MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Categories and homology theories

Jon M. Beck

This is a report on recent work of G. Segal (Oxford) and D. Anderson (La Jolla) which is an interesting application of the work on coherence of natural isomorphisms done by Mac Lane and his school.

Let C be a topological category, that is, a category object in the category of spaces. Let C_n be the space of sequences of n composible morphisms in C. Face operators $\delta_i : C_n \to C_{n-1}$ are given by

$$\delta_i(f_1,\ldots,f_n) = \begin{cases} (f_2,\ldots,f_n) & i = o \\ (f_1,\ldots,f_{i+1},\ldots,f_n) & i = 1,\ldots,n-1 \\ (f_1,\ldots,f_{n-1}) & i = n \end{cases}$$

and degeneracies by inserting identity maps; in particular C_o is the space of objects of C. The classifying space BC of C is the geometric realization of this simplicial space:

$$BC = \coprod_{n \geq o} \Delta_n \times C_n \Big/ \text{(simplicial identifications)}$$

The functor B : (categories) → (spaces) preserves coproducts, filtered colimits and finite limits. It converts functors into continuous maps and natural transformations into homotopies.

For example, if C is a monoid M, BM is the steenrod-Rothenberg-Milgram-Eilenberg-MacLane classifying space of M. If C is all finite-dimensional inner product spaces over the complex numbers with isometries as morphisms, then C is a union of unitary groups, and $BC = \coprod BU_n$.

C is a <u>multiplicative</u> category if there is a coherently unitary and associative $\oplus : C \times C \to C$. Since B commutes with finite products, $\oplus$ induces an infinitely homotopy unitary and associative H-space structure $\oplus : BC \times BC \to BC$. Indeed let A(n) be the component of the free multiplicative category generated by objects $X_1,\ldots,X_n$ which contains the word $(\ldots(X_1 \oplus X_2) \oplus X_3) \oplus \ldots X_n)$. The A(n) form a categorical operad (without permutations) over which C is an algebra $\oplus : A(n) \times C^n \to C$. By MacLane's work, $A(n) \to 1$ is a categorical equivalence. Thus the BA(n) form a contractible operad in spaces, and the classifying space functor gives the true H-space structure $BA(n) \times BC^n \to BC$.

It is well known that, on the basis of MacLane's work, the multiplicative category C can be replaced by an equivalent C' whose equivalent multiplicative structure is strictly unitary and associative. Thus BC can be replaced by the topological monoid BC', where of course "monoid" means space acted upon by the operad 1. However, commutativity isomorphisms can not be chased from the scene so easily.

As is current in algebraic topology, we call C permutative if C is multiplicative and there is a coherent commutativity isomorphism $c : X \oplus Y \cong Y \oplus X$. This means that there are actions $\oplus : E(n) \times C^n \to C$ of the permutative operad (prop) $\{E(n)\}_{n \geq 0}$, where E(n) is the component of the free permutative category generated by $X_1, \ldots, X_n$ containing $X_1 \oplus \ldots \oplus X_n$. By MacLane, $E(n) \to 1$ is an equivalence. Thus BC is not merely a homotopy commutative H-space. It is in fact operated on by the contractible operad BE(n), and is thus a homotopy everything H-space in the sense of Boardman-Vogt, having infinitely many higher order commutativity homotopies.

Permutative categories give rise to homology theories. Let BC also denote the functor (finite sets with base point) $\to$ (spaces) for which $(BC)(n) = B(C * \ldots * C)$; the coproduct is that of permutative categories, $(BC)(S^o) = BC$, S^o being the o-sphere. Putting arguments = o defines a canonical permutative functor $C * \ldots * C \to C \times \ldots \times C$. By MacLane, this is again an equivalence of categories. Thus BC transforms sums into products up to homotopy. This homology-like behavior is expressed by Anderson in saying that BC is a chain functor (Segal: Γ-space). Let us also write BC : (spaces) $\to$ spaces) for the topological direct limit Kan extension. By continuity there are always maps $A \wedge BC(X) \to BC(A \wedge X)$, hence $BC(X) \to \Omega BC(SX)$ (the suspension of X). To avoid complications of left derived functors we adhere in the sequel to cell complexes and categories with similar good topological properties.

The functor $\pi_* BC(X)$ is a homology theory on the category of connected spaces. Its "group completion" $\pi_* BC(X) \to \pi_* \Omega BC(SX)$ gives a homology theory on all spaces. The sequence $BC(S^n)$ is an Ω-spectrum, that is, the first arrow

$$BC = BC(S^o) \to \Omega BC(S^1) \to \Omega^2 BC(S^2) \to \ \ldots\ldots$$

is group completion and the others are weak homotopy equivalences.

Note that $\pi_i BC(S^n) = o$ for $i < n$, which is useful. This spectrum gives the same homology theory as the Kan extension BC, and of course gives a cohomology theory as well.

As an eample, let C be finite dimensional inner product spaces with direct sum as the permutative structure. Then $BC \to \Omega BC(S^1)$ maps $\coprod BU_n$ into a space of the homotopy type of $Z \times$ stable BU. The latter space is familiar in complex K-theory. The map on π_o is additive completion $N \to Z$. If instead the permutative structure arises from the tensor product, then $\pi_o BC \to \pi_o \Omega BC(S^1)$ is multiplicative completion $N_+^* \to Q_+^*$ of the positive integers. In fact $\Omega BC(S^1)$ is the rational completion of $\coprod BU_n$. Other K-theories can be obtained by using PL or topological maps of euclidean spaces.

The category C of finite sets and isomorphisms with disjoint union as permutative structure yields $BC = \coprod BS_n \to \Omega BC(S^1)$. The latter space has the homotopy type of $QS^o = \lim_{k\to\infty} \Omega^k S^k$ by the theorem of Barratt-Priddy-Quillen. The (co)homology theory determined by finite sets is stable (co)homotopy. Using the cartesian product structure of finite sets J. Tornehave (Aarhus) has obtained categorical models of BSG(localized).

On the other hand, ordinary homology $\Sigma_{p+q=n} H_p(X,\pi_q A)$ comes from an abelian topological group or monoid A used as a category without non-identity morphisms and the addition as permutative structure.

If coherent commutativity isomorphisms could always be made strict (i. e. into identity maps), then the classifying spaces BC of such categories would have the homotopy types of commutative topological monoids. A famous theorem of J. C. Moore's would then imply that they were products of Eilenberg-MacLane spaces, which is known not to be the case for most of the above examples. Thus commutativity is essentially more serious than associativity. It is reflected in the existence of extraordinary homology theories in topology.

The work of Anderson and Segal will soon be published. In the meantime Anderson's announcement may be referred to in the proceedings of Nice, 1970. Related work on algebraic K-theory due to Quillen may be found in a recent volume of the Annals of Mathematics.

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Homotopy categories by J. Beck

Catégories et logiques faibles

J. Benabou

I Langage.

Soit $\mathbb{P}$ la catégorie des ensembles préordonnés. A tout couple $(\underline{C},R)$ ou $\underline{C}$ est une catégorie à produits finis et $R: \underline{C}^{op} \to \mathbb{P}$ est un fonctur on associe un langage:

1) Variables: Pour chaque object X de $\underline{C}$ on se donne un ensemble V_X de "variables de Type X". Si $X \neq Y$, $V_X \cap V_Y = \emptyset$. On note $V = \cup V_X$, et $P_f(V)$ l'ensemble des parties finies de V.

2) Termes: L'ensemble T des termes, muni de deux applications "signature" $\sigma: T \to P_f(V)$, et "type" $\tau: T \to ob\underline{C}$ se définit inductivement par:

(i) Si $x \in V_X$, x est un terme et $\sigma(x) = \{x\}$; $\tau(x) = X$.

(ii) Si t_1 et $t_2 \in T$, alors $(t_1,t_2) \in T$ et $\sigma(t_1,t_2) = \sigma(t_1) \cup \sigma(t_2)$ et $\tau(t_1,t_2) = \tau(t_1) \times \tau(t_2)$ (produit cartesien dans $\underline{C}$)

(iii) Si t est un terme et $f: \tau(t) \to X$ une flêche de $\underline{C}$, alors $f(t)$ est un terme, $\sigma(f(t)) = \sigma(t)$ et $\tau(f(t)) = X$.

3) Formules: Les formules atomiques sontles $r(t)$ ou t est un terme et $r \in R(\tau(t))$. L'ensemble F de formules est construit à partir des formules atomiques de constantes logiques V_X et F_X $(X \in Ob(\underline{C}))$ dites "vrai" et "faux" dans X, et de formules $t_1 \underset{X}{=} t_2$ ou $\tau(t_1) = \tau(t_2) = X$ en utilisant les opérateurs $\wedge, \neg, \rightarrow$, V, $\exists x$, $\forall x$ $(x \in V)$.

4) Interprétation: Un terme t de signature $\sigma(t) = \{x_1,..,x_n\}$ et de type $\tau(t) = X$ est interprêté inductivment par une flêche $|t|: X_1 \times ... \times X_n \to X$.

On dêfinit le sous ensemble I de F des formules interprêtables, et une interprêtation qui a toute formule $A(x_1,..,x_n) \in I$ $(x_i \in V_{X_i})$ associe un élêment $|A(x_1,..,x_n)| \in R\ (X_1 \times ... \times X_n)$.

II Classification

Suivant la nature des symboles logiques interprêtables on a une "classification" des couples $(\underline{C},R)$ dont nous donnons les exemples les plus saillants.

1. Si toutes les formules ou ne figurent que $\wedge$, les V_X et les $\overset{=}{X}$ sont interprêtables, on a une theorie êquationnelle (an sens de Joyal), â laquelle on peut associer de manier universelle une catêgorie â lim finies.

2. Si en outrê les $\exists_X$ sont interprêtables, on a une catêgorie rêguliêre formelle, â laquelle correspond de maniêre universelle une catêgorie rêguliere.

3. Si toutes formules sont interpretables, on a une catêgorie logique formelle.

4. Si pour chaque $X \in$ ob $\underline{C}$ le foncteur $R(X_X -)$ est reprêsentable, et si $(\underline{C},R)$ est êquationelle, on a un "topos formel".

A chacune des situations correspond en outre un systeme dêductif qui perment d' avoir une notion syntaxique de consêquence, au moyen de laquelle on "transporte" dans chacun des cas "une partie" des calculs valables dans le catêgorie des Ensembles.

Algêbres non dêtermistique
par Elisabeth Burroni

Nous gênéralisons la notion de $\mathbb{T}$-algèbre, où $\mathbb{T}$ est un triple, par celle de $\mathbb{D}$-algèbre ou $\mathbb{D}$: $\mathbb{T} \to \mathbb{T}'$ est une "distributive law" de Beck: c'est morphisme $TA \xrightarrow{b} T'A$ tel que les diagrammes suivant commutent

$$\begin{tikzcd} A \arrow[r, "IA"] \arrow[dr, "I'A"'] & TA \arrow[d, "b"] \\ & T'A \end{tikzcd} \qquad \begin{tikzcd} TTA \arrow[r, "KA"] \arrow[d, "T'b \cdot DA \cdot Tb"'] & TA \arrow[d, "b"] \\ T'T'A \arrow[r, "K'A"'] & T'A \end{tikzcd}$$

où $\mathbb{T} = (T,I,K)$, $\mathbb{T}' = (T',I',K')$ et $D: T \circ T' \to T' \circ T$ est la transformation naturelle dêfinissant $\mathbb{D}$.

On obtient des cas particuliers intêressants lorsque:

- $\mathbb{T}'$ est le triple des parties: les <u>algêbres non dêterministiques</u> (les automates non dêterministique est le triple des actions d'un moniide).

- $\mathbb{T}$ et $\mathbb{T}'$ sont tous deux des triples d'actions de monoids: <u>les automates avec "input" et "output"</u>.

Si maintenant, on remplace le morphisme $b: TA \to T'A$ par un span $TA \xleftarrow{a} \pi \xrightarrow{b} T'A$, on peut gêneraliser à la fois les $\mathbb{D}$ - algèbres et les $\mathbb{T}$-catégories d'Albert Burroni (et donc aussi les "relational algebras" de Barr): on obtient la notion de $\mathbb{D}$-catègorie. On peut donner une dêfinition plus agrêable des $\mathbb{D}$-catêgories: les $\mathbb{D}$-spans sont des couples: $TA \xleftarrow{a} \pi \xrightarrow{b} T'B$, ce sont les 1-cells d'une pseudo-catêgorie Span ($\mathbb{D}$). Les monades dans Span ($\mathbb{D}$) ne sont autres que les $\mathbb{D}$-categories. (Une pseudo-catêgorie est une $\mathbb{N}$ -"lax"-algèbre au sens d' Albert Burroni ou $\mathbb{N}$ est la thêorie des monoides dans <u>Cat</u>: "lax"vent dire "a morhisme près, avec cohérence").

Applications of Category Theory to Higher Dimensional Formal Languages

Hartmut Ehrig (co authors M. Pfender, H.J. Schneider)

Formal languages and grammars have been generalized to higher dimensions in order to solve generation and recognition problems of flowcharts, proram-shemes and networks and to formalize translations of programming languages.

We give a categorical construction of graph-grammars using morphisms and pushouts which generalizes several other (non-algebraic) concepts and is in fact more powerful. Many problems arising in that field can be solved using category theory, e.g.

- construction of direct derivations (using coproducts and pushouts)
- recognition (decomposition of pushouts)
- embedding of derivations (composition of pushouts)
- series and parallel composition of derivations (derivation categories being a generalization of free strict monoidal categories which are used in the classical case)
- formal translations (special functors)

In this abstract we can only give the construction of direct derivations together with an example, first for the case of unlabeled graphs:
A production $p = ({}'B \xleftarrow{\;'p\;} K \xrightarrow{\;p'\;} B')$ in our sense is not only a pair of graphs $({}'B, B')$ but also a specification of vertices and edges of $'B$ resp. B' given by an auxiliary graph K and a graph morphism $'p\colon K \to {}'B$ resp. $p'\colon K \to B'$.

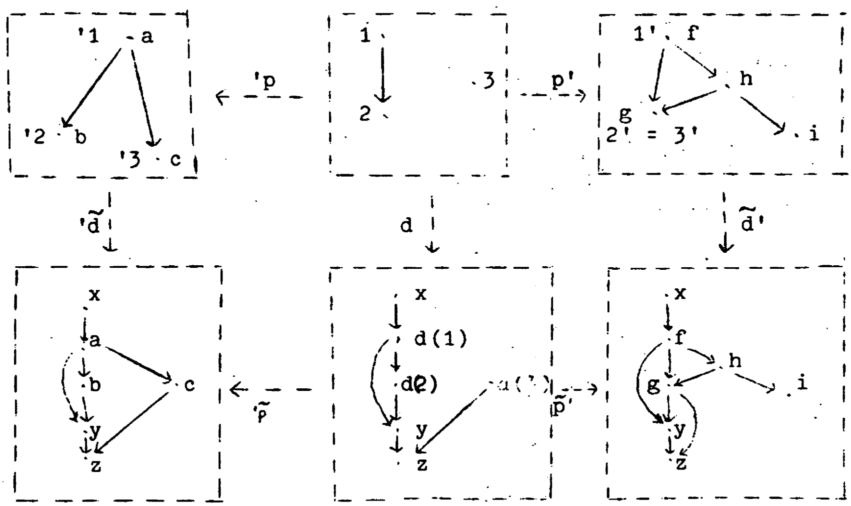
Given a production $p = ({}'B \xleftarrow{\;'p\;} K \xrightarrow{\;p'\;} B')$ and an "enlargement" $e = (K \xrightarrow{\;d\;} D)$, i.e. a graph D with injective graph morphism d, with the same K we get a direct derivation

$$(p,e)\colon D \amalg_{d,'p} {}'B \longrightarrow D \amalg_{d,p'} B'$$

from the pushout $D \amalg_{d,'p} {}'B$ to $D \amalg_{d,p'} B'$. Conversely a graph H can be directly derivated from a graph G if there is a production p and an enlargement e such that G is isomorphic to $D \coprod_{d,'p} {'B}$ and H is isomorphic to $D \coprod_{d,p'} B'$.

We don't want to distinguish isomorphic graphs, because the pushout-construction is only unique up to isomorphism.

Given a production p = $({'B} \xleftarrow{'p} K \xrightarrow{p'} B')$ (first row of the diagram below) and an enlargement e = $(K \xrightarrow{d} D)$ (Middle columm) we have the graphs $G = D \coprod_{d,'p} {'B}$ and $H = D \coprod_{d,p'} B'$ in the second row of the left and right hand side respectively:



In fact we have treated the case of labeled graphs. In this example only vertices are labeled by symbols of the set {a,b,c,f,g,h,i,x,y,z}. We have given labeled graphs ('B,'m) and (B',m') but unlabeled K and only partially labeled $(D,\widehat{m})$, so that we get unique labels on G and H.

