \to (\underline{C},\underline{K})$ is adjoint (i.e. has a left adjoint)

<u>Theorem 1.:</u> Let $\underline{K}$ be a bicomplete, universal-algebraic category. Then

1. Every $\underline{K}$-algebraic functor is adjoint,
2. Every evaluation functor $V_C : Alg(\underline{C},\underline{K}) \to \underline{K} : A \longmapsto AC$ is adjoint.

For example every locally presented category in the sense of Gabriel-Ulmer is bicomplete and universl-algebraic.

<u>Theorem 2.:</u> Let $F: \underline{K} \to \underline{L}$ be an initial functor over a complete category $\underline{L}$. Then the canonical induced functor

$$\mathbb{F} := Alg(\underline{C},\underline{K}) \to Alg(\underline{C},\underline{L}) : A \longmapsto FA \text{ is again an intialfunctor.}$$

Especially it is adjoint and coadjoint and preserves and reflects monos and epis.

<u>Corollary</u>: Let F be as in Theorem 2.

1. $Alg(\underline{C},\underline{K})$ is (co)-complete, (co)-wellpovered iff $Alg(\underline{C},\underline{L})$ has these properties.

2. $Alg(\underline{C},\underline{K})$ possesses generators, cogenerators,..., or is a (cokernel, mono-) bicat. iff the same is true for $Alg(\underline{C},\underline{L})$.

Theorem 3: Let F: $\underline{K} \to \underline{L}$ be an initial functor over a universal-alg. category $\underline{L}$. Then $\underline{K}$ is universal-algebraic.

Examples:

1. Every initial cat. over a locally presented cat. is universal-alg., as e.g. every initial cat. over $\underline{S}$, the cat. of sets. (Top, Meas, Unif,...)

2. Every coreflective subcategory of Top, Meas, Unif, Locconv is universal-algebraic, (Examples are the categories of compactly-generated, locally path-connected, finite generated spaces...)

The restriction of a more general theorem yields the following

Theorem 4: Let $\underline{K}$ be an initial category over $\underline{S}$. Then every epireflective subcategory of $\underline{K}$ is universl-algebraic.

Examples: The categories of T_o, T_1, T_2, T_3 - spaces, completely regular, or zero dimensional spaces.

Since the notion of an initial category is selfdual, F: $\underline{K} \to \underline{L}$ is an initial functor iff $F^{op} : \underline{K}^{op} \to \underline{L}^{op}$ is an initial functor. Since the algebras A: $\underline{C} \to \underline{K}^{op}$ are exactly the $\underline{C}$-coalgebras over $\underline{K}$, the results can immediately be applied for coalgebras.

Wraith, G.C.: Enrichment of Algebras over Coalgebras

Let k be a field, and let $\underline{A}$ be the category of either i) k vector spaces, ii) graded k vector spaces or iii) differential graded k vector spaces. Then $\underline{A}$ is a closed monoidal category with its appropriate Hom and ⊗ . The category Coalgebra of coassociative cocomuutative coalgebras over $\underline{A}$ is Cartesian closed, and the category Alg of associative commutative algebras over A is enriched over Coalg, i.e. is a tensored, cotensored Coalg-category. I heard the following non-constructive proof of the existence of the exponential in Coalg from Jon Beck: Sweedler proves in his book on Hopf algebras that every coalgebra is a direct limit of its finite type subcoalgebras; these form a set. It is easy to see that products distribute over colimits, so one may apply Freyd's adjoint functor theorem. The coalg-valued hom functor on Alg is just Sweedler's "measuring" functor.

If A is an algebra, Alg/A is also enriched over Coalg. The category Ab(Alg/A) of abelian group objects in Alg/A is equivalent to the category of A-modules, and is strongly tripleable over Alg/A. Hence, it too is enriched over Colag. Unfortunately, the enrichment is trival. If M, N are A-moduls, the coalgebra hom from M to N is just the coproduct of the terminal coalgebra k over the set $\mathrm{Hom}_A(M,N)$.

Let HoCoalg, HoAlg denote the categories of fractions obtained by inverting those morphisms which are chain homotopy equivalences in A. So long as we consider positive (cochain type) coalgebras and negative (chain type) algebras we may construct derived functors $\pitchfork^R$, $\overset{L}{\otimes}$ of the structural functors $\pitchfork$, $\otimes$, by usual method of taking free resolutions of algebras and cofree coresolutions of coalgebras. Thus HoCoalg is almost Cartesian closed and HoAlg is almost enriched over it. The interpretation of $H^n(X \pitchfork^R Y)$ and $H^n(X \otimes^L Y)$ remains an open problem.

In a Cartesian closed category (this condition is stronger then necessary) call an object D a tangent object if it has a map $1 \to D$ which is a vector space object in the dual category over 1, and if there is a commutative associative map $D \times D \to D$ making the diagram

$$\begin{tikzcd} D \vee_1 D \arrow[r] \arrow[d] & D \times D \arrow[d] \\ 1 \arrow[r] & D \end{tikzcd}$$

a pushout.

A category having such an object admits a formal interpretation of many notions of differntial geometry (see Lawvere "categorical dynamics"). For an object M, $D \pitchfork M \to M$ is the tangent bundle of M.

For a point $1 \xrightarrow{e} M$, the pullback T_e

$$\begin{tikzcd} T_e \arrow[r] \arrow[d] & 1 \arrow[d, "e"] \\ D \pitchfork M \arrow[r] & M \end{tikzcd}$$

is the tangent space at e. If M is a monoid with unit e, T_e has a natural Lie algebra structure. For any object A, we get a Lie algebra object $T_{\ulcorner 1_A \urcorner}$ of vector fields on A, by considering the monoid $A \pitchfork A$. The object A is Euclidean if $D \pitchfork A \simeq A \times A$, and so on. Coalg is a category with a tangent object. Cofree coalgebras are Euclidean. H is an open question whether the converse is true. It seems likely that the derived functors mentioned above will have a geometrical interpretation in the light of the above ideas of "formal differential geometry".

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