MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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UNIVERSAL ALGEBRA

1.7. to 6.7.1973

The chairmen of this meeting were: G. Grätzer (Winnipeg), W. Felscher (Tübingen), and R. Wille (Darmstadt).

Participants:

P.D. Bacsich (Blechtley)
R. Baer (Zürich)
K.A. Baker (Los Angeles)
B. Banaschewski (Hamilton)
G. Bruns (Tübingen-Hamilton)
P. Burmeister (Darmstadt)
S. Burris (Waterloo)
H. Christiansen (Bremen)
S.D. Comer (Nashville)
B.A. Davey (Winnipeg)
A. Day (Thunder Bay)
T. Evans (Atlanta)
M. De Finis (Rom)
R. Franci (Siena)
E. Fried (Budapest)
B. Ganter (Darmstadt)
S.R. Givant (Berkeley)
M.I. Gould (Nashville-Winnipeg)
J. Hagemann (Darmstadt)
C. Herrmann (Darmstadt)
D. Kelly (Winnipeg)
M. Kolibiar (Bratislava)
R. Kühl (Hamburg)
W.A. Lampe (Honolulu)
W. Luxemburg (Pasadena)
A. Mitschke (Darmstadt)
G. Mitschke (Darmstadt)
E. Nelson (Hamilton)
B.H. Neumann (Canberra)
W. Neumann (Berkeley-Bonn)
A.F. Pixley (Claremont)
R.W. Quackenbush (Winnipeg)
M. Resmini (Rom)
I. Rival (Darmstadt-Winnipeg)
E.T. Schmidt (Budapest)
H. Schubert (Düsseldorf)
J. Schulte-Mönting (Tübingen)
J. Sichler (Winnipeg)
M.G. Stone (Calgary)
W.F. Taylor (Boulder)
J. Timm (Bremen)
H. Werner (Winnipeg-Darmstadt)

Abstracts

BACSICH, P.D.: Amalgamation Properties and Interpolation Theorems

Let T be an equational theory of algebras of type τ, $\Gamma_1,\Gamma_2,\Gamma_3$ be sets of formulas in the first order language for τ. T is said to have the $(\Gamma_1,\Gamma_2,\Gamma_3)$-separation property if whenever $\phi\varepsilon\Gamma_1$ and $\psi\varepsilon\Gamma_2$ are formulas such that $[\phi]\wedge[\psi]=0$, there is $\theta\varepsilon\Gamma_3$ with $[\phi]\leq[\theta]$ and $[\psi]\wedge[\theta]=0$ (where $[\alpha]$ is the image of α in the boolean algebra of formulas module T-equivalence).

Let O be the class of quantifier-free formulas and E [respectively U] be the class of existential [universal] quantifications of formulas in O . Every theory T has (U,U,O)-separation.

Theorem T has the Amalgamation Property iff T has (E,E,O)-separation. Similar characterisations can be given for related properties such as Congruence Extension.

BAER, R.: Vielfachenketten in verallgemeinerten Ringen mit assoziativen Ungleichungen

Ein [verallgemeinerter] Ring R ist eine abelsche Gruppe bezgl. der Addition, genügt den beiden Distributivgesetzen und den Ungleichungen

$$x(yR) + x(Ry) \subseteq (xy)R + (xR)y$$

$$(xR)y + (Rx)y \subseteq x(Ry) + R(xy)$$

$$(xy)R + (yx)R \subseteq x(yR) + y(xR)$$

Sowohl assoziative wie auch Liesche Ringe fallen unter diesen Begriff. Wir befassen uns mit Fragen, die die Endlichwertigkeit von Vielfachenketten

$$v,\ vx_1,\ (vx_1)x_2,\ [(vx_1)x_2]x_3,\ldots$$

betreffen.

BAKER, K.A.: Lattice properties preserved under formation of ideals

The following two theorems provide partial answers to questions of G. Grätzer: Theorem A: Any positive universal sentence true in a lattice L is also true in its lattice of ideals I(L). Theorem B: If a finite projective lattice P is embeddable in I(L), then P must already be embeddable in L. These theorems explain various known examples. They are both corollaries of Theorem C: I(L) is a homomorphic image of a sublattice of an ultrapower of L.

BANASCHEWSKI, B.: A categorical view of equational compactness

An extension E of an algebra A is shown to be pure iff E is an updirected colimit over A of extensions E_α of A each of which has a retraction $E_\alpha \to A$. This provides a characterization of pure extensions, and consequently of equational compactness, in entirely categorical terms and permits a discussion of some of the basic facts concerning equational compactness of algebras in a general categorical setting.

BURRIS, St.: Congruence lattices of subdirect products

Our basic result says that the global property of 'the congruences of a subdirect product of algebras being determined by the congruences on the individual factors' reduces to the local problem of looking at two factors at a time (in several interesting cases). As a special case of these results we generalize a theorem of T. Tanaka to characterize those algebras (in a variety with modular congruence lattices) which have Boolean congruence lattices. (If the variety has permutable congruences we obtain a sharper formulation.)