The general construction is the following:

Given a pair of alphabets $\Omega = (\Omega_E, \Omega_V)$ a labeled graph (G,m) is a graph $G = (E \rightrightarrows V)$ (E: edges, V: vertices) together with a pair of functions $m = (m_E, m_V)$ $m_E\colon E \to \Omega_E$ $m_V\colon V \to \Omega_V$, i.e. morphism $m\colon UG \to \Omega$ in the category Sets^2 of pairs of sets and mappings, U being the forgetful functor $U\colon$ Graph $\to$ Sets^2.

Given a injective graph morphism $d\colon K \to D$ UD is the coproduct (disjoint union) of UK and $\tilde{D} := UD \setminus U(d(K))$ in the category Sets^2.

Now in the labeled case a production

$$p = (('B,'m) \xleftarrow{\ 'p\ } K \xrightarrow{\ p'\ } (B',m'))$$

has labeled graphs ('B,'m), (B',m') and an enlargement

$$e = (K \xrightarrow{\ d\ } (D,\tilde{m}))$$

has a partial labeled graph $(D,\tilde{m})$, i.e. $\tilde{m}\colon \tilde{D} \to \Omega$ in Sets^2.
We construct the labeled graph (G,m_G) in two steps (similar for (H,m_H)):

1. Pushoutconstruction in the category Graph

$$\begin{tikzcd}
K \arrow[r, "'p"] \arrow[d, "d"'] \arrow[dr, phantom, "p.o."] & 'B \arrow[d, dashed] \\
D \arrow[r, dashed] & G
\end{tikzcd}$$

2. Construction of the labels in the category Sets^2:

$$\begin{tikzcd}
UK \arrow[r, "U'p"] \arrow[d, "Ud"'] \arrow[dr, phantom, "p.o."] & U'B \arrow[d, dashed] \arrow[ddr, "'m"] & \\
UD \arrow[r] \arrow[drr, dashed, "m_D"'] & UG \arrow[dr, dashed, "m_G"] & \\
D \arrow[u, "\tilde{\neq}"] \arrow[rr, "m"'] & & \Omega
\end{tikzcd}$$

First we get a unique m_D by the universal property of the coproduct $UD = \tilde{D} \amalg UK$ induced by m and $'m \cdot U'p$. By the universal property of the pushout we then get a unique m_G induced by m_D and 'm.

Moreover the result can be stated as a pushout in the category Graph$_\Omega$ of labeled graphs. On the other hand this construction gives the labeling in the example above.

The Brauer group of a closed category

J. Fisher - Palmquist

For a finitely bicomplete closed category $\underline{V}$ we define the Brauer group of $\underline{V}$, $B(\underline{V})$. This definition depends heavily on the calculus Morita contexts given by J. and P. Palmquist which can be summarized as follows. A $\underline{V}$-Morita context is a cospan $M = (\underline{C}_1 \xrightarrow{d_1} \underline{C} \xleftarrow{d_o} \underline{C}_o)$ in the category of small $\underline{V}$-categories in which the set of objects of $\underline{C}$ is the disjoint union of the set of objects of $\underline{C}_o$ and $\underline{C}_1$, and a map of Morita contexts is a map of cospans. Morita contexts and their maps form a category denoted by $\underline{M}$ which is equipped with four endofunctors. For $M = (\underline{C}_1 \xrightarrow{d_1} C \xleftarrow{d_o} \underline{C}_o)$ we have the transpose of M, $M^t = (\underline{C}_o \xrightarrow{d_o} \underline{C} \xleftarrow{d_1} C_1)$; the opposite of M, $M^o = (\underline{C}_1^o \longrightarrow \underline{C}^o \longleftarrow \underline{C}_o^o)$; the left identity of M, [illegible] $= \underline{C}_1$ $1(M) = \underline{C}_1 \longrightarrow \underline{C}_1 \otimes \underline{G} \longleftarrow C_1)$ where $\underline{G}$ is a $\underline{V}$-category with two objects o and 1 and $G(i,j) = I$, the unit object of $\underline{V}$, for all i and $j = o, 1$; and the right identity of M, $r(M) = 1(M^t)$.

If M and M' are Morita contexts such that $r(M) = 1(M')$, we define $M * M'$ by first taking the composite of M and M' as cospans $(\underline{C}_1 \longrightarrow \underline{C} \sqcup_{C_o} \underline{C}' \longleftarrow \underline{C}_o')$ and then letting the middle category of $M * M'$ be the full subcategory of $\underline{C} \sqcup_{C_o} \underline{C}'$ with objects the disjoint union of $|\underline{C}_1|$ and $|\underline{C}_o'|$. A Morita context M is defined to be a Morita equivalence if has a right and left $*$-inverse. This inverse will necessarily be M^t i.e., $M * M^t \cong 1(M)$ and $M^t * M = r(M)$.

In $\underline{M}$ there is another composition which we might call vertical composition which is always defined and is associative and commutative up to isomorphism. Namely, if $M = (\underline{C}_1 \to \underline{C} \leftarrow \underline{C}_o)$ and $N = \underline{D}_1 \to \underline{D} \leftarrow \underline{D}_o)$ are Morita contexts then $M \square N = (\underline{C}_1 \otimes \underline{D}_1 \to X \leftarrow \underline{C}_o \otimes \underline{D}_o)$ where $\underline{X}$ is the full subcategory of $\underline{C} \otimes \underline{D}$ with fibers $\underline{C}_1 \otimes \underline{D}_1$ and $\underline{C}_o \otimes \underline{D}_o$.

The two compositions are related by a middle four interchange law. $\underline{M}$ becomes a symmetric monoidal category with $\square$ as the tensor product.

We use the one-to-one correspondence between monoids Λ in $\underline{V}$ and one object $\underline{V}$-categories $[\Lambda]$. The enveloping monoid of Λ, Λ^e, corresponds to the $\underline{V}$-category $[\Lambda] \otimes [\Lambda]^o$. We denote the $\underline{V}$-functor category $\underline{V}^{[\Lambda]}$ by $\underline{V}^{\Lambda}$. The <u>center</u> of Λ is then the center of the $\underline{V}$-category where the center of a $\underline{V}$-category $\underline{A}$ is defined by $Z(\underline{A}) = \int_A \underline{A}(A,A)$. We have $Z(\Lambda) = \underline{V}^{\Lambda e}(\Lambda,\Lambda)$. If Mon ($\underline{V}$) is the monoidal category with objects the monoids of $\underline{V}$ and maps the monoid isomorphisms, there is a map of symmetric monoidal categories ψ: Mon $(\underline{V}) \to \underline{M}$ given by $\psi(\Lambda) = ([\Lambda^e]^o \to \underline{C}_\Lambda \leftarrow [Z(\Lambda)])$ where $\underline{C}_\Lambda$ is the full subcategory of $\underline{V}^{\Lambda e}$ with objects Λ and Λ^e. The monoid Λ is said to be <u>Azumaya</u> if $\psi(\Lambda)$ is a Morita equivalence and Λ is central. This is equivalent to the functor $\Lambda\otimes : \underline{V} \to \underline{V}^{\Lambda e}$ being an equivalence of categories. Another equivalent condition is that Λ is a $\underline{V}$-atom, the functor $[\Lambda^e] \to \underline{V}$ induced by Λ is dense and the canonical map $\Lambda^e \to \underline{V}(\Lambda,\Lambda)$ is an isomorphism. We consider objects B in $\underline{V}$ such that the functor $[\underline{V}(B,B)] \to \underline{V}$ induced by B is $\underline{V}$-dense and B is a $\underline{V}$-atom and call these objects $\underline{V}(B,B)$ - I-prodense. We let $A_o(\underline{V})$ be the class of objects $\underline{V}(B,B)$ where B is $\underline{V}(B,B)$ - I - prodense, while $A(\underline{V})$ is the class of Azumaya monoids. Both $A(\underline{V})$ and $A_o(\underline{V})$ are closed under $\otimes$. Define an equivalence relation $\sim$ on $A(\underline{V})$ by saying $\Lambda_1 \sim \Lambda_2$ if and only if $\underline{V}^{\Lambda 1}$ is equivalent to $\underline{V}^{\Lambda 2}$.

This is equivalent to the existence of Λ_1 and Λ_2 in $A_o(\underline{V})$ such that $\Lambda_1 \otimes \Omega_1 \cong \Lambda_2 \otimes \Omega_2$. Let $B(\underline{V})$ the Brauer group of $\underline{V}$ be the class of equivalence classes with group operation $\otimes$. The inverse of cl(Λ) is cl(Λ^o) and the identity is cl(I).

Let $\underline{V}'$ be another bicomplete closed category and T: $\underline{V} \to \underline{V}'$ a strict map of monoidal categories. If T preserves atoms and coequalizers and there exists an isomorphism $K_B : T(\underline{V}(B,I)) \to V'(TB, TI)$ for each atom B in $\underline{V}$ which is natural in B, then T induces a group homomorphism $B(\underline{V}) \to B(\underline{V}')$. As a consequence we see that B induces a functor from the category of commutative monoids in $\underline{V}$ to the category of abelian groups where a map of commutative monoids $t: R \to S$ in $\underline{V}$ goes to $B(\underline{V}^R) \to B(\underline{V}^S)$ by the map induced by the left-adjoint of V^t.

If $\underline{V}$ is the category of left R-modules, $B(\underline{V})$ is the Brauer group of R as defined by M. Auslander and Goldman [Trans. Amer. Math. Soc. 97 (1960), 367-409]; if $\underline{V}$ is the category of sheaves of modules over a commutative ringed space (X, U), we have $B(\underline{V})$ is the Brauer group of (X, U) as defined by B. Auslander [J. of Algebra 4 (1966), 367-409]; if $\underline{V}$ is the category of $Z/2Z$ graded R-modules M with R a commutative ring concentrated at zero and symmetric homomorphism

$c: M \otimes M' \to M' \otimes M$ defined by $c(m \otimes m') = (-1)^{\delta m \ \delta m'} (m' \otimes m)$ for homogeneous elements $m \in M$ and $m' \in M'$ of degrees δm and $\delta m'$, respectively, then $B(\underline{V})$ is the Brauer-Wall group of R as defined in Bass [Lectures on Topics in Algebraic K-theory, Tata Institute, 1967].

Some properties cocerning the natural number object in a topos

Godelieve - Van de Wauw - De Kinder

We consieder a topos $\underline{E}$ with a natural number object (NNO) $1 \xrightarrow[o]{} N \xrightarrow[s]{} N$.

We use the concept: "Internal lanquage of a topos", and interpretation of formulas and terms of that lanquage on morphisms of the topos. This is a special case of what Benabou set up about: "Deductive Categories". When the interpretation $|A(x)|$ of $A(x)$ factors through $1 \xrightarrow{true} \Omega$, we will denote this by $\vdash_X A(x)$ and say: $A(x)$ is internally true in X (or simply: is internally true).

For each X and Y $\in$ Ob (E) the following formulas can be interpreted on morphisms: $\Omega^{X \times Y} \times \Omega^X \times \Omega^Y \to \Omega$: "R is a relation (resp. function, monomorphisms, epimorphism, isomorphism) from X' to Y' ".

I will denote the corresponding morphisms: $\Omega^X \times \Omega^X \to \Omega^{\Omega^{X \times Y}}$ by Rel (resp. Ft, Mono, Epi, Iso).

$\vdash_{\Omega^{X \times Y} \times \Omega^X \times \Omega^Y}$ R $\in$ Rel (X',Y') $\iff$ R is a relation from X' to Y', and analogously for Ft, Mono, Epi, Iso.

We have the property: For $R \rightarrowtail X \times Y$, $X' \rightarrowtail X$, $Y' \rightarrowtail Y$ we have $\vdash \ulcorner R \urcorner \in$ Rel $(\ulcorner X' \urcorner, \ulcorner Y' \urcorner)$ if and only if R is the graph of a relation between X' and Y'.

The last result holds also for Ft, Mono, Epi, Iso.

Concerning the NNO we have: $N \times N \xrightarrow[\langle pr_1, s+\rangle]{} N \times N$ is the strict order relation.

It's caracteristic map is $N \times N \xrightarrow{|p<q|} \Omega$

We can consider the corresponding map $N \xrightarrow[seg]{} \Omega^N$.

For each $1 \xrightarrow{p} N$, seg' $\circ$ p is the name of the subobject $Seg_p \rightarrowtail N$.

Results:

1) $N \xrightarrow[\text{seg}]{} \Omega^N$ is a monomorphism.

2) Concerning first order intuitionistic aritmetic I proved:
   - all axioms are internally true in E
   - for all rules (R) of deduction of the form:

$$\frac{A_1, \dots A_n}{B} \quad (R)$$

we have: if $\vdash_{X_1} A_1, \dots \vdash_{X_n} A_n$, then $\vdash_X B$ for a suitable X.

We express this by: R is internally valid in the topos.

Using the theorem of Gentzen that each furmula which is a theorem in first order classical aritmetic and does not contain $\exists$ or v can be deduced in the intuitionistic system, we have the result that such a formula is internally valid in the topos.

3) I deduced that certain rules of induction of classical aritmetic, concerning formula's with one or more variables, are internally valid in the topos.

   Using these tools I derived further (between other results):

4) $\vdash_{N \times N} p \leq q \wedge q \leq p \Longrightarrow p = q$

5) $\vdash_{N \times N} p < q \vee p = q \vee q < p$ ($<$ is a total order)

6) $\vdash_{N \times N \times N} x < p + q \Longleftrightarrow [x < p \vee (\exists_y)(y < q \wedge x = p + y)]$

   Consequence: for $1 \xrightarrow{p} N$ we have: $\bar{\bar{p}} \sqcup \bar{\bar{q}} \simeq \overline{\overline{p+q}}$

7) I defined a morphism $N \times N \xrightarrow[\alpha]{} \Omega^{N \times N \times N}$ such that $\vdash_{N \times N} \alpha \in \mathrm{Iso}$ (seg p x seg q, seg p.q)

   Consequence: for $1 \xrightarrow[q]{p} N$ we have: $\bar{\bar{p}} \times \bar{\bar{q}} \simeq \overline{\overline{p \cdot q}}$

8) $\vdash_{\Omega^{N \times N}} {}_{N \times N}\sigma \in \mathrm{Mono}$ (seg p, seg q) $\Longrightarrow p \leq q$

9) $\Omega^{\vdash N \times N} {}_{N \times N}\sigma \in \mathrm{Iso}$ (seg p, seg q) $\Longrightarrow p = q$

In fact, 9) is a consequence of 8) und 4). Consequence of 8):

If $1 \xrightarrow[q]{p} N$ and $\mathrm{Seg}_p \xrightarrow[\text{mono}]{\alpha} \mathrm{Seg}_q$ then $p \leq q$ for the canonical order relation on $\mathrm{Hom}_E(1,N)$. Consequence of 9): If $1 \xrightarrow[q]{p} N$ and $\mathrm{Seg}_p \xrightarrow{\sim} \mathrm{Seg}_q$ then p and q are the same morphisms.

Distributors and Kan-extensions
by Roswitha Harting

Let $\underline{U}$ be a complete and cocomplete symmetric monoidal closed category. A $\underline{U}$-distributor $\Phi\colon \underline{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \underline{B}$ is a $\underline{U}$-bifunctor $\underline{B}^{op} \otimes \underline{A} \longrightarrow \underline{U}$. If $J\colon \underline{A} \longrightarrow \underline{B}$ is a $\underline{U}$-functor denote by $\Phi_J\colon \underline{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \underline{B}$ and $\Phi^J\colon \underline{A} \leftarrow\joinrel\mapstochar\joinrel\relbar \underline{B}$ the $\underline{U}$-distributors given by $\Phi_J(B,A) = \underline{B}(B,JA)$ and $\Phi^J(A,B) = \underline{B}(JA,B)$. If $\Phi\colon \underline{A} \longrightarrow \underline{B}$ is a $\underline{U}$-distributor, call Φ representable if $\Phi \simeq \Phi_J$ for some $J\colon \underline{A} \longrightarrow \underline{B}$ and corepresentable if $\Phi \simeq \Phi^J$ for some $J\colon \underline{B} \longrightarrow A$. In the situation

$$\begin{tikzcd} \underline{A} \arrow[r, "J"] \arrow[d, "F"'] & \underline{B} \\ C & \end{tikzcd}$$

we ask whether there are $\underline{U}$-distributors $\underline{B} \relbar\joinrel\mapstochar\joinrel\rightarrow \underline{C}$ or $\underline{C} \relbar\joinrel\mapstochar\joinrel\rightarrow \underline{B}$ whose representability or corepresentability would provide the $\underline{U}$-Kan-coextension $(E_J(F)\colon \underline{B} \longrightarrow \underline{C},\ \eta\colon F \longrightarrow E_J(F)\cdot J)$.

In the bicategory $\underline{D\ ist}(\underline{U})$ if $\Phi\colon \underline{A} \longrightarrow \underline{B}$ is a $\underline{U}$-distributor, let $\Phi^{\searrow}$ denote the right adjoint to $\Phi\cdot\_$

Theorem 1:

${\Phi^J}^{\searrow\Phi^F}\colon \underline{C} \relbar\joinrel\mapstochar\joinrel\rightarrow \underline{B}$ is corepresentable, i. e. ${\Phi^J}^{\searrow\Phi^F} \simeq \Phi^H$ for $H\colon \underline{B} \longrightarrow \underline{C}$, iff H is the $\underline{U}$-Kan-coextension $E_J(F)$ and $E_J(F)$ is preserved by all $\underline{U}$-functors $\underline{C}(-,C)\colon \underline{C} \longrightarrow \underline{U}^{op}$

Theorem 2:

If $\Phi_F\cdot\Phi^J$ is representable, i. e. $\Phi_F\cdot\Phi^J \simeq \Phi_H$ for $H\colon \underline{B} \longrightarrow \underline{C}$, then H is the $\underline{U}$-Kan-coextension $E_J(F)$.

Corollary:

In the above situation, $E_J(F)$ is absolute, i. e. for any $\underline{U}$-functor $G\colon \underline{C} \longrightarrow \underline{D}$, $E_J(G\cdot F) = G\cdot E_J(F)$.

Theorem 3:

Suppose $E_J(F)$ exists and is preserved by the $\underline{U}$-functor $\underline{C}(C,-)\colon \underline{C} \longrightarrow \underline{U}$ for each $C \in \mathrm{Ob}\ \underline{C}$.

Then $\Phi_F\cdot\Phi^J \simeq \Phi_{E_J(F)}$.

The above theorems have the following corollaries:

Corollary 1:

If $\Phi_F \cdot \Phi^J \simeq \Phi_H$, then $\Phi^J \backslash \Phi^F \simeq \Phi^H$

Corollary 2: Let

$$\begin{tikzcd} \underline{B} \arrow[rr, "\Phi"] & & \underline{C} \\ & \underline{C}^{\Phi} \arrow[ul, "J"] \arrow[ur, "F"'] & \end{tikzcd}$$

be the standard factorization of an arbitrary $\underline{U}$-distributor $\Phi: \underline{B} \dashrightarrow \underline{C}$

Then Φ is representable iff the $\underline{U}$-Kan-coextension $E_J(F)$ exists and is absolute.

For example, if $\underline{U}$ = Ens, $\underline{B} = \underline{1}$, Then Φ is simply a functor $\underline{C}^{op} \longrightarrow$ Ens. In this case Corollary 2 says Φ is representable iff the comma category (Y,Φ) has a terminal object.

Corollary 3: $E_J(F)$ is absolute iff it is preserved by every $\underline{U}$-functor $\underline{C}(C,-): \underline{C} \longrightarrow \underline{U}$.

Let $\underline{C}$ be $\underline{U}$-cocomplete and suppose $F: \underline{A} \longrightarrow |\underline{C}|$ is an ordinary functor. F induces a $\underline{U}$-functor $F': \hat{\underline{A}} \longrightarrow \underline{C}$, where $\hat{\underline{A}}$ is the free $\underline{U}$-category generated by $\underline{A}$. Denote by $T: \hat{\underline{A}} \longrightarrow \underline{1}$ the $\underline{U}$-functor induced by $\underline{A} \longrightarrow \underline{1}$. Then we have

Corollary 4:

The following are equivalent:

(i) colim F is absolute

(ii) colim F is preserved by each $\underline{U}$-functor $\underline{C}(C,-): \underline{C} \longrightarrow \underline{U}$

(iii) $\Phi_{F'} \cdot \Phi^T$ is representable

For $\underline{U}$ = Ens, $\Phi_F \cdot \Phi^T(C,*)$ is the set of the connected components $\Pi_0(C,F)$ of the comma-category (C,F). In thisway we obtain Paré s description of absolute colimits by means of connected components.

New results in category theory in Prague

Z. Hedrlin.

The research in the category theory continued in Prague in the last years - among others - in the following ways:

a) investigation of systems of morphisms which need not be closed under composition,

b) more detailed questions about connections between category theory and set theory were studied

c) application of categorical methods to other fields.

a) Define a prestructure $\underline{P}$ in the same way a structure is defined in [1] omitting the assumption that identity morphisms and composition must belong to the class. Obviously, the definition of extension does not depend on these assumptions and we can define "$\underline{P}_1$ has an extension into $\underline{P}_2$" which we abbreviate by $\underline{P}_1$ e $\underline{P}_2$. Define a prestructure $\underline{A}(2)^{bi}$ by: the objects are triples $<X,o_1,o_2>$ where o_1,o_2 are binary operations on the set X. $f: X \to X'$ carries a morphism from $<X,o_1,o_2>$ into $<X',o_1',o_2'>$ iff it is a homomorphism from $<X,o_1,o_2>$ into $<X',o_1',o_2'>$ iff it is a homomorphism from the algebra $<X,o_1>$ into the algebra $<X',o_2'>$.

Theorem. Under (M), $\underline{P}$ e $\underline{A}(2)^{bi}$ for every prestructure $\underline{P}$. A theory of prestructures which can replace $\underline{A}(2)^{bi}$ in the theorem was developed. Moreover, similar theorems were proved for morphisms which are carried by relations.

b) The remaining implication in the equivalence

$$\underline{A}(2) \text{ is a superstructure} \iff (M) \text{ holds}$$

was proved in [2] by L. Kučera and A Pultr.

Define a sequence $\{i_a(\underline{S})\}$ for a structure $\underline{S}$ by: $i_a(\underline{S})$ is the "cardinal" of the class of all non-isomorphic objects in $\underline{S}$ whose underlying set has cardinal a .

Theorem. In finite set theory, there is superstructure, whose $\{i_a(\underline{S})\}$ is bounded by 1.In the infinite set theory, $\{i_a(\underline{S})\}$ is not bounded for any superstructure $\underline{S}$.

Investigations of the assertion "The big discrete category has a full embedding into category of algebras" viewed as a set theoretical axiom has been carried by V. Koubek and A. Pultr.

Methods were seeked how to work in finite algebra if the absence of free objects does not permit us to use adjoint functor methods (V. Müller).

c) Many new algebraic and topological structures have been proved or disproved to be superstructures in the seminar of V. Trnková.

Closely connected result: J. Sichler proved that the structure of commutative rings with unit is a superstructure.

Every m-factorization of the category of sets and all mappings was proved to be concrete by V. Koubek and J. Reiterman (see [1]).

[1] Z. Hedrlín, Extensions of structures and full embeddings of categories, Proc. Intern. Congr. of Mathematicians, Nice 1970 (Gaithier-Villars, Paris, 1971).

[2] L. Kučera, A. Pultr, Non-algebraic concrete categories, J. of Pure and Applied Algebra 3 (1973), 95 - 102.

Topological Structures
Horst Herrlich

Part I: Relative topological functors

A functor $T: \underline{A} \to \underline{X}$ is called absolutely topological iff each class-indexed cone (= source) of the form $(X, X \xrightarrow{f_i} TA_i)$ can be lifted T-initially (in the sense of Bourbaki). Such functors have many properties (e. g. they are faithful, absolutely cotopological, detect and preserve limits and colimits, preserve and reflect monos, epis, and bimorphisms, have a full and faithful lari and a full and faithful rari). But the forgetful functors $\underline{Haus} \to \underline{Set}$ and $\underline{Unif} \to \underline{Top}$ are not of this kind. $\underline{X}$ is called a $(\underline{E}, \underline{M})$ - category provided $\underline{E}$ is a class of $\underline{X}$-epis and $\underline{M}$ is a class of cones in $\underline{X}$ such that every cone in $\underline{X}$ has a $(\underline{E}, \underline{M})$ - factorization and the $(\underline{E}, \underline{M})$ - diagonalization property holds in $\underline{X}$. A functor $T: \underline{A} \to \underline{X}$ is called $(\underline{E}, \underline{M})$ - topological iff each cone of the form $(X, X \xrightarrow{f_i} TA_i)$ in $\underline{M}$ can be lifted T-initially. Such functors are faithful, detect limits and colimits, have a left adjoint and induce on $\underline{A}$ the structure of a $(\underline{E}_T, \underline{M}_T)$ - category with $E_T = T^{-1}(\underline{E})$, $M_T = T^{-1}(\underline{M}) \cap \{$T-initial cones$\}$. Equivalent are: (1) T is $(\underline{E}, \underline{M})$ - topological, (2) $\underline{A}$ is a $(\underline{E}_T, \underline{M}_T)$ - category and T has a left adjoint with front adjunctions in $\underline{E}$, (3) there exists an absolutely topological functor $S: \underline{B} \to \underline{X}$ and a full embedding $E: \underline{A} \to \underline{B}$ of $\underline{A}$ as a $\underline{E}_S$-reflective subcategory of $\underline{B}$. An embedding $T: \underline{A} \to \underline{X}$ of an isomorphism - closed subcategory is $(\underline{E}, \underline{M})$ - topological iff $\underline{A}$ is a full, $\underline{E}$ - reflective subcategory of $\underline{X}$. A $(\underline{E}, \underline{M})$ - topological functor reflects isomorphisms iff it is equivalent to the embedding of a full $\underline{E}$ - reflective subcategory.

Part II: A concrete topological category over Set

A nearness structure on a set X is an element ξ of P^3X satisfying the following axioms (where $\mathcal{A}$ and $\mathcal{B}$ denote subsets of PX):

(N1) $\bigcap \mathcal{A} = \emptyset \Rightarrow \mathcal{A} \in \xi$

(N2) if $\mathcal{A}$ corefines $\mathcal{B}$ (i. e. for each $A \in \mathcal{A}$ exists $B \in \mathcal{B}$ with $B \subset A$) then $\mathcal{B} \in \xi$ implies $\mathcal{A} \in \xi$

(N3) $\emptyset \in \mathcal{A} \Longrightarrow \mathcal{A} \notin \xi$

(N4) Axiom of infinity ($\mathcal{A} \notin \xi$ and $\mathcal{B} \notin \xi$) $\Rightarrow \{A \cup B \mid A \in \mathcal{A}$ and $B \in \mathcal{B}\} \notin \xi$.

(N5) $\{cl_\xi A \mid A \in \mathcal{A}\} \in \xi \Longrightarrow \mathcal{A} \in \xi$ where $cl_\xi A = \{x \in X \mid \{A, \{x\}\} \in \xi\}$

A map $f: (X,\xi) \to (Y,\eta)$ between nearness spaces is nearness preserving iff $\mathfrak{A} \in \xi \Longrightarrow \{fA \mid A \in \mathfrak{A}\} \in \eta$. The corresponding concrete category <u>Near</u> has the following properties: (1) the forgetful functor U: <u>Near</u> → <u>Set</u> is absolutely topological. (2) <u>Near</u> contains the categories <u>Unif</u>, <u>Contig</u>, and <u>Prox</u> as full bireflective subcategories. (3) <u>Near</u> contains the category of topological R_0-spaces as full bicoreflective subcategory (a slight enlargement of <u>Near</u> contains <u>Top</u> as full bicoreflective subcategory). Equivalent descriptions of <u>Near</u> can be obtained by axiomatizing the concepts of "uniform covers" (cf. Isbell and Morita) or of collections of sets containing arbitrary small members (Katětov)

A Categorical Concept of Connectedness.

R.-E. Hoffmann.

Summary

An object C of a category $\underline{C}$ is called a Z-object, iff Hom(C,-) preserves coproducts, C is called Z-generated, iff $C = \coprod_{i\in J} C_i$ for Z-objects C_i, the class of Z-objects is denoted by B($\underline{C}$), the class of Z-generated objects by Z($\underline{C}$)(for the corresponding subcategories the same symbols are used), $\underline{C}$ is called based, iff $\underline{C}$ has coproducts and every object of $\underline{C}$ is Z-generated.

In Top (topological spaces and continous maps) Z-objects are the non-void connected spaces (non-void, because an initial object in a category $\underline{C}$ is not a Z-object), analogously in Cat (small categories and functors) and Graph (directed graphs) the non-void connected categories, resp. graphs are the Z-objects. For a group G [G,Ens] denotes the category of G-sets: the transitive (or simple) G-sets are exactly the Z-objects, in the category ^{q}Met of quasi-metric spaces (two different points may have infinite distance or null distance) and non-expansive maps Z-objects are those non-void spaces with finite distances only. Cat, Graph, [G,Ens], ^{q}Met, and especially Ens is a based category (Z-objects in Ens are final objects), but Top is not. There are negative criteria (existence of a null object or of a strictly final object) that exclude the existence of Z-objects in the categories of groups, etc., and of unital rings too ({0} is strictly final), in particular if both $\underline{C}$ and $\underline{C}^{op}$ are based categories, then $\underline{C}$ is equivalent to the final category $\underline{1}$.

A based category $\underline{C}$ is exactly the universal completion of its basis B($\underline{C}$) with respect to coproducts, the functor B($\underline{C}$) → 1 obviously induces a coproduct-preserving functor $\underline{C}$ → Ens (because B(Ens) ≃ 1), and if, in addition, $\underline{C}$ has a final object t, one has the following adjunction S ⊣ L where L is the left adjoint to Hom(t,-): $\underline{C}$ → Ens.

A cone $\{f_i:A_i \to A\}_{i\in J}$ is called a Z-system, iff every A_i is a Z-object and for every Z-object C $\{Hom(C,f_i)\}_{i\in J}$ is a coproduct, Z-systems are uniquely determined (up to ...). In Top a Z-system is the

family of connected components, in a based category it is the (unique!) representation of an object as a coproduct of Z-objects. Let $\underline{C}$ have coproducts, then $Z(\underline{C}) \to \underline{C}$ is coreflective, iff $\underline{C}$ has Z-systems (for every object). If $\underline{C}$ has colimits, $Z(\underline{C}) \to \underline{C}$ is closed under colimits. This provides an idea for an existence criterion for Z-systems: if $\underline{C}$ has an ($\underline{E}$,$\underline{M}$-Mono)-factorization (diagonal condition), if the class of Z-objects is closed under $\underline{E}$-"quotients", if $\underline{C}$ is $\underline{M}$-well-powered, and if $\underline{C}$ has colimits for connected diagrams $T:\Sigma \to \underline{C}$ with M-transition morphisms (i. e. $Tp \in \underline{M}$ for $p \in \Sigma$), then every object $A \in \underline{C}$ has a Z-system $\{f_i:A_i \to A\}_{i\in J}$ and every $f_i \in \underline{M}$.-
A category $\underline{C}$ with coproducts is based, iff $\underline{C}$ has Z-systems and $B(\underline{C}) \to \underline{C}$ is dense.

If $\langle T,\eta,\mu\rangle = \Pi$ is a monad in a based category $\underline{C}$ such that T preserves coproducts, then $\underline{C}^{\Pi}$ is based too (e. g. Cat over Graph, $[G,Ens]$ over Ens), since for a small category $\underline{A}$ $[\underline{A},\underline{C}] \to [\text{Ob }\underline{A},\underline{C}] \equiv \underline{C}^{\text{Ob }\underline{A}}$ is monadic, and a power of a based category is based too (its basis is a co-power of the original basis), $[\underline{A},\underline{C}]$ is based (because $\underline{C}$ is) (e. g. Graph = $[\cdot \rightrightarrows \cdot,Ens]$). In a power of Ens one has the following characterization of "coproduct-preserving" triples:

If $V: \underline{C} \to Ens^M$ (for a set M) is monadic and preserves coproducts, then there is exactly one (up to ...) category $\underline{A}$ with Ob $\underline{A}$ = M and an isomorphism $\underline{C} \cong [\underline{A},Ens]$ such that the following square commutes:

$$\begin{tikzcd} \underline{C} \arrow[r, "\cong"] \arrow[d, "V"'] & {[\underline{A},Ens]} \arrow[d, "\text{Evaluation}"] \\ Ens^M \arrow[r, "\cong"'] & {[\text{Ob }\underline{A},Ens]} \end{tikzcd}$$

Affine Parts of algebraic theories
by J. R. Isbell, M. I. Klun, and S. H. Schanuel

1. The affine part of a varietal theory $\mathbb{T}$ is the subtheory $\underline{A}$ on all the idempotent operations of $\mathbb{T}$.