COMER, S.D.: CANCELLATION LAWS FOR FREE PRODUCTS

This talk will survey some results dealing with two cancellation problems for free products of BA's and lattices. The first problem has a general form. Let V denote a variety in which every two members have a V-free product. For a subclass K of V, a member A of V cancels for K if for all B,C in K, $A*B \cong A*C$ implies $B \cong C$. The first problem is to give conditions for the cancellation of small algebras. Let V_{fin} denote the class of all finite members of V. Here is one such result. If A is finitely generated and for every B in V_{fin}, $Hom(A,B) \neq 0$, then A cancels for V_{fin}. Examples can be given for BA's to show that finite algebras do not have to cancel for the class of all denumerable BA's. The second problem concerns the concrete form of the cancellation law for varieties where it makes sense. For example, if V denotes the variety of BA's or some variety of lattices, then find conditions for $A*B = A*C$ to imply $B = C$. This concrete cancellation always holds for distributive lattices, necessary and sufficent conditions can be given for BA's, and partial answers for bounded distributive lattices.

DAY, A.: Filter Monads, Continuous Lattices, and Closure Systems

We examine two monads(=triples) determined by filters. Firstly, the filter set monad $\underset{\sim}{F}=(F,\eta,\mu)$ over Sets where Fx is the set of all (notnecessarily proper) filters on x, and secondly $\underset{\sim}{\Phi}=(\Phi,\eta,\mu)$ the filter space monad over Top_0, the category of all T_0 spaces and continuous maps where ΦX is the filter space of the lattice of open sets of X. We determine the algebras of both these monads and show that they are both naturally equivalent to the category of

Scott's continuous lattices with arbitrary meet and (non-empty) directed join preserving functions as maps. Finally, using Barr's notion of a relational algebra for a monads over Sets, we extend his result by showing that Clos, the category of all closure systems and continuous maps is naturally equivalent to a suitable reflective subcategory of all relational algebras over $\underset{\sim}{F}$.

EVANS, T.: APPROXIMATING ALGEBRAS BY FINITE ALGEBRAS

Let V be a variety defined by a finite number of finitary operations. Some examples of the following problem are considered. If A is a V-algebra and P a property of elements, subalgebras, etc. of A, if P holds in all finite homomorphic images of A, does P hold in A? It is known that residual finiteness is preserved under free products for groups, semigroups and lattices. It is shown that it is preserved also for free products of loops, quasigroups, Steiner quasigroups and other varieties of non-associative systems for which we have a finite embedding property. Finite separability with respect to finitely generated subalgebras is also shown to hold for these varieties. If two f.g. subalgebras of a free semigroup have empty intersection, then there is a finite homomorphic image of the free semigroup in which this property is preserved. The problem is raised of whether, if an element w of a free V-algebra on n generators maps onto a primitive element in every finite relatively free homomorphic image of $F_n(V)$, then w is primitive in $F_n(V)$. The answer is affirmative for the corresponding problems for sets if n elements in $F_n(V)$ which map onto free generating sets in finite relatively free homomorphic images of $F_n(V)$.

FRANCI, R.: A NOTE ON IDEAL CLASSES

Let $(A_i)_{i\varepsilon I}$ be a family of algebras. A be a subalgebra of $\prod_{i\varepsilon I} A_i$. A has idealized congruences if each $R\varepsilon C(A)$ is linked to an ideal of $\prod_{i\varepsilon I} C(A_i)$. A class X of algebras is ideal iff each $B\varepsilon P_S X$ has idealized congruences. An algebra A is k-principal iff for each compact congruence the least cardinal of its sets of generators is less or equal to k. A is super-k-principal iff for each compact congruence R on A there exists at most k elements of A: $y^0, y^1, \ldots, y^{k-1}$ suh that $R = \overline{\{(x,y^r):r\varepsilon k\}}$ (the bar means the congruence generated).

We have the following theorems: Theor. If X is an ideal and super-k-principal class then each algebra in V X is regular (an algebra is regular if each congruence of each subalgebra is extendible to a congruence of the algebra). Theor. If X is semi-ideal and k-principal then HP X is semi-ideal and k-principal. Theor. Let X be an ideal and k-principal class such that (i) each algebra in H X is semisimple (ii) C(A) is finitely equilimited for each $A\varepsilon X$. Then $V X = SP(\Sigma \cap V X)$ THEOR. If X is a finite class of finite ideal algebras such that H X is semisimple then $V X = SPH X$.

FRIED, A.: SOME EQUATIONAL CLASS OF WEAKLY ASSOCIATIVE LATTICES

A weakly associative lattice is an algebra with two idempotent and commutative binary operations for which the absorption lows hold and we have a weakened form of the associativities. We have an alternative definition by join and meat similarly to the case of lattices but the "partial Order" need not be transitive. Some equational classes of weakly associative lattices are investigated where the congruence extension property hold. This is a generalisation of a result of Grätzer and myself. One can prove, also, that these classes are finitely based.

GANTER, B.: VARIETIES OF STEINER SYSTEMS

A variety $\mathfrak{A}$ of algebras is said to have property (k,m), iff every k-generated algebra in $\mathfrak{A}$ has exactly m elements.
Theorem: A variety with property (k,m) exists for

(i) $k = m$;

(ii) $k = 0,\ m = 1$;

(iii) $k = 2$, m a primepower;

(iv) $k = 3,\ m = 4$;

and for no other pair (k,m) of natural numbers.
The subalgebras of an algebra in a class with property (k,m) form a Steiner System of type (k,m). For the values of (k,m) which are allowed by the theorem, we can give defining equations of varieties such that each Steiner System of the given type is "coordinatized", that the word problem is solvable and that the variety has the strong amalgamation property.