Theorem. For $\mathbb{T}$ finitary, every $\underline{A}$-algebra is embeddable in the $\underline{A}$-algebra underlying some $\mathbb{T}$-algebra.

The proof proceeds by representing any A-algebra as F/R for some free $\underline{A}$-algebra F and congruence relation R. F embeds (since $\underline{A} \to \mathbb{T}$) in a free $\mathbb{T}$-algebra G. Extending $R \subset F \times F$ to a $\mathbb{T}$-congruence $\hat{R}$ on G, one wants $a, a' \in F$, $< a,a' > \in \hat{R}$, to imply $< a,a' > \in R$. From the form of congruences it suffices to prove that given $\omega(y_1,\ldots,y_{p+q}) \in F$,

$z_i = y_i$ for $i \leq p$, $< y_i,z_i > \in R$ for $p < i \leq p + q$, then $\omega(z_1,\ldots,z_{p+q}) \in F$ and $< \omega((y_i)), \omega((z_i)) > \in R$.

One expresses $y_1,\ldots,y_p$ in terms of finitely many free generators $x_1,\ldots,x_r$, by $y_i = \eta_i((x_j))$. The $r + q$ - ary operation ψ defined by $\omega(\eta_1(x_1,\ldots,x_r),\ldots,\eta_p(x_1,\ldots,x_r), y_{p+1},\ldots,y_{p+q})$ is readily checked to be idempotent, whence the result follows.

The theorem fails for infinitary $\mathbb{T}$, specifically for $\mathbb{T}$ presented by three operations $\alpha(x_1,x_2)$, $\beta(x_1)$, $\sigma(x_1,x_2,\ldots)$, and three laws

$$\alpha(x,x) = x,$$
$$\beta(\alpha(x,y)) = \beta(\alpha(y,x)),$$
$$\sigma(x,\beta(x),\beta(x),\ldots) = x.$$

We have not determined the affine part $\underline{A}$ of that $\mathbb{T}$, but it has a specialization presented by two operations $\alpha(x_1,x_2)$, $\omega(x_1,x_2,\ldots)$, and laws as follows. (1) Two laws: α and ω are idempotent. (2) One law: $\omega(x_0, \alpha(y_1,z_1), \alpha(y_2,z_2),\ldots) = \omega(x_0, \alpha(z_1,y_1), \alpha(z_2,y_2),\ldots)$. (3) Uncountably many laws: $\omega(x_0,x_1,x_2,\ldots) = \omega(x_0,x_{f(1)}, x_{f(2)},\ldots)$ whenever $\{0\} \cup \{\text{all } f(n)\} = \{\text{all } n\}$. The specialization takes α to α and $\sigma(\ \ , \beta(\ \ ), \beta(\ \ ),\ldots)$ to ω. It has a model on $A = \{a_1,a_2,\ldots,b_1,b_2,\ldots,0\}$ with idempotent operations α and ω such that $\alpha(a_i,a_{i+1}) = b_i$, $\alpha(a_{i+1},a_i) = b_{i+1}$, $\omega(b_{f(0)}, b_{f(1)},\ldots) = b_{f(0)}$ when the range of f is finite, and all other values of operations are 0.

If this A were embeddable in the $\underline{A}$-algebra underlying a $\mathbb{T}$-algebra one would have

$$\beta(b_i) = \beta(\alpha(a_i,a_{i+1})) = \beta(\alpha(a_{i+1},a_i)) = \beta(b_{i+1}),$$

whence $\beta(b_i)$ is constant and $0 = \omega(b_1,b_2,b_3,\ldots) = \sigma(b_1,\beta(b_2),\ldots) =$ $\sigma(b_1,\beta(b_1),\ldots) = b_1$, a contradiction.

2. For finitary $\mathbb{T}$, $\underline{A} \to \mathbb{T}$ determines adjoint functors $F \dashv U$ $(S_{\underline{A}}, S_{\mathbb{T}})$. $A \in |S_{\underline{A}}|$ is embedded in FA as $(Ft)^{-1}(x_o)$ where $t: A \to A^o$, FA^o is (like A^o in $S_{\underline{A}}$) free on one generator x_o, and the diagrams $Y \to Z$ isomorphic with $FA \to FA^o$ can be characterized, but not conveniently.

If $\mathbb{T}$ is the theory of groups, or any other theory having exactly one constant and admitting an interpretation of the theory of loops in it, the canonical representations of $\underline{A}$-algebras $Y \to Z$ are all surjective $Y \to Z$ (Z free on one generators) and also $0 \to Z$ (0 being $F\emptyset$ for the empty affine algebra $\emptyset$). Up to isomorphism these diagrams, exzept $0 \to Z$, are the same as short exact sequences $0 \to K \to Y \to Z \to 0$ (Z fixed as before, K, Y arbitrary). The underlying set of the $\underline{A}$-algebra A is equipotent with K. For the theory of groups, these extensions of Z by K may be described as K together with an arbitrary automorphism σ of K. If σ is inner, the $\underline{A}$-algebra is just UK.

Portions of the description in case of groups generalize. More economical representations of A as a fibre,

$$0 \to K \to Y \to Z \to 0,$$

correspond (when $\mathbb{T}$ has just one constant and interprets loop theory) to ideals J of Y orthogonal to K. J an ideal means the commutator $[J,Y] \subset J$ (commutator after Kuroš, here defined as done by Livšic, Calenko, and Šulgeifer in Mat. Sb. in 1964). In case of groups, $J \cap K = 0$, $J \rightarrowtail Z$ imply J is Abelian, and whenever J is Abelian it is central in Y. Hence for groups (since $J \rightarrowtail Z$ also makes J cyclic) these J correspond to central elements of Y not in K. The smallest positive value of $\in : Y \to Z$ on the center of Y is the index n(A), A can be represented as a fibre $h^{-1}(1)$ for $h: Y' \to Z_m$, if and only if n/m.

For any such theory $\mathbb{T}$, if Z can always be reduced to 0, i. e. every A has the form UK, then $\mathbb{T}$ is a theory of modules.

For theories of modules $\mathbb{T}$, $\mathbb{T}$ and $\underline{A}$ are reversibly related by simple constructions, $\underline{A}$ is the affine part of $\mathbb{T}$, $\mathbb{T}$ is the coproduct of $\underline{A}$ with the theory $\underline{P}$ of pointed sets. Hence both (and the ring of scalars) or neither are finitely generated -- and the same for finitely related. The minimum number of generators is not interesting, if finite, it is 2 for $\mathbb{T}$ and 1 for $\underline{A}$. The minimum number of relations is very difficult. It is the same for $\underline{A}$ as for $\mathbb{T}$, and for the ring Z or Z_n Tarski announced in Hannover 1966 that the number is 1. The minimum arity for generating operations is 2 for $\mathbb{T}$. For $\underline{A}$, it is 3 if Z_2 is a homomorphic image of the ring of scalars, otherwise 2.

The affine part of a finitely presented theory need not be finitely generated or finitely related. Two non-isomorphic affine theories (even non-synonymous in the sense of K. de Bowère, Indag. Math. 27 (1965) 622-629) $\underline{A}_1$, $\underline{A}_2$ can have isomorphic coproducts with $\underline{P}$.

The example for the last assertion is found in the theory $\underline{A}$ on operations α^k of arities $k = 2,3,\ldots$, with axioms

$$\alpha^k(x_1,\ldots,x_{k-1},x_{k-1}) = x_{k-1},$$

$$\alpha^k(x_1,\ldots,x_k) = \alpha^k(x_1,\ldots,x_{k-2},x_k,x_{k-1}),$$

$$\alpha^{k+2}(x_1,\ldots,x_i,x_i,x_{i+2},\ldots) = \alpha^k(x_1,\ldots,x_{i-1},x_{i+2},\ldots)$$

for $i \leq k-1$. The subtheories $\underline{A}_i$ on operations α^k with $k \equiv i \pmod 2$ are non-synonymous since $\underline{A}_1$, has a symmetric two-element model but $\underline{A}_2$ has not. However, in $\underline{A} \sqcup \underline{P}$

$$\alpha^k(x_1,\ldots,x_k) = \alpha^{k+1}(0,x_1,\ldots,x_k)$$

and

$$\underline{A}_1 \sqcup \underline{P} \cong \underline{A} \sqcup \underline{P} \cong \underline{A}_2 \sqcup \underline{P}.$$

On coherence for distributivity

G.M. Kelly

1) A doctrine D is a 2-monad on a 2-category, usually on Cat. Our doctrines are "strict": D is a 2-functor, the unit $j: 1 \to D$ and the multiplication $m: D^2 \to D$ are 2-natural (= Cat-natural) transformations, and the usual equations are satisfied on the nose. A D-algebra (or D-category) is a category $\underline{A}$ with an action $\theta: D\underline{A} \to \underline{A}$ strictly satisfying the usual axioms. A D-morphism (or D-functor), however, is a functor $\phi: \underline{A} \to \underline{B}$ together with a natural transformation $\bar{\Phi}$:

$$(1) \qquad \begin{tikzcd} D\underline{A} \arrow[r, "\theta"] \arrow[d, "D\phi"'] & \underline{A} \arrow[d, "\phi"] \arrow[dl, phantom, "\bar{\Phi}\Nearrow"] \\ D\underline{B} \arrow[r, "\theta'"'] & \underline{B} \end{tikzcd}$$

satisfying the two obvious axioms (one for m and one for j). With a D-natural transformation between D-functors defined in the obvious way, we get a 2-category D-Cat. There is a sub-2-category D-Cat_o obtained by taking only the strict D-functors, i.e. those for which $\bar{\Phi}$ is the identity.

2) If in (1) $\underline{A}$ is the free D-category D1, with $\theta = m1$, we can compose (1) with $Dj1: D1 \to DD1$; since $m. Dj = 1$, the top leg is just ϕ, while the bottom leg is the strict D-functor $\theta'. D\phi. Dj1, = s$ say. The 2-cell $\bar{\Phi}\ .\ Dj1$ is a D-natural transformation $\eta: s \Longrightarrow \phi$, and it is easily seen that this is the coreflexion of the D-functor ϕ into the strict D-functors $D1 \to \underline{B}$.

3) Let $\lambda: MD \to DM$ be a distributive law between two doctrines on Cat. It is well known that D lifts to a doctrine on M-Cat_o ; in fact it lifts to a doctrine $\tilde{D}$ on M-Cat. A $\tilde{D}$-algebra is a category $\underline{A}$ which is both a D-category and an M-category, the two structures being related by a two - cell subject to four axioms. Such $\tilde{D}$-algebras are in fact the K-categories for a certain doctrine K on Cat; those $\tilde{D}$-algebras in which the 2-cell is the identity are the algebras

for the doctrine DM on Cat. Since every DM - category is a $\mathcal{K}$ - category, there is a map of doctrines ϕ: K → DM.

Theorem 1. In the 2 - category of endofunctors of Cat, ϕ: K → DM is a reflection onto a "full subthing".

Proof outline Let ψ: DM → K be the unique strict morphism of D-algebras induced by the obvious doctrine - map M → K. By the definition of $\tilde{D}$ and K, ψ is in fact a (non - strict) K - morphism. Clearly $\phi\psi$ = 1. It is easly seen that the non - strict K - morphism $\psi\phi$: K → K (with domain the free K - algebra K) has strict coreflection 1: K → K. Thus we have the comparison 2 - cell η: 1 ⟹ $\psi\phi$, and the proof is completed by the easy verifications that $\phi\eta$ = 1 and $\eta\psi$ = 1.

4.) Commutative monads (= monoidal monads) have been discussed by A. Kock. For doctrines one relaxes the commutativity condition, allowing the appropriate diagram to commute only up to an isomorphism, subject to two coherence conditions.

Theorem 2. Let D be a commutative doctrine and let M be the doctrine whose algebras are symmetric monoidal categories. Then there is a canonical distributive law λ: MD → DM.

We can therefore apply Theorem 1 in this case. A K - category now admits an alternative description: it is a D - category A which is also a monoidal category, and each A ⊗ -: A → A is enriched to be an an op - D - functor (i.e. the 2 - cell goes in the opposite sense to that for a D - functor). Moreover (A ⊗ B) ⊗ - is (isomorphic to) the composite of A ⊗ - and B ⊗ - as op - D - functors, and I ⊗ - is (isomorphic to) the identity as an op D - functor. The final axiom is that, if one makes - ⊗B into an op - D - functor via the symmetry, then (A⊗-)⊗B and A⊗(-⊗B) are suitably related; this "suitably" refers to the isomorphism occuring in the commutativity condition for D.

There are many examples of commutative doctrines: for example those whose algebras are a category - with - a - monad, or a category - with - finite - coproducts, or a category with - all - colimits, or a symmetric monoidal category.

In this last case D is M itself; and a K - category has two monoidal structures $\otimes$ and $\oplus$, together with natural transformations (not necessarily isomorphisms)

$$d : A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C), \qquad e : A \otimes N \to N,$$

where N is the identity for $\oplus$. Such structures have been considered by Laplaza Coherence in categories, Lect. Notes in Math. 281, and the above axioms reduce precisely to those he imposes.

5.) Let Cat/Cat be the 2 - category of categories over Cat, so that an object is a category A with a functor Γ: A $\to$ Cat. Let [Cat, Cat] be the 2 - category of endofunctors of Cat. There is a non - symmetric monoidal closed structure on Cat/Cat, whose tensor product is denoted by $\bullet$. The well - known 2 - functor F : Cat/Cat $\to$ [Cat, Cat] carries $\bullet$ into composition. Call a $\bullet$-monoid D in Cat/Cat a club; it is carried into a doctrine D by F. In fact Cat is embedded in Cat/Cat, and D is nothing but D$\bullet$-. The doctrines which arise in this way from clubs therefore have very concrete descriptions.

The doctrine M for symmetric monoidal categories arises from a club M; and the original coherence theorem of MacLane may be seen as asserting an equivalence of categories M $\to$ P, where P has the natural numbers as objects and permutations as maps. If D in theorem 2 comes from a club D, then K comes from a club K; and theorem 1 then takes the more concrete form that ϕ: K $\to$ D $\bullet$ M is a reflexion onto a full subcategory. In Laplaza's case we have ϕ: K $\to$ M$\bullet$M $\simeq$ P $\bullet$ P; being a strict K-functor, this is immediately calculable on the generators of K.

Since ϕ is a reflexion, two maps f,g: T $\to$ S in K have equal realisations in a model A (i.e. the diagram commutes) if and only if $\phi f = \phi g$: provided that d and e, and hence the realisation of η, are isomorphisms in A. The same is true for any model A if S actually lies in the full subcategory M $\bullet$ M of K. This is essentially the result of which Laplaza gives a combinatorial proof.

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Formal Theory of free-completion monads

Anders Kock

Monads of the said kind are examples of the kind of abstract structure (T, y, m, λ) described below, their "weak algebras" (in the sense below) are, in the application, categories equipped with a <u>not necessarily associative</u> choice of colimits. We display the data:

$$1 \xrightarrow{y} T \xleftarrow{m} TT$$

$$\begin{tikzcd} T \arrow[r, shift left=1.5, "yT"] \arrow[r, shift right=1.5, "Ty"'] \arrow[r, phantom, "\Downarrow\lambda"] & TT \end{tikzcd},$$

where T is a 2-endofunctor on a 2-category (for the application: on Cat), y and m are 2-natural transformations, and λ a natural 2-cell, satisfying three equations (see (1) - (3) below), whose essential content is that $T\lambda * mT$ provides a coreflexive adjunction between m and Ty.

Out of these data, one can construct an invertible 2-cell

$$mT * m \overset{\alpha}{\Longrightarrow} Tm * m$$

which makes (T,y,m) into a monad "up to isomorphism". One can prove many coherence statements about α. One can then for this monad talk about ("weak") algebras (in the Eilenberg-Moore sense, but only associative up to coherent isomorphism). <u>Or</u> one can discuss objects A with an action

$$AT \xrightarrow{x} A$$

which is left adjoint to the unit $A \xrightarrow{Ay} AT$ by an adjunction which arises out of λ in much the same way as the adjunction between m and Ty. Such $AT \longrightarrow A$ we call λ-modules (precise definition in (4) and (5) below). Formal computations on 2-cells then give:

<u>Theorem</u> $AT \xrightarrow{x} A$ is a weak algebra iff it is a λ-module, i. e. x is left adjoint to Ay in the way specified by λ if and only ifx makes A into a weak algebra (the associativity 2-cell for such one being uniquely determind).

In the application (T,y,m,λ) is one of the free-completion-monads of the author (see Limit Monads in Categories, Aarhus Prepint 1967). The weak algebras for these are categories equipped with some choice $x\colon AT \longrightarrow A$ of colimits. That this is so follows from x being left adjoint to Ay (which we know from the Theorem). The maps f between the underlying objects of free algebras, for which a certain canonically constructed 2-cell

$$\begin{tikzcd} AT \arrow[r, "fT"] \arrow[d, "x"'] \arrow[dr, phantom, "\swarrow"] & A'T \arrow[d, "x'"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

is invertible, are in the application to the free-completion monads precisely the cocontinuous maps (not necessarily preserving the <u>chosen</u> colimits).

We give here the precise equations assumed for (T, y, m, λ), as well as for the $x: AT \longrightarrow A$.

(1) $\lambda * m = 1_{1_T}$

(2) $T\lambda * mT * m = 1_m$

(3) $Ty * T\lambda * mT = 1_{Ty}$

and for x:

(4) $Ay * x = 1_A$

(5) $A\lambda * xT * x = 1_x$

It is possible that one can get away with fewer equations, and that one may replace λ by other elements of structure. Recent work of Zöberlein seems to indicate this (he worked independently along similar lines). However, for the free-completion monads, λ is a very natural thing, namely in some sense the generic inclusion of colimit diagrams into their colimit.

The full details can be found in my:
<u>Monads for which structures are adjoint to units</u>, Aarhus Universitet prepint 1972/73 No. 35 (January 1973).

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Homologisation de Categories Pre-Regulieres

R. Lavendhomme

Dans des catêgories comme celles des groupes, des groupes topologiques, des espaces vectoriels topologiques, on sait que la notion de conoyau n'est pas bonne en ce sens qu'elle ne mesure pas adêquatement l'écart au caractêre épimorphique. Dans le cas des groupes, on peut dire que le "vrai" conoyau de $f: A \to B$ est l'espace homogêne $(B,f(A))$. Nous avons montré en [1] que cela fournit une algèbre homologique acceptable pour les groupes.

Dans quelles catêgories peut-on avoir une telle construction et cela conduit-il à une bonne notion d'homologie?

1. Idêaux

L idêe consiste â travailler modulo un idêal. Soit α un idêal de $\mathcal{C}$, on dit que $i: K_x \to A$ est un α-noyau de $x: A \to B$ si i est un mono et un α-mono et si tout y tel que $xy \in \alpha$ se factorise par i. (On dit que i est un α-mono si $iu \in \alpha \Longrightarrow u \in \alpha$). On dêfinit dualement l' α-conoyau $s: B \to Q_x$ de x.

On dêfinit alors l' α-homologie d' un couple (x,y) tel que yx soit dans α par le diagramme suivant:

$$\begin{tikzcd}
 & H_1=Q_{i_3} \arrow[rr, "c"] \arrow[dd, "s_4"] & & H_2=Ks_3 \arrow[dr, "i_6"] \arrow[dd, "i_4"] & \\
H_3=Q_{x'} \arrow[ur, "s_6"] & & & & H_4=K_{y'} \arrow[dl, "i_5"] \\
 & K_y \arrow[ul, "s_5"] \arrow[dr, "i_1"] & & Q_x \arrow[dr, "y'"] \arrow[dd, "s_3"] & \\
A \arrow[ur, "x'"] \arrow[rr, "x"] \arrow[dr] & & B \arrow[ur] \arrow[rr, "y"] \arrow[dr] & & C \\
 & K_{s_1} \arrow[uu, "i_3"] \arrow[ur, "i_2"] & & Q_{i_1} \arrow[ur] &
\end{tikzcd}$$

Quelles qualités doit vérifier une telle homologie?

-a) On dit que l' α-homologie est <u>nette</u> si s_6,c et i_6 sont des isos.

b) On dit que l' α-homologie est <u>catégoriquement bonne</u> si (x,y) est α-exact (i.e. i_3 et s_3 sont des isos) ssi les H_i sont dans α. Cette exactitude catégorique ne suffit généralement pas. On dit que (x,y) est fortement exact si K_y est une image de x.

c) Soit $\mathcal{D}$ une sous-catégorie de $\mathcal{C}$. On dit que l' α-homologie est $\mathcal{D}$ -bonne si pour tout x de $\mathcal{D}$,(x,y) est fortement exact ssi les H_i sont dans α.

<u>2. Problème d'homologisation</u>

On peut maintenant poser le problème d'homologisation. Une solution du problème d'homologisation pour une catégorie $\mathcal{C}$ avec zéros consiste en la donnée d'un foncteur f : $\mathcal{C} \to \mathcal{C}'$ et d'un idéal α de $\mathcal{C}$ de telle manière que:

a) F soit pleinement fidèle et injectif;

b) $\alpha \cap F(\mathcal{C}) = F(\omega)$ (où ω est l'idéal nul de $\mathcal{C}$);

c) l' α-homologie de $\mathcal{C}'$ soit nette et catégoriquement bonne;

d) l' α-homologie de $\mathcal{C}'$ soit $\mathcal{D}$-bonne pour une sous-catégorie $\mathcal{D}$ de $\mathcal{C}'$ contenant $F(\mathcal{C})$.

Supposons que $\mathcal{C}$ soit une catégorie avec object nul pré-régulière (i. e. $\mathcal{C}$ a des limites et colimites finies et a la propriété de factorisation de tout morphisme en un épi régulier suivi d'un mono). On a alors une solution du problème d'homologisation pour $\mathcal{C}$; elle se construit comme suit:

a) Pour $\mathcal{C}'$ on prend la catégorie $\mathcal{C}_p$ des paires de $\mathcal{C}$ dont les objects sont les monos de $\mathcal{C}$ et les morphismes de i vers i', les carrés commutatifs fi = i'f'.

b) L'idéal α de $\mathcal{C}_p$ est formé des morphismes (f,f') de i_1:A' $\to$ A vers i_2:B' $\to$ B tels qu'il existe f_1:A $\to$ B' avec $i_2 f_1$ = f.

c) Le foncteur F : $\mathcal{C} \to \mathcal{C}_p$ associe à f:A $\to$ B le carré

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ 0 \arrow[u] \arrow[r, equal] & 0 \arrow[u] \end{tikzcd}$$

d) Comme sous-catégorie $\mathcal{D}$ de $\mathcal{E}_p$ il suffit de prendre la classe des morphismes (f,f') où f' est un iso (où même un épi régulier).-

3. Suite longue d' α-homologie.

Dans la situation décrite en (2) ci-dessus, on vérifie que l' α-homologie est fonctorielle et que pour toute suite exacte courte d' α-complexes de $\mathcal{E}_p$ on peut construire les morphismes connecteurs et obtenir une suite longue d' α-homologie.

G. Van den Bossche a obtenu des conditions nécessaires et suffisantes sur la catégorie $\mathcal{E}$ pour que, pour toute suite exacte courte de complexes de $\mathcal{E}$, la suite longue d' α-homologie de $\mathcal{E}_p$ soit α-exacte. Indiquons seulement ici des conditions suffisantes.

Théorème

Si 1) $\mathcal{E}$ est pré-régulière avec zéros à factorisations conormales,

2) tout morphisme d'extension dans $\mathcal{E}$ est un iso, i. e. si on a

$$\begin{tikzcd}
 & & B \arrow[dd, "f"] \arrow[dr] & & \\
0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\
 & & B' \arrow[ur] & &
\end{tikzcd}$$

avec des lignes exactes, alors f est un iso;

3) les images commutent aux changements de base monomorphiques,

alors, pour toute suite exacte courte de complexes de $\mathcal{E}$, la suite longue d' α-homologie correspondante de $\mathcal{E}_p$ est α-exacte.

Outre dans les catégories abéliennes, ces hypothèses sont vérifiées dans les catégories des groupes, des groupes topologiques, des groupes topologiques séparés, des espaces vectoriels topologiques, des espaces vectoriels topologiques séparés, etc.

Le présent travail a été fait en collaboration avec G. Van den Bossche.

[1] R. Lavendhomme, "Un plongement pleinement fidèle de la catégorie des groupes" Bull. Soc. Math. Belgique, (1965) 153-185.

Coherent Topos: Algebraic Geometry = Geometric Logic

F. W. Lawvere

In a base topos S with natural-number-object, the fundamental contradiction between finite $\varprojlim$ and (internally) filtered $\varinjlim$ gives rise to two interesting classes of adjoint pairs of functors between large categories / S: Pairs in which the left adjoint preserves finite $\varprojlim$ ("geometric") and pairs in which the right adjoint preserves filtered $\varinjlim$ ("algebraic"). Topoi over S are the categories between which geometric adjoint pairs appear most naturally while locally finitely presentable categories over S (in sense of Gabriel - Ulmer) are appropriate for algebraic adjoint pairs. Adjoint pairs wich belong to both classes go most naturally between large categories over S which are topos, locally finitely presentable and in which the finite presentability is compatible in a certain sense with finite $\varprojlim$: these are the coherent topoi of SGA 4 § 6 in which Grothendieck and Verdier show that Top coh $/_{S} \cong$ Pretopos $(S)^{op}$ by characterizing the small categories arising as the coherent objects in a coherent topos in one direction and by taking sheaves w.r.t. finite coverings in the opposite direction. A pretopos is just an exact category in the sense of Barr which moreover has finite coproducts which are disjoint and universal. Refining work of G. Reyes (which in turn was based partly on work of Volger and Joyal) we point out that such a pretopos T can be considered as the "Lindenbaum category" of an intuinistic many - sorted first order theory in which $=, \exists, \vee, \wedge$ are used for forming wfs but which may also have axioms of the form $A \vdash B$. The corresponding coherent topos may be considered as the category $TS[M]$ of "variable sets" obtained by adjoining an inderminate model M of the theory T to the category of sets. Models $(T,X) \cong \mathrm{TOP}/_{S}(X,S[M])$ for any topos X over S, so that a continuous family of models of T parametrized by X = a sheaf of models of T over X. See also Wraith's paper in this collection. De Signes theorem that a coherent topos has enough points $\longleftrightarrow$ completeness theorem of logic.

Category Theory in the System Sciences

E. G. Manes (in collaboration with M. A. Arbib)*

The abstract was written after the talk was presented and was influenced by interaction with the participants. We also learned at this conference that H. Ehrig and E. Burroni had done significant work in categorical automata theory.

A <u>sequential</u> <u>machine</u>: Q (set of states), X_0 (set of inputs), $\delta : Q \times X_0 \to Q$ (next-state function; the <u>dynamics</u> of the system), $\tau : 1 \to Q$ (starting state), $P : Q \to Y$ (output). (For example, a telephone: X_0 = digits $0,\ldots,9$; Y = {ring signal, busy signal,...}). "Sequential machine" is the classical example.

Let K be a category. Given $X : K \to K$, an X-dynamics is (Q,δ) with $\delta : QX \to Q$. The obvious maps, called <u>dynamorphisms</u>, give rise to the category $\mathrm{Dyn}(X)$. X is an <u>input process</u> if $\mathrm{Dyn}(X) \to K$ has a left adjoint. Further, X is <u>state-behavior</u> if there is also a right adjoint. Consider the free dynamics $(IX^{@}, \mu_0 : IX^{@}X \to IX^{@})$ over I and (if X is state-behavior) the cofree dynamics $(YX_{@}, YX_{@}X \to YX_{@})$ over Y. By the universal properties there are dynamorphisms r, σ

$$\begin{tikzcd} I \arrow[r, "I\eta"] \arrow[dr, "\tau"'] & IX \arrow[d, dashed, "r"] \\ & Q \end{tikzcd} \qquad \begin{tikzcd} YX \arrow[r, "Y\epsilon"] & Y \\ Q \arrow[u, dashed, "\sigma"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

whenever $M = (X,Q,\delta,I,\tau,Y,\beta)$ is a <u>machine</u> (i.e., X is at least an input process, (Q,δ) $\mathrm{Dyn}(X)$, $\tau : I \to Q$ and $\beta : Q \to Y$). In the classical case, K = <u>Sets</u>, $X = - \times X_0$, $I = 1$; the <u>object</u> <u>of</u> <u>inputs</u> $IX^{@}$ is the free monoid X_0^* and r is the <u>reachability</u> <u>map</u> in the sense that $r(x_1 \ldots x_n)$ is the state reached if inputs $x_1,\ldots,x_n$ are applied in that order. Thus $f_M = r\beta : IX^{@} \to Y$ is the <u>behavior</u> of M (as seen externally). Classically, $YX_{@} = Y^{X_0^*}$ and $\sigma(q)$ is the behavior resulting from starting in state q.

In automata theory it is important to know a specific (preferably effective) construction of $IX^{@}$. The recent work of Dubuc in constructing free monads is useful because $X \to X^{@}X \to X^{@}$ is the free monad over X.

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Physical systems give rise to sequential machines. The differential equation $\dot{q} = f(q,x)$ (e.g. q is the position-momentum vector --the state of the system-- and X is the "input" of environmental and control forces acting on the system) is linearized by truncating the Taylor expansion of f to $Aq + Bx$ and discretized by assuming $q(t + \Delta t) \doteq q(t) + \dot{q}(t)\Delta t$. This leads to the sequential machine $\delta(q,x) = Fq + Gx$ where Q, X_o are finite-dimensional vector spaces and $F : Q \to Q$, $G : X_o \to Q$ are linear. Despite its oversimplified appearance, this algebraic approach (conventionally taught in linear systems engineering at the undergraduate level) leads directly to realization algorithms successfully used in practice by control engineers. The categorical formulation is $X = id_K$, $I = X_o$; if K has countable copowers $I^§$, then $I^§$ is the free dynamics $IX^{@}$, this being the Lawvere simple recursive object $0 : I \longrightarrow I^§$, $s : I^§ \longrightarrow I^§$; dually, X is state-behavior if K has countable powers.

Of course a number of standard theorems can be categorized. Minimal realization theorems have been proved by Arbib-Manes, Bainbridge, E. Burroni, Ehrig-Pfender and Goguen. Non-deterministic automata are studied by Arbib-Manes, Burroni and Ehrig-Pfender; the first two show how Beck distributive laws classify the passage from determinism to non-determinism. A non-deterministic version of Alagić's natural state transformations allows a full treatment of stochastic processes. Hierarchies of natural languages have been related to Lambek style Gentzen cut theorems by Wand.

The crucial question is "will category theory provide new insights?". What has been mentioned so far is not convincing. One small contribution is a clarification of the rôle of duality which has been considered by engineers (a linear map $IR^n \to IR^m$ has a canonical transpose $IR^m \to IR^n$). In fact, assume K has countable products and coproducts and let $X : K \to K$ have a right adjoint $X^{\cdot}$. Then X is state-behavior ($IX^{@} = \coprod IX^n$, $YX_{@} = \prod Y(X^{\cdot})^n$) and we have the dual $M^{op} = (X^{\cdot}, Q, \delta^{\cdot}, Y, \beta, I, \tau)$ in K^{op} of the machine M in K. $(M^{op})^{op} = M$, also, we establish the metaprinciple that reachability and observability are dual.

A more important insight, however, is that the "object of inputs" is not predetermined by intuition. Engineers have assumed without question that an input to a system is just a string of input symbols.

This works in linear system theory because of a coincidence: the countable copower vectorspace $X_o^{\S}$ is in bijective correspondence with the monoid X_o^* via $x_1 \dots x_n \in X_o^* \longmapsto x_1 \dots x_n 00.. \in X_o^{\S}$. In their investigation of "linear systems" of groups, Brockett and Willsky in fact observed that the weak direct sum group built on X_o^* produced reachability maps which were not group homomorphisms, and this is clarified by the categorical approach since the free product $X_o^{\S}$ is not built on X_o^*. To illustrate how this insight works, consider the sightly non-linear system $\dot{q} = f(q,x) \doteq c + Aq + Bx + qCy$. A suitable categorical context was suggested by Goguen. Here, take K to be the category of affine vector spaces (the objects are vectorspaces but the maps satisfy f - f(0) is linear) and define $X : K \to K$ by $AX = A \otimes X_o + A + X_o$ where X_o is a fixed object. Then K has products and coproducts (the usual ones) and X has a right adjoint. An X-dynamics $\delta : QX \to Q$ corresponds to the discretization of the system in question and the object of inputs is much more complicated than X_o^*.

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On the Internal Completeness of Elementary Topoi.

By C. J. Mikkelsen

The reason for studying internal completeness properties in elementary topoi, is that it simplifies the access to constructive proofs.