GIVANT, S.R.: VARIETIES AND QUASI-VARIETIES CATEGORICAL OR FREE IN POWER

A quasi-variety K is free in power κ if it has a member of power κ and every member of power κ is K-free. This notion (due to Tarski) generalizes the notion of categoricity in power κ.

Tarski asked: if a quasi-variety K is free in some infinite power is every non-trivial $\mathfrak{A}\in K$ K-free (i.e. is K a free class)?

Theorem 1: Let K be a quasi-variety free in power κ ($\geq\omega$), and suppose λ is the cardinality of the K-free algebra on ω generators.

(i) If K is not κ-categorical then $\kappa=\lambda$ and K is a free class.

(ii) If K is κ-categorical and $\lambda>\omega$ then $\kappa>\lambda$ and every non finitely generated $\mathfrak{A}\in K$ is K-free; K has a prime member and every $\mathfrak{A}\in K$ has a minimal prime extension (for $\lambda=\omega$ analogous results were established by Morley, and Baldwin-Lachlan). The methods of theorem 1 and suitable extensions of results of Urbanik (Fund.Math. 1963) lead to a complete description of quasi-varieties free (or categorical) in power. We give it here for free varieties.

Theorem 2: If K is a free variety then K is polynomially equivalent to one of the following varieties: (i) the vector spaces over a skew field $\mathfrak{F}$ (ii) the affine algebras over a skew field $\mathfrak{F}$ (iii) $\{\langle A,f\rangle$: f is a constant unary function on A$\}$ (iv) $\{\langle A,f\rangle$: f is the identity function on A$\}$.

GOULD, M.I.: REPRESENTABLE CONGRUENCES ON MONOIDS

Given a universal algebra $\underset{\sim}{A}$, define a congruence $\Theta(\underset{\sim}{A})$ on its endomorphism monoid by identifying two endomorphisms α and β iff $X\alpha = X\beta$ for all subuniverses X of $\underset{\sim}{A}$. Given a congruence Θ on a monoid M, call Θ representable if there is a universal algebra $\underset{\sim}{A}$ whose endomorphism monoid is isomorphic to M in such a way that Θ corresponds to $\Theta(\underset{\sim}{A})$. Theorem. The representable congruences on a monoid M constitute a principal ideal in the congruence lattice of M. The generator of this ideal is the congruence ρ defined by:

$a \rho b$ iff for all $t \in M$, $\{\langle u,v\rangle \in M^2 : tau = tav\} = \{\langle u,v\rangle \in M^2 : tbu = tbv\}$.

The proof that ρ is representable is achieved by modifying the algebra freely generated by a partial algebra that almost has the required properties. As a corollary of the proof every representable congruence can be represented by a finitely generated algebra. Further corollaries are obtained by examining the relation ρ in specific classes of monoids.

GRÄTZER, G.: REFINEMENTS OF FREE PRODUCTS OF LATTICES

For any nontrivial equational class $\underset{\sim}{K}$ of lattices it is proved that

1) any two free $\underset{\sim}{K}$-products have a common refinement ;
2) there are 2^m lattices (in $\underset{\sim}{K}$) of cardinality m that cannot be decomposed into a free $\underset{\sim}{K}$-product of indecomposable lattices.

HAGEMANN, J.: FOUNDATIONS OF UNIVERSAL TOPOLOGICAL ALGEBRA

A theory of preuniform and pretopological universal algebras is presented (preuniform structures and pretopologies are considerably more general than uniform structures and topologies, respectively, and consequently behave well, if combined with universal algebra). Working in this general framework it is possible to characterize equational classes κ by Mal'cev conditions which have special properties, for example:

(a) Every admissible topology on an $\underline{A} \in \kappa$ is uniformizable.
(b) Every admissible topology on an $\underline{A} \in \kappa$ is determined by the neighbourhood system of every $a \in \kappa$.

If κ satisfies (b) then it satisfies also (a).

If κ satisfies (a), then, if $\underline{A}$ is a topological algebra and θ a congruence on $\underline{A}$, then the quotient topology is again compatible with the algebraic structure.

HERRMANN, Chr.: EQUATIONAL THEORY OF SUBMODULE LATTICES

The lattice varieties $\mathcal{N}, \mathcal{A}, \mathcal{C}$ generated by all normal subgroup lattices of groups, all subgroup lattices of abelian groups, all complemented modular lattices resp. are studied. The basic result is that $\mathcal{A}$ is generated by the subgroup lattices $L(\mathbb{Z}_p^n k)$ - p prime, $n,k<\infty$. As a consequence, the word problems for free lattices in $\mathcal{A}$ resp. $\mathcal{C}$ are solved.