Let $\underline{E}$ be a fixed elementary topos. An ordered object $(A, \downarrow seg)$ is complete iff there exists a contravariant orderpreserving morphism $\inf: \Omega^A \to A$ such that inf and $\downarrow$seg are adjoint on the right.

__Proposition.__ (Dedekind) Let $(A, \downarrow seg, \inf)$ be a complete ordered object and $\phi: A \to A$ any orderpreserving morphism then ϕ has a smallest fixpoint.

__Proof.__ Let $B \overset{i}{\rightarrowtail} A$ such that $ch(i) = \langle \phi, id \rangle \downarrow \hat{seg}$ then $\exists_i . \inf . \phi \le \exists_i . \exists \phi \inf = \exists_{i.\phi} . \inf \le \exists_i . \inf$. hence for $x_o = \ulcorner true_B \urcorner . \exists_i . \inf, x_o . \phi = x_o$ and x_o is smallest with the property $y . \phi \le y$.

Observe that there are "enough" complete ordered objects in $\underline{E}$, indeed the morphism $\bigcap_x: \Omega^{\Omega^x} \to \Omega^x$ defined by $\bigcap_x = \downarrow seg . \Omega^{\downarrow seg} . \Omega^{\{\}_x}$ is the inf corresponding to the induced ordering on Ω^x.

__Proposition.__ The functor $\Omega^-: \underline{E}^{op} \to \underline{E}$ is tripable.

__Proof.__ The left adjoint is $\underline{E} \xrightarrow{\Omega^-} \underline{E}^{op}$; clearly $\underline{E}^{op}$ has coequalizers of reflexive pairs and Ω^- preserves these coequalizers. Indeed, by the internal Beck - condition we see that they are taken into contractible coeuqalizers, finally Ω^- reflects isomorphisms as for any morphism $f: A \to B$ $\exists_f \dashv \Omega^f$, where $\exists_f$ is constructed as follows

$\exists_f = \downarrow seg . \Omega^{\Omega^f} . \bigcap_B$ and as the functor $x \longmapsto \Omega^x$, $f \longmapsto \exists_f$ reflects monos and epics.

Observe that this proposition tells us that the algebrae for the double dualization triple $x \longmapsto \Omega^{\Omega^x}$ and $f \longmapsto \Omega^{\Omega^f}$ are exactly the objects of the form Ω^x, $x \in |\underline{E}|$. These objects have the following internal lattice theoretic description:

<u>Theorem</u> (Stone) The objects in $\underline{E}$ of the form Ω^X, $x \in |\underline{E}|$ are exactly the complete atomic Heyting algebra objects.

<u>Proof.</u> Let $(A, \uparrow\mathrm{seg}, \inf, \Rightarrow)$ be a complete Heyting algebra, and call a global section $1 \xrightarrow{a} A$ an atom in A iff

(a) $\downarrow\mathrm{seg} \rightarrowtail A \xrightarrow{<!a,id>} A \times A \xrightarrow{\uparrow\hat{\mathrm{seg}}} \Omega$

is an orderpreserving isomorphism. Equivalently an atom in A is given by a system of three orderpreserving morphisms

$$\begin{tikzcd} A \arrow[rr, bend left, leftarrow, "\phi!"] \arrow[rr, "\phi^*"] \arrow[rr, bend right, leftarrow, "\phi_*"'] & & \Omega \end{tikzcd}$$

$\phi! \dashv \phi^* \dashv \phi_*$ and ϕ^* preserves implication. The equivalence is given by $a \rightsquigarrow \phi^* = \,<!a,id> \Rightarrow$ and $(\phi!, \phi^*, \phi_*) \rightsquigarrow a = true.\phi!$

We can represent the atoms of A by a subobject $(A)\,T \overset{i_A}{\rightarrowtail} A$ such that

$$\mathrm{Hom}_{\underline{E}}(X, (A)T) \simeq \mathrm{Atoms}_{\underline{E}/X}((A) \times X).$$

Explicitely $ch(i_A) = \,<\alpha, !\ulcorner true_\Omega \urcorner > \delta_\Omega \Omega$ where $\alpha = \,<\uparrow\mathrm{seg}.\ str^{\Omega(-)}_{A,\Omega,}\downarrow\mathrm{seg}.$ $(\wedge.ch(\{\}_A)) >$ comp. $\Omega^{\{\}\Omega}$, and as compatible atoms are equal (!) it follows that $i_A.\downarrow\mathrm{seg}.\Omega^{i_A} = \{\}_{(A)T}$.

Let $\mathfrak{A}(\underline{E})$ be the category of complete Heyting algebra objects and morphisms (from A to B)

$$\begin{tikzcd} \phi: A \arrow[rr, bend left, "\phi!"] \arrow[rr, leftarrow, "\phi^*"] \arrow[rr, bend right, "\phi_*"'] & & B \end{tikzcd} \quad \text{where}$$

$\phi! \dashv \phi^* \dashv \phi_*$ and ϕ^* preserves implication. Observe that we have the following functors

$$\begin{tikzcd} \underline{E} \arrow[r, "\underline{\forall}"] \arrow[dr, "\forall"'] & \mathfrak{A}(\underline{E}) \arrow[d] \\ & \underline{E} \end{tikzcd}$$

and that Υ has a right adjoint $T: \mathfrak{A}(\underline{E}) \to \underline{E}$, indeed

$$\frac{\dfrac{\Omega^{X} \xrightarrow{\ \phi\ } Y \text{ in } \mathfrak{A}(\underline{E})}{Y \xrightarrow{\ \phi^{*}\ } \Omega^{X} \text{ is cont. co-cont. and pres.} \Rightarrow}}{X \xrightarrow{\ \overline{\phi^{*}}\ } \Omega^{Y} \text{ factors } i_Y.\uparrow seg}$$

$$\begin{tikzcd}
X \arrow[r, "\overline{\phi^{*}}"] \arrow[ddr, dashed] & \Omega^{Y} \\
& Y \arrow[u, bend left, "\uparrow seg"] \\
& (Y)T \arrow[u, "i_Y"]
\end{tikzcd}$$

Computing the end adjunction

$$I_{(Y)T} \rightsquigarrow ev \doteq (\exists_{i_Y}.sup, \downarrow seg\ \Omega^{i_Y}, ev_*)$$

we see that $\exists_{i_Y}.sup. \downarrow seg.\Omega^{i_Y}$ is co-continous, and as

$$\{\}_{(Y)T}.\exists_{i_Y}.sup. \downarrow seg\ \Omega^{i_Y} = i_Y. \downarrow seg.\Omega^{i_Y} = \{\}_{(Y)T}$$

we have $\exists_{i_Y}.sup. \downarrow seg.\Omega^{i_Y} = id_{\Omega(Y)T}$

As $\downarrow seg.\Omega^{i_Y}$ preserves implication we have that $\exists_{i_Y}.sup.$ is epic iff $\ulcorner true_{(Y)T} \urcorner . \exists_{i_Y}.sup = e_Y$, where $!_Y \dashv e_Y$, thus taking this as the definition of atomic (i. e. "the sup of the atoms in Y is the greatest element in Y") we have established the theorem.

Homological and Homotopical Applications of Barr's Theorem.

D. van Osdol

Let $\underline{E}$ be a fixed (Grothendieck-Giraud) topos. Barr has proved the following conjecture of Lawvere: there exists a complete Boolean Algebra B and a left exact cotriple G on $\tilde{\underline{B}}$ (= sheaves on B for the canonical topology) such that $\underline{E} \simeq \tilde{\underline{B}}_G$. We will write $\underline{E} \rightleftarrows \underline{B}$ for the associated adjoint pair. Since $\tilde{\underline{B}}$ is a Boolean (elementary) topos, Lawvere paraphrases this theorem by saying that any Grothendieck topos has enough Boolean-valued points. One should be able to use this theorem to prove results for $\underline{E}$ which up to now have only been proved under the assumption that $\underline{E}$ has enough points. This paper presents a few such consequences of Barr's theorem.

If $\underline{Th}$ is a finitary theory, then the left exactness of G implies that it lifts to the categories of $\underline{Th}$-algebras:

$$\begin{tikzcd} \mathrm{Alg}\,(\underline{Th},\underline{E}) \arrow[r, shift left, "S'"] \arrow[d, shift left, "U'"] & \mathrm{Alg}\,(\underline{Th},\tilde{\underline{B}}) \arrow[l, shift left, "Q'"] \arrow[d, shift left, "U"] \\ \underline{E} \arrow[u, shift left, "F'"] \arrow[r, shift left, "S"] & \tilde{\underline{B}} \arrow[l, shift left, "Q"] \arrow[u, shift left, "F"] \end{tikzcd}$$

Given an object X, let $(F'U')^* X$ and $(Q'S')^* X$ be the usual simplicial and cosimplicial resolutions induced by the cotriple $F'U'$ and the triple $Q'S'$. As examples of the kinds of homological theorems which can be proved with the help of Barr's theorem we offer the following. If $\underline{Th}$ is the theory of Abelian groups and Γ_V is the functor "sections over V", then for each $\underline{Th}$-algebra P we have that $H_n(\Gamma_V((Q'S')^* P))$ is the $n^{\underline{th}}$ right derived functor of Γ_V at P. If P' is another $\underline{Ab}$-algebra then $H_n(\mathrm{Hom}((F'U')^* P, (Q'S')^* P'))$ is $\mathrm{Ext}^n(P,P')$. The proofs of these assertions consist of verifying the axioms for right derived functors, and ultimately depend on the fact that epimorphisms split in $\tilde{\underline{B}}$. If $\underline{Th}$ is the theory of rings, R is a $\underline{Th}$-algebra, and M is a sheaf of R-modules then $H_1(\mathrm{Der}_R((F'U')^* R, (Q'S')^* M))$ is in one-to-one correspondence with the set of equivalence classes of singular extensions of R by M; here $\mathrm{Der}_R(R,M)$ is the abelian group of global derivations of R into M. The proof of this result is based on the techniques developed in Beck's thesis.

Now let ΣE be the category of simplicial E-objects and let $\Phi \in |\Sigma E|$. For $n \geq 0$, $0 \leq k \leq n+1$, define $\Phi_n^{(k)}$ to be the object of open boxes with k-face missing. This is a certain inverse limit, which for a pointed topos is the set $\{(x_0,\ldots,x_{k-1},x_{k+1},\ldots,x_{n+1}) \mid x_i \in \Phi_n,\ \partial_i x_j = \partial_{j-1} x_i$ for $i < j$ and $i \neq k \neq j\}$. (We will denote face operators by ∂, degeneracy operators by Δ.) Given a morphism $\Phi \xrightarrow{p} \Psi$ in ΣE, say that p is a fibration if in the pullback diagram

$$\begin{tikzcd}
\Phi_{n+1} \arrow[dr, dashed] \arrow[rrrr, "{(\partial_0,\ldots,\partial_k,\ldots\partial_n)}"] \arrow[ddr, "p_{n+1}"'] & & & & \Phi_n^{(k)} \arrow[d] \\
& P \arrow[rrr] \arrow[dd] & & & \Pi\Phi_n \arrow[dd, "\Pi p_n"] \\
& & & & \\
& \Psi_{n+1} \arrow[r, "\Delta"] & \prod_{i\neq k} \Psi \arrow[rr, "\Pi\partial_i"] & & \prod_{i \neq k} \Psi_n
\end{tikzcd}$$

the induced map $\Phi_{n+1} \to P$ is an epimorphism. This definition agrees with that given by Illusie (Springer lecture Notes, vol.239). Define Φ to be a Kan complex if $\Phi \to 1$ is a fibration. These are good simplicial objects for homotopy theory because for them, "homotopy" is an equivalence relation. To define this relation, let $Y_{io} \to \Phi_{n+1} \underset{\Delta_{n-1}\partial_i\partial_n}{\overset{\partial_i}{\rightrightarrows}} \Phi_n$ and $Y_i \to \Phi_{n+1} \underset{\Delta_{n-1}\partial_i\partial_{n+1}}{\overset{\partial_i}{\rightrightarrows}} \Phi_n$ be equalizers and let Y be the intersection of all these Y_{ij}'s. Let $Y \twoheadrightarrow I \rightarrowtail \Phi_n \times \Phi_n$ be the image factorization, if Φ is a Kan complex then $I \rightrightarrows \Phi_n$ is an equivalence relation. Given a basepoint $\phi\colon 1 \to \Phi$, let $\tilde{\Phi}_n$ be the intersection of the equalizers $X_i \to \Phi_n \overset{\partial_i}{\rightrightarrows} \Phi_{n-1}$, and define $\pi_n(\Phi,\phi)$ to be the coequalizer of the diagram $I \cap (\tilde{\Phi}_n \times \tilde{\Phi}_n) \rightrightarrows \Phi_n$.

These are the homotopy sheaves of Φ relative to the basepoint ϕ. They are the same as those defined by Illusie (ibid).

It is clear from the definitions that $S\pi_n(\Phi,\phi) \cong \pi_n(S\Phi,S\phi)$; and thus various theorems concerning simplicial $\underline{E}$-objects can be proved by proving them in $\underline{\Sigma\tilde{B}}$. We will give two examples. Let Hot $(\underline{\Sigma E})_{Kan}$ be the category whose objects are Kan complexes in $\underline{E}$, and whose maps are homotopy classes of simplicial maps. Then the quasi-isomorphisms (maps p: $\Phi \to \Psi$ such that for each V in $\underline{E}$, and each basepoint ϕ: $1 \to \Phi_V$ we have $\pi_n(p_V)$: $\pi_n(\Phi_V,\phi) \to \pi_n(\Psi_V,p\phi)$ is an isomorphism) admit a calculus of left fractions. The proof ultimately depends on the fact the pullback of a fibrant quasi-isomorphism is a quasi-isomorphism. It suffices to prove this in $\underline{\Sigma\tilde{B}}$, and there one has the long exact homotopy sequence of a fibration. This allows one to complete the proof. As another example, one can define the Hurewicz homomorphisms $\pi_n(\Phi,\phi) \xrightarrow{h_n} \tilde{H}_n(\Phi)$, where $\tilde{H}_n$ denotes reduced homology. If $\pi_n(\Phi,\phi) = 0$ for $n \leq k - 1$ then $\tilde{H}_n(\Phi) = 0$ for $n \leq k - 1$ and h_k is an isomorphism. To prove this, one uses the fact that S reflects isomorphisms. Thus it is enough to prove it in $\underline{\Sigma\tilde{B}}$. There one can develop the theory of minimal subcomplexes, and that allows one to complete the proof.

The Internal and External Aspect of Logic and Set Theory in Elementary Topoi

Gerhard Osius

Concerning the relation ship between set theory and elementary topoi we consider two problems: (1) First a general procedure to prove results of set theory "internally" in a topos (internal aspect), (2) Characterize certain topoi as "the category of sets" arising from (very general types of) models of set theory, thus generalizing the well-know result from COLE-MITCHELL-OSIUS (external aspect).

To attack both problems by a common method, let us start off with an elementary topos $\underline{E}$ and recall W. MITCHELL's language $L(\underline{E})$. $L(\underline{E})$ is a many-sorted language having the objects of $\underline{E}$ as "types". The terms of $L(\underline{E})$ are: (a) variables (of each type), (b) for any map $A \xrightarrow{f} B$ a unary operation $f(-)$ from terms of Type A to terms of Type B, (c) "ordered pairs" $< t_A, t_B >$ of type $A \times B$ for all terms t_A, t_B of type A,B.

Predicates of $L(\underline{E})$: For any subobject $A \xrightarrow{M} \Omega$ we have a unary predicate $(\ ) \in_A M$ for terms of type A. The Formulas of $L(\underline{E})$ are given by the atomic ones using the connectives $\neg, \wedge, \vee, \Longrightarrow$ and quantifiers $\exists_A x_A$, $\forall_A x_A$.

In $L(\underline{E})$ we define equality, membership and evaluation as follows

(o) $\quad x_A = x_A' :\Longleftrightarrow \ < x_A, x_A' > \ \in (A \times A \xrightarrow{\Delta} \Omega)$

(i) $\quad x_A \in X_{PA} :\Longleftrightarrow \ < X_{PA}, X_A > \ \in (PA \times A \xrightarrow{ev} \Omega)$

(ii) $\quad f_{B^A}(x_A) = (B^A \times A \xrightarrow{ev} B)\ (x_A)$

Using MITCHELL's internal interpretation of $L(\underline{E})$ in $\underline{E}$ (with slight modifications), which assigns to any formula ϕ of $L(\underline{E})$ with free variables of types $A_1,\ldots,A_n$ a subobject $A_1 x \ldots x A_n \xrightarrow{\|\phi\|} \Omega$ we define internal validity for formulas of $L(\underline{E})$:

ϕ is internally valid iff $\|\phi\|$ facters through $1 \xrightarrow{true} \Omega$

Theorem: Internally valid are: (a) the (properly stated) axioms and deductive rules of intuitionistic logic, (b) the axioms of equality and ordered pair, (c) the following (properly stated!) axioms of many-sorted set theory with spect to $=, \in$ as defined in (o), (i): extensionality, empty set, singleton, union (binary and arbitrary), powerset, separation schema.