For $n\geq 3$ we determine all subdirectly irreducible lattices in $\mathcal{N}\vee\mathcal{C}$ which are generated by an n-frame: $L(\mathbb{Q}^n_{\mathbb{Q}})$, $L(\mathbb{Z}_p^n k)$, $L(\mathbb{Z}_p^n \infty)$ and its dual, and, for n=3, nondesarguesian planes with 4 generators. Applications: For $n\geq 3$, $k<\infty$ $L(\mathbb{Z}_p^n k)$ is splitting in $\mathcal{N}\vee\mathcal{C}$ (but there is an unbounded epimorphism from $L(\mathbb{Z}_p^n \infty)$ on $L(\mathbb{Z}_p^n)$). There is an identity valid in $\mathcal{N}\vee\mathcal{C}$ but not in all Arguesian lattices. No class $\mathcal{L}$ with $\mathcal{C}\subseteq\mathcal{L}\subseteq\mathcal{N}\vee\mathcal{C}$ can be defined by a finite set of axioms. For any n there is an identity χ_n such that χ_n is valid in the submodule lattice $L(M_R)$ iff the g.c.d. of the additive orders of any 3 weakly independent elements divides n.

KELLY, D.: NON-IMPLICATIONS AMONG CERTAIN MAL'CEV CONDITIONS

S and T always denote sets of equations. The "implication" $S \longrightarrow T$ means that every equational class that admits S also admits T. Let n and k be integers with $n\geq 2$ and $k\geq 1$. Lower (upper) case Greek letters denote (sets of) equivalence relations on $\{1,2,\ldots,k\}$. $B_n^k(\alpha_1,\ldots,\alpha_n;\Gamma_1,\ldots,\Gamma_{n-1})$ denotes the following set of equations: (writing (x) for $(x_1,\ldots,x_k)$)

$\{f_1(x)\alpha_1=x_1\}\cup\{f_{i-1}(x)\alpha_i=f_i(x)\alpha_i \mid 2\leq i\leq n-1\}\cup\{f_{n-1}(x)\alpha_n=x_k\alpha_n\}\cup$
$\cup\{f_i(x)\gamma_i=x_1 \mid 1\leq i\leq n-1, \gamma_i\varepsilon\Gamma_i\}$.

This set is of <u>length</u> n . Certain of these sets of equations, called J-sets, are described. A K-set is a union of J-sets without common operational symbols; its length is that of the shortest J-component. <u>Theorem</u>. If S and T are K-sets with S of greater length than T , then $S \not\rightarrow T$. Since the sets P_n, D_n and M_n appearing in the familiar Mal'cev conditions are J-sets of length n, the theorem provides many interesting non-implications. By a different method, we show that $P_4 \not\rightarrow M$ (modularity).

LAMPE, W.A.: <u>THE LATTICE OF CONGRUENCE RELATIONS OF A (FINITARY OR INFINITARY) ALGEBRA</u>

The following theorem extends and generalizes almost every known result on the lattice of congruence relations of a finitary algebra to infinitary algebras. It also gives some new results for finitary algebras. This theorem is the culmination of some joint work done with George Grätzer. We shall say that the join in the congruence lattice of an algebra is of type-2 if for any congruence relations Θ and Φ we have that $\Theta \vee \Phi = \Theta \cdot \Phi \cdot \Theta$, is of type-3 if $\Theta \vee \Phi = \Theta \cdot \Phi \cdot \Theta \cdot \Phi$, etc. The $\cdot$ is relation product.
<u>Theorem</u>: Suppose $\mathfrak{G} = \langle G;\cdot\rangle$ is any group, and m is a regular cardinal and, $\mathcal{L}_0$ and $\mathcal{L}_1$ are any two m-algebraic lattices

having two or more elements, and $3 \leq n \leq \omega$. Then there is an algebra $\mathfrak{A}$ such that: (i) $\mathrm{Aut}(\mathfrak{A})$ is isomorphic to $\mathfrak{G}$; (ii) $\mathrm{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}_1$; (iii) $\mathrm{Sub}(\mathfrak{A})$ is isomorphic to $\mathcal{L}_0$; (iv) $\mathfrak{A}$ is of characteristic m; (v) the join in $\mathrm{Con}(\mathfrak{A})$ is of type n and not type n-1. Moreover, if $|G| = 1$ and $\mathcal{L}_1$ is modular and $n = 2$, then an $\mathfrak{A}$ satisfying (i)-(v) exists.

MITSCHKE, A.: FREE MODULAR LATTICES $FM({}_DM_3)$

We consider modular lattices freely generated by the partial lattice ${}_DM_3$ which is $D \cup M_3$ where M_3 is the five element modular nondistributive lattice $\{0,a_2,a_3,a_5,1\}$ and D is a bounded distributive lattice with least element 0 and greatest element a_2, $D \cap M_3 = \{0,a_2\}$ and the joins (respectively meets) $d \vee a_3$, $d \vee a_5$ ($d \wedge a_3$, $d \wedge a_5$) are defined only for elements $d \in D \cap M_3$.

Theorem. Let M be a modular lattice and ${}_DM_3$ a relative sublattice of M . Then the following conditions are equivalent:

(1) M is generated by ${}_DM_3$.

(2) M is isomorphic to the free modular lattice generated by ${}_DM_3$.

(3) M is isomorphic to the subdirect power of M_3 consisting of all quasi-proper, continuous functions from the Stone space over D to the T_0-space M_3 with $\{[a) \mid a \in M_3\}$ as subbasis.

MITSCHKE, G.: COMBINATORIALLY COMPLETE ALGEBRAS

The notion of combinatorial completeness plays a fundamental role in Combinatory Logic, λ-calculus and in the theory of partial recursive functions and it's generalizations and various axiomatic treatments.

Definition: Let $(A,F)=\underline{A}$ be a partial algebra (F a family of finitary partial operations) and for each natural number n be Φ_n a $(n+1)$-ary algebraic operation of $\underline{A}$. $\underline{A}$ is called combinatorially complete with respect to the family $(\Phi_n)_{n\varepsilon\omega}$ iff the following conditions are satisfied.

(i) (Enumeration)

For every $n\varepsilon\omega$ and every n-ary algebraic function of $\underline{A}$ there is an $a\varepsilon A$ such that for all $x_1,..,x_n \varepsilon A$ $f(x_1,..,x_n) \doteq \Phi_n(a,x_1,...,x_n)$

(ii) (s-m-n-property)

For all $n,m\varepsilon\omega$ there is a total $(m+1)$-ary algebraic operation s_n^m such that for all $y,y_1,...,y_m,x_1,...,x_n \varepsilon A$ $\Phi_{n+m}(y,y_1,..,y_m,x_1,.x_n) =$
$= \Phi_n(s_n^m\ (y,y_1,..,y_m)x_1,...,x_n)$

Here and in the following an equation means that whenever one side is defined then is the other and both are equal.

Theorem: Every partial combinatorially complete Algebra $\underline{A}$ (with respect to $(\Phi_n)_{n\varepsilon\omega}$) can be embedded as a relative algebra into a complete algebra $\underline{\hat{A}}$ with the properties

(i) In $\underline{\hat{A}}$ exactly the same equations are valid as in $\underline{A}$.

(ii) $\underline{\hat{A}}$ is combinatorially complete with respect to the completions of the Φ_n .

This theorem is a generalization and extension of a theorem of Dana Scott concerning models of combinatory logic.

NELSON, E.: INFINITARY EQUATIONAL COMPACTNESS

The concepts of purity and equational compactness are defined for infinitary algebras in a way that takes into account the infinitariness of the operations. Several of the results on equational compactness for finitary algebras are then obtained, including:

1. An algebra is equationally compact iff it is a pure-absolute retract.
2. If an equational class $\underset{\sim}{A}$ has "enough" equationally compact members, then every algebra in A has only a set of essential extensions, and thence $\underset{\sim}{A}$ has only a set of subdirectly irreducibles.

The converse of 2. no longer holds; the class of $\aleph_0$-Boolean algebras has only one subdirectly irreducible member but not every algebra in the class can be embedded in an equationally compact one.

PIXLEY, A.F.: EQUATIONAL CLASSES GENERATED BY FINITE ALGEBRAS

For a non-trivial variety V conditions A) and B) below are equivalent:

A) $V = IP_S HS(A)$

B) a) V is equationally semi-complete

b) V is locally finite

c) For each finite subdirectly irreducible algebras B in V there are at most finitely many subdirectly irreducible algebras C in V which are not isomorphic with B and for which Id C = Id B.

For congruence distributive varieties, condition c) of B) may be omitted. Conditions a), b) and c) are shown to be independent, Hence an equationally semi-complete congruence distributive variety need not be generated by a finite algebra unless the variety is locally finite. The special case of a semi-complete variety of lattices (originally posed by R.McKenzie) is still open, though this result sharpens the question.

QUACKENBUSH, R.W.: NEAR VECTOR SPACES OVER GF(q)

Def.1: Let V_q be the variety of all vector spaces over GF(q); then K_q, the variety of all near vector spaces over GF(q), is the variety of all algebras satisfying all two variable laws holding in V_q.

Def.2: Let S be a set and B a set of n element subsets of S; then <S,B> is a Steiner (2,n) system if for every $a_1, a_2 \in S (a_1 \neq a_2)$ there is a unique $b \in B$ with $\{a_1, a_2\} \in b$.

Theorem 1: Let $\mathfrak{A} \in K_q$; then the 1-generated (but not zero-generated) subalgebras of $\mathfrak{A}$ form a Steiner (2,q+1) system. Let <S,B> be a Steiner (2,q+1) system; then there is an $\mathfrak{A} \in K_q$ whose 1-generated subalgebras form a Steiner (2,q+1) system isomorphic to <S,B>.

Corollary: If there are Steiner (2,q+1) systems on v, and v_2 elements then there is a Steiner (2,q+1) system on $(q-1)v_1v_2+v_1+v_2$ elements.

Def.3: A variety V is binary if V has a binary polynomial which depends on both variables.

Def.4: A variety V has property (1,m,n) if every 0-generated algebra in V has exactly 1 element, if every 1-generated (but not 0-generated) algebra in V has exactly m elements and if every 2-generated (but not 1-generated) algebra in V has exactly n elements.

Theorem 2: Let V be a binary variety with property (1,m,n). Then $n=m^2$ and m is a prime power.