The importance of internal validity lies in the fact that the morphisms $A \xrightarrow{f} B$ in $\underline{E}$ are in 1-1-correspondence to formulas $\phi(x_A, y_B)$ of $\underline{E}$ for which $\forall_A x_A \; \exists_B^! y_B \; \phi(x_A, y_B)$ is internally valid (via $\| \phi(x_A, y_B)\| =$ graph (f)). This enables us to prove results in $\underline{E}$ (e. g. existence of maps, equality of maps etc.) by showing that some formula of L($\underline{E}$) is internally valid, and this contributes to the problem (1).

As to the external aspect, let us from now on suppose that $\underline{E}$ is well-opened i. e. the subobjects of 1 (called open objects) generate. Then we can define an external interpretation of L($\underline{E}$) in $\underline{E}$ in the following way: terms of type A range over partial maps from 1 to A (called A-elements) which can be considered as maps $1 \xrightarrow{a} \tilde{A}$ or $U \xrightarrow{u} A$ (U open) which are related by the pullback:

$$\begin{array}{ccc} U & \xrightarrow{u} & A \\ \downarrow & & \downarrow \eta_A \\ 1 & \xrightarrow{a} & \tilde{A} \end{array}$$

(the operations on terms f(-), < -, - > are interpretated in the obvious way) For any A-element a resp. u we define, its support $1 \xrightarrow{|a|=|u|} \Omega$ as the character of $U \xrightarrow{u} 1$.

Given any formula $\phi(x_{A_1}, x_{A_2}, \dots)$ with free variables $x_{A_1}, \dots x_{A_n}$ and given A_i-elements a_i $(i=1,\dots,n)$ we define the external value $1 \xrightarrow{|\phi(a_1,\dots a_n)|} \Omega$ in the external HEYTING-algebra $\underline{E}(1,\Omega)$ by induction on the length of ϕ as follows. In the atomic case $|a \in_A M|$ is $1 \xrightarrow{a} \tilde{A} \xrightarrow{\tilde{M}} \Omega$ where $\tilde{M}$ is the existential image of $A \xrightarrow{M} \Omega$ under $A \xrightarrow{\eta_A} \tilde{A}$. In the non-atomic cases we mention only two in detail

$$|\neg \phi(a_1,\dots,a_n)| := \neg |\phi(a_1,\dots,a_n)| \cap |\langle a_1,\dots,a_n\rangle|$$

$$|\forall_A x_A \; \phi(x_A,a_1,\dots,a_n)| := \inf_{1 \xrightarrow{a} \tilde{A}} (|a| \Rightarrow |\phi(a,a_1,\dots,a_n)|) \cap |\langle a_1,\dots,a_n\rangle|$$

and similar for $\wedge, \vee, \Longrightarrow, \exists$. Notes: a) the "restriction" through the term $|\langle a_1,\dots,a_n\rangle| = |a_1| \cap \dots \cap |a_n|$ is essential, since we want $|\phi(a_1,\dots a_n)| \subset |a_1| \cap \dots \cap |a_n|$, b) the "inf" is an "external" infinum in $\underline{E}(1,\Omega)$ which can be shown to exist (since $\underline{E}$ is well-opened).

We are now able to define external validity for formulas of $L(\underline{E})$: $\phi(x_{A_1},\dots x_{A_n})$ is <u>externally valid</u> iff for all A_i-elements a_i $(i=1,\dots,n)$: $(|\langle a_1,\dots a_n\rangle| \Rightarrow |\phi(a_1,\dots a_n)|)$ = true. Between the internal and external interpretation of $L(\underline{E})$ we have the following connection:

<u>Theorem:</u> For any formula $\phi(x_{A_1},\dots x_{A_n})$ of $L(\underline{E})$ and A_i-elements a_i $(i=1,\dots,n)$ we have:

$$|\phi(a_1,\dots a_n)| = |\langle a_1,\dots a_n\rangle \in \| \phi(x_{A_1},\dots x_{A_n})\||$$

As an easy consequence we note, that the notion of internal and external validity coincide, so that we can concentrate on internal validity, which we already have studied. Let us finally indicate how the external interpretation contributes to our problem (2). Using the same method (of "transitive set-objects") which we introduced in the case that $\underline{E}$ is well-pointed, we are to identify certain A-elements with certain B-elements (under certain conditions) and thus obtain from the external interpretation an actual HEYTING-valued model for an appropriate subsystem of ZF set theory. Throwing in more axioms on E (e. g. existence of a natural number object etc.) gives raise to such characterizations as described in our problem (2).

Categories with involution and isometric kernals

Paul H. Palmquist

Baer*- categories are a convenient context to answer the three questions: (A) - Is the passage from a unitary representation of a subgroup to the induced representation left (or right) adjoint to restriction?; (B) - The wellknown assignments from a Baer*- Semigroup S to its lattice L(S) of projections, and from an orthomodular lattice L to its (universal) coordinatizing Baer - semigroup S(L), yeild isomorphisms $L(S(L)) \simeq L$, why not $S(L(S)) \simeq S$? (C) - Express naturally facts about Hilbert spaces over the <u>reals</u> in terms of Hilbert spaces over the complexs; and similarly, for any two of the list: reals, complexs, and quaternions.

A Baer*- category has involution $(\ )^*: B^{op} \to B$, $B^* = B$, $f^{**} = f$ and isometric kernals $k(f) : K(f) \to \mathrm{dom}\,(f)$, $k(f)^* \circ k(f) = \text{identity}$. Examples are Hilbert spaces over $\Lambda = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and continuous Λ-linear maps, $H(\Lambda)$; unitary representations of a locally G in $H(\Lambda)$, $U(G,\Lambda)$; orthomodular lattices and residuated maps, RO; sets and relations, Rel.; and *-functors into a Baer*- category and natural transformations.

A Baer*- functor preserves kernals and involutions. Examples are the five natural functors between categories of Hilbert spaces displayed in the noncommutative diagram below.

$$
\begin{tikzcd}
 & H(\mathbb{C}) \arrow[dd, "t"] & \\
H(\mathbb{R}) \arrow[ur, "c"] & & H(\mathbb{H}) \arrow[ul, "c'"'] \\
 & H(\mathbb{C}) \arrow[ul, "r"] \arrow[ur, "h"'] &
\end{tikzcd}
$$

Others are restriction functors $U(G,\Lambda) \to U(H,\Lambda)$ when H is a subgroup of G, the evaluation $F \longmapsto F(c)$ of *-functors F into a Baer*- category, and the canonical representation of a Baer - category into RO, sending B into its lattice R(B) of normal subobjects.

In general, if a $*$-functor F has a left adjoint L, then it has a right adjoint $(\ )^* \circ L \circ (\ )^*$. Thus the answer to A is "both or neither". If L is also a $*$-functor, we have a Frobenius adjoint triple = $\mathbb{T} = (FL,\mu,\eta) \longrightarrow \mathbb{G} = (FL,\mu^*,\eta^*)$. Thus the answer to C is that the five functors diagrammed above are $*$-tripleable (and $*$-cotripleable).

For example, the adjoint situation $r \underset{\varepsilon}{\overset{\eta}{\longrightarrow}} c$ generates a Frobenius triple $\mathbb{T}'$ on H(C) yeilding a $*$-Eilenberg-Moore category which is the full subcategory of the ordinary E-M category with objects the self-adjoint algebras with respect to the involution $(A,\xi: TA \to A) \longrightarrow (A,\xi^*)$ where ξ^* is the $\mathbb{T}$-algebra structure map corresponding to the $\mathbb{G}$-coalgebra $(A,\xi^*: A \to TA)$. Problem B is put in perspective by observing that every endomorphism monoid in a Baer$*$-category is a Baer$*$-semigroup, and that the canonical representation of RO is isomorphic to the identity functor. Thus L(S(L)) is isomorphic to L. Conversely, every Baer$*$-semigroup is embedded in its projection-completion Baer$*$-category. While the canonical representation takes central automorphisms to identity maps. Thus in general the homomorphism $S \to S(L(S))$ is neither injective nor surjective.

Limits of Profunctors and Representability

Robert Paré

Let $\underline{I}$ and $\underline{A}$ be small categories. The $\varprojlim$ of a profunctor $\underline{I}$ ..... > $\underline{A}$ generalizes the usual notion of $\varprojlim$ when the profunctor in question represents a functor $\underline{I} \to \underline{A}$ and the $\varprojlim$ of this functor exists in $\underline{A}$. The $\varprojlim$ in the profunctor sense always exists. This paper studies the relationship between representability (or partial representability) of profunctors and their $\varprojlim$ preservation properties.

A profunctor $\underline{A}$ ..... > (i. e. a functor $\underline{A} \to \underline{S}$) preserves all small $\varprojlim$ (in the profunctor sense) if and only if it is a retract of a representable. If $\underline{A}$ has split idempotents, then a retract of a representable it itself representable. So assuming, for now, that $\underline{A}$ has split idempotents, it follows that any profunctor with domain $\underline{A}$ which has a left adjoint is induced by a functor, and any functor with domain $\underline{A}$ which preserves all $\varprojlim$ (in the profunctor sense) has a left adjoint (functor). This last statement is essentially Freyd's "More General Adjoint Functor Theorem".

Several "partial representability" theorems are obtained of the same nature as the following nonadditive version of Lazard's theorem on flatness: A profunctor $\underline{A}$ ..... > $\mathbf{S}$ preserves all finite $\varprojlim$ if and only if it is a filtered $\varinjlim$ of representables.

A characterization of those categories $\underline{J}$ for which $\underline{J}^{op}$ - $\varprojlim$ commute with $\underline{I}$-$\varprojlim$ in $\underline{S}$ is given in terms of profunctors.

Free finitary equationally defined algebras in an elementary topos containing a natural number object.

D. Schumacher

The construction of free algebras presented here was obtained by translating R. Harkloff's paper "Eine Konstruktion absolut freier Algebren" Math. Ann. Bd. 158 (1965) into the language of topoi. According to given a finitary type $\tau = (n_i)\ 1 \leq i \leq k$ and finitely many equations first Peanoalgebras of type τ are constructed, then Peanoalgebras are shown to be free and finally the free equationally defined algebras are obtained by factoring the Peanoalgebras through the given equations. (Note that it is enough to show that free algebras of a type not containing the arity 0 exist.)

Some details of the free algebra construction:
Given a type $\tau = (n_i)\ 1 \leq i \leq k$ these is a functor K from $\underline{E}$ into the category Alg $(\tau, \underline{E})$ of algebras of type τ and a natural transformation $id_{\underline{E}} \xrightarrow{\eta} UK$ (U the underlying functor) such that for every object X of $\underline{E}$ η_X and the operations of K(X) are monomorph and mutually disjoint. This is shown by (given $X \in ob(\underline{E})$) defining an object of $\underline{E}$ and mutually disjoint monomorphisms $X \rightarrowtail{x} B$, $n_i \cdot B \xrightarrow{g_i} B$ ($n_i \cdot B$, n_i - fold coproduct of B) and then passing from coproduct to products by roughly speaking turning to power sets. More precise: If m is a monomorphism, then so is $\exists_m (\Omega^m \exists_m = id)$; if m and n are disjoint monomorphisms then not just $\exists_m$ and $\exists_n$ but their values under the non empty direct image functor P are. P is thereby the subfunctor of $\exists_{(\ )}$ the natural transformation from which into $\exists_{(\ )}$ at any object A, μ_A, is an image of $\eta_A \xrightarrow{\epsilon_A} A \times \Omega^A \xrightarrow{p_2} \Omega^A$. Furthermore if $C \xrightarrow{m} A$ and $Z \xrightarrow{z} A$ ase disjoint monomorphisms $\{\tilde{\ }\}_A z$ and P(m) are where $\{\tilde{\ }\}_A$ is the morphism for which $\{\ \}_A = \mu_A \{\tilde{\ }\}_A$. And finally for every object A $P(A)^n \xrightarrow{(\mu_A)^n} \Omega^{n \cdot A}$ factors through $\mu_{n \cdot A}$. Thus the mutually disjoint monomorphisms $X \rightarrowtail B$, $n_i \cdot B \xrightarrow{g_i} B$ give rise to equally mutually disjoint monomorphisms $X \xrightarrow{x} B \xrightarrow{\{\tilde{\ }\}_B} P(B)$, $P(B)^{n_i} \rightarrowtail P(n_i \cdot B) \xrightarrow{P(g_i)} P(B)$.

To find the above monomorphisms x and g_i in the first place let Y denote the coproduct X + 1 + ... + 1 of X and n + k times 1 where n is the highest arity in the type and let i_x and i_e $1 \leq e \leq n + k$ its injections. Then B will be Y^N, x the morphism

$X \xrightarrow{i_x} Y \xrightarrow{\text{const.}} Y^N$ (const. = exponential adjoint of $N \times Y \xrightarrow{p_2} Y$),

and the g_i: will be defined as follows: For every $1 \xrightarrow{c} C$ let sh_c

be the morphism $C^N \xrightarrow{c \times C^N} C^{1+N} \xrightarrow{C^g} C^N$ where g is the isomorphism from N into 1 + N defined by

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "q"] & N \arrow[d, "g"] \\ 1 \arrow[ur, "o"] \arrow[r, "i_1"'] & 1+N \arrow[r, "i_2 \binom{o}{s}"'] & 1+N \end{tikzcd}$$

(see P. Freyd: Aspects of topoi, Bull. Austral. Math. Soc. Vol. 7 (1972).). In sets sh_c shifts every sequence in C^N one place to the right putting c into the zeroth place thus becoming free. Then for $c \neq d$ $sh_c \cap sh_d = 0$. Let now in particular for $1 \leq i \leq k$, $1 \leq j \leq n_i$ $m_{ij} = sh_{\iota_j} sh_{\iota_i}$. Then $m_{ij} \wedge m_{lk} = 0$ for $(i,j) \neq (l,k)$ and hence the $g_i =_{def} [m_{ij}]$, $1 \leq i \leq k$ are monomorph, mutually disjoint and disjoint from x.

In the paper (I presented on the conference) I go now on defining recursively the subalgebra X generated by a subobject $X \xrightarrow{x} A$ of an Algebra $\underline{A}$ and I use again the natural numbers to show that any morphism from X into an algebra $\underline{B}$ can by uniquely extended to an homomorphism from the subalgebra $[X]$ of K(X) generated by $X \xrightarrow{n_x} K(X)$ and finally again in defining the congruent hull of an relation. This excessive use seems to be redundant in the wake of Ch. J. Mikkelson's fixpoint theorem, more precise, N is likely to be used only in the definition of the above functor K.

Strong Universality and Testing Categories

J. Sichler

Let $\underline{K}_1$, $\underline{K}_2$ be concrete categories with fixed forgetful functors $\underline{U}_i : \underline{K}_i \to \underline{Set}$. A full embedding $\Phi : \underline{K}_1 \to \underline{K}_2$ is strong if there is a functor $\underline{F} : \underline{Set} \to \underline{Set}$ such that $\underline{U}_2 \circ \Phi = \underline{F}\,\underline{U}_1$ (the embedding Φ is carried by $\underline{F}$).

Call a category $\underline{K}_2$ universal (or binding) if every category of algebras can be fully embedded into it, $\underline{K}_2$ is strongly binding (strongly universal) if every category of algebras can be strongly embedded into $\underline{K}_2$.

Let $\underline{k}$ be a small category and let $\underline{Set}^{\underline{k}}$ denote the category of all functors $\underline{k} \to \underline{Set}$ and all their natural transformations.

The symbol $X \vee Y$ denotes the disjoint union of X,Y.

Theorem 1: Let $\underline{Set}^{\underline{k}}$ be universal and let $\underline{C}$ be a category of algebras. Then there are sets a,b,c, and a strong embedding $\Phi : \underline{C} \to \underline{Set}^{\underline{k}}$ carried by the functor $\underline{F}(x) = (x^a \times b) \vee c$. In particular, universality is equivalent to strong universality for any $\underline{Set}^{\underline{k}}$.

Theorem 2: There is a finite category $\underline{t}$ such that the following conditions are equivalent:

(a) $\underline{t}$ is isomorphic to a full subcategory of $\underline{Set}^{\underline{k}}$

(b) $\underline{Set}^{\underline{k}}$ is universal.