RIVAL, I.: DISMANTLABLE LATTICES

An element x of a lattice L is doubly irreducible in L if $x \neq y \vee z$ and $x \neq y \wedge z$ whenever y and z are elements of L distinct from x . A lattice L is dismantlable if every sublattice S of L contains an element that is doubly irreducible in S . Sublattices and homomorphic images of dismantlable lattices are dismantlable. A (lower) <u>fence</u> is a partially ordered set $\{x_i | 0 \le i < n\}$, $(n \le \omega)$, for which the comparabilities that hold are precisely: (*) $x_i < x_{i+1}$ (i even), $x_i > x_{i+1}$ (i odd). For finite even $n \ge 6$, a <u>crown</u> is a partially ordered set $\{x_i | 0 \le i < n\}$ for which $x_0 < x_{n-1}$ and (*) are precisely the comparabilities that hold.

<u>Theorem</u>. Let L be a lattice which contains no infinite chains and no infinite fences. Then L is dismantlable if and only if L contains no crowns.

<u>Theorem</u>. Let L be a modular lattice of finite length. L is dismantlable if and only if L contains no crown of order 6.

SCHMIDT, E.T.: <u>CONGRUENCE RELATIONS OF MODULAR LATTICES</u>

Jeder endliche distributive Verband ist dem Kongruenzverband eines modularen Verbandes isomorph. Eine Verallgemeinerung dieses Satzes wird noch vorgeführt.

SCHULTE MÖNTING, J.: <u>ON THE ALGEBRAIC MEANING OF CUT ELIMINATION</u>

The property of a logical calculus to admit cut elimination cannot be characterized by one universal algebraic property of its matrices There are different possibilities to formulate a cut rule for one class of algebras. It turns out that for each of some interesting classes of lattices there is one adequate form of a cut rule, which implies most of the inconstructive algebraic properties of that class. Using this adequate rule a cut elimination algorithm can be given. Extending the proof of cut elimination to a calculus for derivable rules instead of provable formulas one can solve the general word problem for the classes mentioned above.

SICHLER, J.: <u>GROUP-UNIVERSAL EQUATIONAL CLASSES</u>

Call an equational class $\mathcal{E}$ of algebras group-universal if every group G is isomorphic to the full automorphism group of an algebra $E \varepsilon \mathcal{E}$.

<u>Theorem 1</u>: Every regular equational class R with at least one at least binary essential operation is group-universal.

<u>Theorem 2.</u> An equational class of unary algebras is group-universal if and only if there are two nonconstant polynomials m and n such that (m=pn or n=pm) for no polynomial p.

TAYLOR, W.: <u>THE FINE SPECTRUM OF A VARIETY</u>

The <u>fine spectrum</u> of a variety V is the function f_V where $f_V(n)$ = the number of nonisomorphic models of V of power n .

Various scattered results were mentioned.

<u>Problem</u>. Is $\{f_V\}$ a closed subset of ω^ω ?

<u>Theorem</u>. If $f_V(2^k) = 1$ and $f_V(n) = 0$ for n not a power of 2, then V is one of exactly seven varieties.

TIMM, J.: <u>ORDERED UNIVERSAL ALGEBRAS</u>

The concept of (quasi)ordered universal algebras, the generalized smothness-problem of FUCHS (Coll.Math. XIV, 1966) and some results of CHRISTIANSEN and TIMM giving specialized sulutions have been discussed. The main results concern with the class of bidual quasi-ordered algebras, where polynomials p,t exist such that $p(x,x)$ is constant and $t(p(x,y),y)=x$, t being isoton in both, p in the first and antiton in the second component. Such algebras have the following properties:

(1) $x\theta y \quad p(x,y)\varepsilon N_\theta=[p(x,x)]_\theta$ (θ congruence)

(2) $x\leq y \quad p(x,y)\varepsilon P=\{x\varepsilon A|x\geq p(x,x)\}$

(3) congruences permute and preserve the order in both directions in a specified way

(4) The theorem of Lagrange holds for finite A (GEWERS)

The problem of Fuchs is solved in this class by

Theorem: If the formulas describing the monotony-sets are conjunction of atoms $q_1(x) = q_2(x)$ or $q_1(x) \leq q_2(x)$ with q_2 inducing orderisomorphisms and constant $q_2^{-1}q_1$, then the monotony-sets of o-subalgebras o-direct products and o-homomorphic images are described by the same formulas.

WERNER, H.: <u>WHICH PARTITION LATTICES ARE CONGRUENCE LATTICES?</u>

For every set X we have the complete graph C_X with vertices X and edges $E=\{\{x,y\}|x,y\varepsilon X,x\neq y\}$. This graph defines an $|E|$-ary operation F_X on the partition lattice $\Pi(X)$ on X as follows: distinguish two elements $a,b\varepsilon X$ and for $\psi:E\longrightarrow\Pi(X)$ define

$$R_\psi := \{(x,y)|\exists f:X\longrightarrow X \quad f(a)=x\&f(b)=y\&\forall u,v\varepsilon X\ u\neq v\rightarrow$$
$$\Rightarrow(f(u),f(v))\varepsilon\psi(\{u,v\})\} .$$

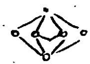
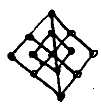
$F_X(\psi)$ is defined as the equivalence relation generated by R_ψ

<u>THEOREM</u>: A complete sublattice $\mathcal{L}$ of $\Pi(X)$ is a congruence lattice of a suitable algebra $A=(X,\mathfrak{F})$ iff $\mathcal{L}$ is closed under F_X.