The small category $\underline{t}$ thus "tests" the universality of $\underline{Set}^{\underline{k}}$. Question: is there a small category testing the universality of an arbitrary equational class of algebras?

Monoidal Universal Algebra in Initialstructure Categories.

Manfred Bernd Wischnewsky

Initialstructure functors $F : K \to L$, the categorical generalization of Bourbaki's notion of an "initial object", equivalent to Kennison s pullback stripping functors ([5]), which Wyler calles topological functors ([10]), are extremely suitable for lifting algebraic properties from the base category to the INS-category K. ([3,8,9,10]). This article now deals with monoids in monoidal INS-categories. It is shown, that "all" categories properties, which are valid for the category of monoids in an arbitrary monoidal category, hold for the category of monoids in any monoidal INS-category K over L. The results stated here are even valid for arbitrary S-monoidal algebraic categories over S-monoidal INS-categories in the sense of ([11]).

1. Def.: Let $F : K \to L$ be a faithful, fibresmall functor. F is called an INS-functor and K an INS-category over L, if for all small categories D and all functors $T \in [D,K]$ the functor

$$F_T : (\Delta,T) \to (\Delta,FT)$$

$$(\Psi : \Delta k \to T) \longmapsto (F\Psi : \Delta Fk \to FT)$$

has an adjoint with counit id, where (Δ,T) resp. (Δ,FT) denote the comma categories of all cones from K to T resp. from L to FT.

If K and L are complete, then F is an INS-functor if and only if F preserves limits and has a full and faithful right adjoint. The most wellknown examples of INS-categories are the categories of topological, measurable, limit, compactly generated, or locally convex spaces, groups, rings, sheaves....

Let now $L = (L,\Box,\text{can})$ be a monoidal category. Monoidal functor shall always mean strict monoidal functor.

2. Lemma. Let $F : K \to L$ be an INS Functor over a monoidal category $L = (L,\Box,\text{can})$. Then there exists at least one monoidal structure on K, such that $F : K = (K,\Box,\text{can}) \to L = (L,\Box,\text{can})$ becomes a monoidal functor.

In general there are a lot of monoidal structures on K with this property as the case of the forgetful functor from the category of topological spaces to the cartesian closed category of all sets shows.

Assume now, that the INS-category K carries any monoidal structure, such that $F : \mathcal{K} = (K,\Box,\mathrm{can}) \to \mathcal{L} = (L,\Box,\mathrm{can})$ becomes a monoidal functor.

Denote by MonK resp. MonL the categories of all monoids over K resp. over L.

3. Theorem. Let $F : (K,\Box,\mathrm{can}) \to (L,\Box,\mathrm{can})$ be a monoidal INS-functor, then the following assertions are valid:

1) The canonically induced functor MonF : MonK $\to$ MonL is again an INS-functor. In particular MonF is adjoint and coadjoint and preserves and reflects monos and epis.
2) The forgetful functor V_K : MonK $\to$ K is adjoint if and only if the forgetful functor V_L : MonL $\to$ L is adjoint. If V_K is adjoint, then V_K is monadic.
3) Let $H : K \to K'$ be a monoidal INS-morphism over L ([9,10]). Then the induced functor MonH : MonK $\to$ MonK' is again an INS-morphism and hence in particular adjoint.
4) MonK is complete, cocomplete, wellpowered, cowellpowered if and only if MonL has these properties.
5) MonK has generators, cogenerators, projectives, injectives... if and only if MonL has these objects.
6) MonK is a (coequalizer,mono)-bicategory if and only if MonL is a (coequalizer,mono)-bicategory.

Let $k = (k,\mu,\varepsilon)$ MonK. Denote by Lact(k,K) the category of all K-objects, on which k acts on the left.

4. Theorem. Let $F : (K,\Box,\mathrm{can}) \to (L,\Box,\mathrm{can})$ be a monoidal INS-functor, and let k be a monoid in $(K,\Box,\mathrm{can})$. Then the induced functor

LactF : Lact(k,K) $\to$ Lact(Fk,L)

is again an INS-functor. Furthermore all assertions of theorem 3 are valid.

5. Example: Let K be an INS-category over S e. g. the category of topological, measurable, compactly generated, zero dimensional, or uniform spaces. The category K(Ab) of all abelian groups in K is a closed monoidal INS-category over Ab, the category of abelian groups. The closed monoidal structure is given by the "inductive tensorproduct". The monoids in K(Ab) are just the rings in K. The category Lact(r,K(Ab)) is the category of all K-modules over the K-ring r. Hence the forgetful functor K(r-mod) $\to$ r-mod into the category of all r-modules in Ens is an INS-functor. The category K(r-mod) is complete, cocomplete, wellpowered, cowellpowered, has generators, cogenerators, projectives, injectives and a cononical (coequalizer, mono) bicategory structure. But K(r-mod) is in general not abelian, since bimorphisms need not to be isomorphisms.

Recursion schemes and elementary Toposes

G. C. Wraith

Let $\underline{E}$ be an elementary topos with natural number object (NNO) $1 \xrightarrow{o} N \xrightarrow{\sigma} N$.

Proposition. (Global section criterion). Given $X = (A \xrightarrow{x} N)$ in $\underline{E}/N$ and maps

(i) $1 \xrightarrow{x_o} o^*(X)$ in $\underline{E}$,

(ii) $X \xrightarrow{t} \sigma^*(X)$

in $\underline{E}/N$, then there is a unique map $1 \xrightarrow{x} X$ in E/N such that $o^*(x) = x_o$ and $\sigma^*(x) = tx$.

Proposition. (Uniqueness criterion). Let $T: \underline{E}/N \to \underline{E}/N$ be a strong functor. Given X_o in $\underline{E}$, if there exists an object X in $\underline{E}/N$ such that $o^*(X) = X_o$ and $T(X) = \sigma^*(X)$ then X is unique up to an isomorphism f which is unique with the properties $o^*(f) = 1_{X_o}$, $T(f) = \sigma^*(f)$.

The uniqueness criterion may be proved by applying the global section criterion to the object $Iso(X,X')$ in $\underline{E}/N$ where X,X' both satisfy the same conditions.

A natural number in $\underline{E}$ is a global section of N. In $\underline{E}/N$ there is a natural number n given by the diagonal map $N \to N \times N$. Let $[n]$ denote the object $N \times N \xrightarrow{+} N \xrightarrow{\sigma} N$ in $\underline{E}/N$ (this notation is due to Bénabou), and for any natural number $1 \xrightarrow{x} N$ in $\underline{E}$, let $[x] = x^*([n])$. The object $[n]$ is unique with the property $o^*[n] = \emptyset$, $\sigma^*[n] = [n] \amalg 1$.

Proposition. If $\underline{F} \xrightarrow{f} \underline{E}$ is a geometric morphism, then $f^*(A^{[x]}) = f^*(A)^{[f^*(x)]}$.

This proposition may be proved by considering the pullback f/N of f along $\underline{E}/N \to \underline{E}$, and proving the analogous result with $[n]$ in place of $[x]$, using the uniqueness criterion. As a corollary, if $\underline{E}_{fin}$ denotes the internal full subcategory generated by the map $N \times N \xrightarrow{+} N \xrightarrow{\sigma} N$ (notion due to J. Bénabou), then $f^*(\underline{E}_{fin}) \simeq \underline{F}_{fin}$.

Let $\underline{E}[U]$ denote the topos of internal functors $\underline{E}_{fin} \to \underline{E}$, where U denotes the internal inclusion functor. Then $\underline{E}[U]$ is an object classifier for $\underline{E}$-Toposes, i. e. for any $\underline{E}$-topos $\underline{F} \xrightarrow{f} \underline{E}$ and object Y of $\underline{F}$ there is a unique geometric morphism $\underline{F} \xrightarrow{g} \underline{E}[U]$ over $\underline{E}$, such that

$Y \simeq g^*(U)$. That is to say,

$$\underline{\mathrm{Top}}_{\underline{E}}\,(\underline{F},\underline{E}[U]) \simeq \underline{F}.$$

It follows that any geometric morphism $\underline{E}[U] \xrightarrow{T} \underline{E}[U]$ determines a natural endomorphism of $\underline{\mathrm{Top}}_{\underline{E}}$, the 2-category of $\underline{E}$-toposes. Such a natural endomorphism gives a strong endofunctor on each $\underline{E}$-topos. Consequently, given a diagram

$$\underline{E} \xrightarrow{x_0} \underline{E}[U] \xrightarrow{T} \underline{E}[U]$$

if there exists a morphism $\underline{E}/N \xrightarrow{X} \underline{E}[U]$ such that the diagram

$$\begin{tikzcd}
 & \underline{E}/N \arrow[rr, "\sigma"] \arrow[d, "X"] & & \underline{E}/N \arrow[d, "X"] \\
\underline{E} \arrow[ur, "\upsilon"] \arrow[r, "x_0"'] & \underline{E}[U] \arrow[rr, "T"] & & \underline{E}[U]
\end{tikzcd}$$

commutes, then X is unique up to a unique isomorphism. It is conjectured that X always exists.

Multiple Functor Categories

Oswald Wyler

n-tuple categories generalize the double categories of Ehresmann, n-categories which generalize 2-categories are a specialization. The consideration of n-tuple categories is motivated by the observations that adjunctions and monads in a 2-category are functors, and form in a natural way the objects not of 2-categories but of double categories, and that 4-dimensional cubes in B are needed to construct the functor 2-category $\underset{\sim}{\mathrm{Fun}}(A,B)$ of Gray, for 2-categories A and B. $\underset{\sim}{\mathrm{Fun}}(A,B)$ and the conjugate 2-category are also embedded into a double functor category.

We generalize 2-categories rather than the bicategories of Bénabou since this avoids very complicated coherence diagrams, and n-categories suffice for our applications. All functors will be strict in the sense of Bénabou, more general functors would inevitably lead to profunctors. The "morphisms only" approach is forced on us, otherwise an n-tuple category would have 2^n distinct kinds of elements.

Thus we define an n-tuple category $\mathbb{C}$ as a class, also denoted by $\mathbb{C}$, of things called morphisms, with n-category actions each of which is functorial with respect to the others. We denote by $d_i^0, d_i^1, *_i$ the domain, codomain and composition of action no. i; with laws like

$$d_j^\alpha(g *_i f) = (d_j^\alpha g) *_i (d_j^\alpha f),$$

for $i \neq j$, and middle four interchange laws.

An n-tuple category $\mathbb{C}$, with actions indexed by natural numbers $i < n$, is called an n-category if always

$$d_j^\beta d_i^\alpha = d_i^\alpha = d_i^\alpha d_j^\beta,$$

for $i < j < n$ and α, β in $\{0, 1\}$.

For an interval (or "box") B in $\mathbb{Z}^n$, with $a_i < x_i < b_i$, denote by $d_i^\alpha B$ the interval of all $x \in B$ with $x_i = a_i$ or $x_i = b_i$ respectively. With these domain and codomain operations, and with set unions of intervals as compositions, intervals in $\mathbb{Z}^n$ form an n-tuple category. We denote by [B] the subcategory of all intervals in $\mathbb{Z}^n$ which are contained in a given "box" B.

Let K be the unit cube in $\mathbb{Z}^n$, and let $\mathbb{C}$ be an n-tuple category. For $f \in \mathbb{C}$, there is a unique functor $\hat{f} : [K] \to \mathbb{C}$ with $\hat{f}(K) = f$, we call $\hat{f}$ the name of f. If $p_i^\alpha : [K] \to [K]$ is the name of the face $d_i^\alpha K$ of K,

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then we have in general

$$(d_i^\alpha f)^\wedge = \hat{f} \circ p_i^\alpha,$$

with composition at right. Let $K *_i K$ be the box with $0 \leq x_i \leq 2$, and $0 \leq x_j \leq 1$ otherwise. The "shift" s_i^h with $x_i \longmapsto h + x_i$ defines functors s_i^0 and s_i^1, and $\gamma_i(K) = K *_i K$ defines a functor γ_i, all from $[K]$ to $[K *_i K]$. In the diagram

$$\begin{tikzcd}
 & {[K]} \arrow[dl, "p_i^1"'] \arrow[dr, "s_i^0"] \arrow[drrr, "\hat{f}"] & & & \\
{[K]} \arrow[dr, "p_i^0"'] & & {[K *_i K]} \arrow[dl, "s_i^1"] \arrow[rr, "\langle g,f \rangle"] & & \mathbb{C} \\
 & {[K]} \arrow[urrr, "\hat{g}"'] & & &
\end{tikzcd}$$

the square at left is a pushout, thus $\langle g,f \rangle$ is defined iff $g *_i f$ is. If this is the case, then

$$(g *_i f)^\wedge = \langle g,f \rangle \circ \gamma_i.$$

The laws of an n-tuple category can be treated in similar fashion, this defines the <u>basic theory of</u> n-<u>tuple categories</u>.

If $\mathbb{A}, \mathbb{B}$ are n-tuple categories, the morphisms of the n-<u>tuple functor category</u> $[\mathbb{A}, \mathbb{B}]$ are functors $F : [K] \times \mathbb{A} \to \mathbb{B}$. The functor $- \times \mathbb{A}$ preserves the pushouts of the basic theory, thus

$$d_i^\alpha F = F \circ (p_i^\alpha \times \mathbb{A})$$

and $\quad G *_i F = \langle G,F \rangle \circ (\gamma_i \times \mathbb{A}),$

with $\langle G,F \rangle$ obtained from a pushout diagram if $d_i^1 F = d_i^0 G$, defines the operations of an n-tuple category. Composition of functors of n-tuple categories can be extended to functors $[\mathbb{B},\mathbb{C}] \times [\mathbb{A},\mathbb{B}] \longrightarrow [\mathbb{A},\mathbb{C}]$; thus (small) n-tuple categories and their functors are objects and arrows of a cartesian closed (n+1)-tuple category.

The embedding functor from n-categories and their functors to n-tuple categories has a right adjoint C, with $C\mathbb{C}$ the subcategory of all $f \in \mathbb{C}$ such that $d_i^\alpha f = d_j^\beta d_i^\alpha f$ for $i < j$. With $[\mathbb{A},\mathbb{B}]$ replaced by $C[\mathbb{A},\mathbb{B}]$, (small) n-categories and their functors are the objects and arrows of a cartesian closed (n+1)-category.

From a 2-category $\mathbb{A}$, a double category $\Gamma\mathbb{A}$ of squares in $\mathbb{A}$ can

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be constructed in the obvious and easy way, we omit details. This can be generalized to n-dimensional cubes in an n-category in the obvious, but by no means easy way. The construction involves lengthy combinatorial considerations, we omit all details.

Adjunctions in a 2-category $\mathbb{C}$ are considered as objects of a double functor category $[\mathcal{A}, \ulcorner\mathbb{C}]$, where $\mathcal{A}$ has two objects a, b, two arrows f and g, one of each kind, and two squares η, ε. The following diagrams show domains, codomains and non-trivial compositions.

$$b \xrightarrow{f} a \;, \qquad \begin{array}{ccc} & a & \\ u & \boxed{\varepsilon} & a \\ & f & \end{array} \;, \qquad \boxed{\eta}\boxed{\varepsilon} = f \;,$$

$$\begin{array}{c} a \\ \downarrow u \\ b \end{array} \;, \qquad \begin{array}{ccc} & f & \\ b & \boxed{\eta} & u \\ & b & \end{array} \;, \qquad \begin{array}{c} \boxed{\varepsilon} \\ \boxed{\eta} \end{array} = u \;.$$

With composition of adjunctions as third composition, adjunctions in $\mathbb{C}$ are the objects of a triple category Adj $\mathbb{C}$ of adjoint cubes in $\mathbb{C}$.

We regard finite ordinals n = {0,...,n-1} and their order-preserving mappings as arrows and cells of a 2-category Δ with a single object, with composition of mappings as "weak" composition and ordinal additions as "strong" composition. Monads in a 2-category $\mathbb{C}$ are then functors from Δ to $\mathbb{C}$, and onjects of a full double subcategory Mon $\mathbb{C}$ of the double functor category $[\ulcorner\Delta, \ulcorner\mathbb{C}]$. Adj$\mathbb{C}$ acts from the left on Mon$\mathbb{C}$. The Eilenberg-Moore and Kleisli constructions can also be defined in this context and become functors of 2-categories, right inverse strongly right and left adjoint respectively to "forgetful" functors.

Quasi-Kan extensions for 2-categories

J. Gray

Lecture notes Springer New York

Necessary and sufficient conditions for exactness of Kan extensions.

F. Ulmer

Inventiones Mathematicae

R.-E. Hoffmann
H. Lindner
Th. Thode

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