P R O B L E M S   S E S S I O N

(R.WILLE) Does every finitely generated modular lattice contain a prime quotient?
(If yes, then there is no finitely generated simple modular lattice of infinite length).

(I.RIVAL) L lattice, $J(L)$, $M(L)$ are join, meet irreducibles except 0, 1 . L distr. $\Rightarrow J(L) \cong M(L)$ as poset and there is a <u>maching</u> i.e. $f: J(L) \cup \{0\} \longrightarrow M(L) \cup \{1\}$ bij. s.t. $x \leq f(x)$.
DILWORTH: For modular lattices $|J(L)| = |M(L)|$
<u>Problem</u>: For a modular lattice L of finite length is there always a matching?

(R.WILLE & C.HERRMANN) Is the following list of 4-generated subd. irred. arguesian lattices complete?
•, M_4, $S(n,4)$, FM(), FM(),
$\underset{\sim}{S}(\mathbb{Z}^m_{p^k})$ p prime, $1 \leq k \leq \infty$, $m \geq 3$; $S(1,4)$ =  ,
$S(2,4)$ =  , $S(3,4)$ =  , $S(4,4)$ =  ,...

(G.GRÄTZER) It's little known about finitely presented lattices.
$\mathfrak{E}$ equ. class of lattices, $n \in \mathbb{N}$ s.t. $|F_{\mathfrak{E}}(n)| = \aleph_0$
<u>Conjecture</u>: $F_{\mathfrak{E}}(n)$ is not finitely presented.
$\mathfrak{E}$ modular lattices, n=4 sittled by EVANS & HONG (Alg.Univ. 2,3).

(S.BURRIS) Does Π_∞ belong to the models of the 1^{st}-order theory of $\{\Pi_n | n \varepsilon N\}$?

(K.BAKER) Is the variety of all modular lattices generated by its members of finite length?

Is the variety generated by the projective planes generated by its finite members?

If L is a sublattice of a proj. geometry of dim n, do its homomorphic images have the same property?

(G.GRÄTZER & H.LAKSER) Have an identity which holds for proj. geom. but not for all modular lattices.

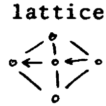
Can one "describe" the equations of the variety generated by all projective geometries.

(G.GRÄTZER) For a class $\mathfrak{K}$ of lattices $A \varepsilon \mathrm{Amal}(\mathfrak{K})$ iff <u>over A</u> we can amalgamate ROBINSON: $\mathrm{Amal}(\mathfrak{K})$ is cofined in $\mathfrak{K}$ (i.e. every $A \varepsilon \mathfrak{K}$ can be embedded in some $\bar{A} \varepsilon \mathrm{Amal}(\mathfrak{K})$ of the "same" cardinality).

Find a modular lattice with more than one elt. in Amal (Mod).

(E.NELSON) If G is a torsion-free group, each of whose proper subgroups is free, is G free?

(true for abelian groups)

(E.FRIED) For weakly associative lattices, the eq. classes gen. by  ,  ,  ,  cover distr. lattices.

Are there any others?

(R.QUACKENBUSH) P.KRAUS: A categorical if it is finite and is the only s.i. in HSP(A).

Problem: If V is equ. complete and has a non-trivial finite algebra, is V generated by a categorical algebra?

If A is categorical, is A injective in HSP(A)? or does HSP(A) have CEP?

(H.WERNER) A group G is funct. complete iff it is finite, simple, non-abelian.

Problem: Prove that every functionally complete group has even order! Give the ternary discriminatur function on it!

(S.COMER) Do there exist two non isomorphic finite subd.irred. rings with 1. Which generate the same variety? (B.H.NEUMANN: Try $\mathbb{Z}_2[D_8]$ and $\mathbb{Z}_2$ [Quat])

(A.DAY) Is CEP for a variety V characterized by some properties on the subd. irr. algs in V ? Conjecture: No.

(R.QUACKENBUSH) Characterize the spectra of varieties of groupoids!

(G.GRÄTZER) If A is an alg. lattice with type 1 ($\vee = \circ$) representation, is it the congruence lattice of an algebra with permutable congruences?

(E.T.SCHMIDT) Every finite dist. lattice is congr. lattice of a lattice. Every finite group is the aut. group of a lattice. What about independence?

(W.LAMPE) Conjective: Any alg. lattice is congr. lattice of a groupoid.

-Suggestion - get rid of H-invertibility in Lampe's Lemma.

(R.WILLE) Can one define by 1^{st} order sentences a class $\mathfrak{K}$ of algebras (of type <2,2>) such that

(1) $\forall n \in N \ \exists! \ A \in \mathfrak{K}$ (up to isom.) $\therefore$ $|A|=n$

(2) $A \in \mathfrak{K}$ and $|A|=p^k \Rightarrow$ A is a field

(3) all finite algebras in $\mathfrak{K}$ are functionally complete.

(A.PIXLEY) THM.: If $\Theta(A)$ is finite then $\Theta(A)$ is arithmetical iff $f: A^3 \longrightarrow A$ which preserves all congr. on A and satisfies $f(x,x,z)=z$, $f(x,y,x)=x = f(x,z,z)$

Questions: 1. What is going on? (i.e. is there a Mal'cev theory for single algebras?)

2. Can we generalize to infinite $\Theta(A)$?

(J.HAGEMANN) Is every alg. in an n-perm. class, when equipped with a 0-dim compact T_2 topology, a profinite top. algebra?

(S.GIVANT) Given a 1^{st} order universal theory which is ω-categorical. Does this imply it is ω_1-categorical (hence categorical in everey infinite power)?

Known for universal Horn theories and for universal classes of unary algebras.

Can we describe all universal theories which are ω-categorical?

An algebra is strictly λ-generated if it is λ-generated but not $<\lambda$-generated.

What are the <u>λ-unique</u> varieties (i.e. $\exists\,!A$ strictly λ-generated up to isom.)?

(A.DAY) (NATION-McKENZIE-CONJECTURE) If a variety has the congr. (p=q)-property is it congr.-modular? (p=q nontrivial lattice equation)

If p=q is a Mal'cev condition $\Rightarrow$ p=q implies modularity for congr. lattices?

(R.QUACKENBUSH) V variety, V^n all algebras satisf. all n-var. identities of V

V_n variety generated by $F_V(n)$

$V_0 \subset V_1 \subseteq \ldots \subseteq V \subseteq \ldots \subseteq V^1 \subseteq V^0$

What can be said about these chains?

For $V_0 \subseteq V_1 \subseteq \ldots V$ and $V \subseteq \ldots \subseteq V^1 \subseteq V^0$ each pattern of equalities is possible, how about the whole chain?

(M.GOULD) For Σ closed set of identities, $T(\Sigma)=\{\underset{\sim}{S}(L) \mid \Sigma \models L\}$ define $\Sigma_1 < \Sigma_2$ iff $T(\Sigma_1) \subseteq T(\Sigma_2)$.

Then $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \Sigma_2 < \Sigma_1$.

Does the converse hold?

Quackenbush sais <u>no</u>, take two independent primal algebras on the same set.

(R.WILLE) Are the finite projective planes over a prime field splitting in the class of all modular lattices? Furthermore, are these bounded homomorphic images of a free lattice?

(G.GRÄTZER) Does there exist a class $\mathfrak{K}$ of modular lattices s.t.

 ε Amal($\mathfrak{K}$)?

(W.FELSCHER) Let A be an algebra A^{A^n} contains F(A,n)(gen. by the projections) and A. What can be said about the subalgebra G(A,n) generated by A and F(A,n)?

1. Is G(A,n) the free extension of A by n in Eq(A) (*) if every first order definable polynomial is equational definable?
2. When does (*) hold non-trivially.
3. Is there a theory of how to solve equations for algebras of a given variety, which comprises a) fields, b) groups (alg.closed!), c) cylindric algebras?

(B.GANTER) A variety $\mathfrak{V}$ has property (k) iff $\mathfrak{V}$ contains a nontrivial finite k-generated algebra and if every finite k-generated (not k-1-generated) is generated by every k of its elts.

<u>Problem</u>: 1) for which $k \in N$ does there exist a variety with property (k)?

2) Which combinatorial structures can be "coordinatized" by algebras in such a variety?

The following two abstracts were sent in but the authors could not attend the meeting.

CSÁKÁNY, B. (Budapest): AFFINE MODULES.

The idempotent reduct of a module is called affine module. Varieties of affine modules over some commutative ring may be characterized by the following conditions: idempotency and permutability of all operations, normality (permutability of congruences) and the Hamiltonian property (all subalgebras are congruence classes). Hence it follows that varieties of idempotent medial quasigroups are equivalent to varieties of affine modules. Systems of axioms are derived for varieties of quasigroups which are equivalent to the varieties of affine modules (=affine vector spaces) over finite fields. Especially, the variety of medial Steiner triple systems is equivalent to the variety of affine vector spaces over the field of order 3 (i.e., to the variety of idempotent reducts of abelian groups in the sense of J. Płonka).

IWANIK, A. (Wrocław): THE NOTION OF LINEAR INDEPENDENCE IN COMMUTATIVE SEMIGROUPS.

The notion of linear independence in commutative semigroups is proposed. Let S be a commutative semigroup with identity 1. Elements $a_1,\dots,a_n \in S$ are linearly independent if for any k_i, $m_i \geq 0$, the equality $a_1^{k_1}\dots a_n^{k_n}=a_1^{m_1}\dots a_n^{m_n}$ implies $a_1^{k_i}=a_i^{m_i}$ for $i=1,\dots,n$.
A linearly independent set of generators is called a basis. A decomposition of a semigroup with a basis into its Archimedean components can be explicitely obtained. A semigroup with a basis has a kernel iff it is periodic and has finitely many idempotents. In semigroups with identity the linear independence is weaker then the algebraic independence and is equivalent to G-independence in the suitable monoid.

H. Werner, Darmstadt