MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Algebraische K-Theorie

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Vortragsauszüge

J.R.STROOKER: Die Fundamentalgruppe der GL_2 für einen Körper

In einer gemeinsamen Arbeit mit Herrn Villamayor haben wir zu einem links-exakten Funktor F von Ringen nach Gruppen eine Fundamentalgruppe $\pi_1 F$ eingeführt, und wir zeigten, daß für die allgemeine lineare Gruppe GL diese $\pi_1 GL$ gerade der Milnorsche K_2, also der Schursche Multiplikator $H_2(E-,\underline{Z})$ der elementaren Gruppe E ist.

Hier wird der Fall $F = GL_2$ diskutiert, und zwar modifiziert für Gruppenschemata über einen Körper k.

Satz: Für $k \neq \underline{F}_2, \underline{F}_4, \underline{F}_9$

gilt $\pi_1 GL(2,k) \simeq H_2(E(2,k),\underline{Z})$

W.L.J. VAN DER KALLEN, H.MAAZEN, J.STIENSTRA:
The $\langle\ ,\ \rangle$-presentation for K_2

Let R be a commutative ring with unit. By $D(R)$ we denote the abelian group with generators $\langle a,b\rangle$, where $a,b \in R$ are such that $1 + ab \in R^*$, subject to the relations

(D_1) $\quad \langle a,b\rangle \langle -b,-a\rangle = 1$

(D_2) $\quad \langle a,b\rangle \langle a,c\rangle = \langle a,b+c+abc\rangle$

(D_3) $\quad \langle a,bc\rangle = \langle ab,c\rangle \langle ac,b\rangle$

There always is a homomorphism $D(R) \to K_2(n,R)$ for $n \geqslant 3$. This is even an isomorphism for semilocal rings, whose residue classfields have $\geqslant 16$ elements, for local rings whose residue classfield is a primefield and some other cases.

Similarly, if I is a radical ideal of R, such that $R \to R/I$ splits, then the kernal of $K_2(n,R) \to K_2(n,R/I)$ is, for $n \geqslant 3$, isomorphic to the abelian group with generators $\langle a,b\rangle$ where a or b is in I, subject to the relations (D_1), (D_2) and (D_3).

M.KAROUBI: The exact-sequence of a localisation in hermitian K-theory

Let A be a ring with involution with $1/2 \in A$. We define $L_n(A) = \pi_n(B^+_{O(A)})$ where $O(A/$is the infinite orthogonal group. Then one proves an exact sequence of the type

$$L_{n+1}(A) \to L_{n+1}(A_S) \to U_n(A,S) \to L_n(A) \to L_n(A_S) \qquad n \in \mathbb{Z},\ A_S = S^{-1}A$$

where S is a multiplicative set of non zero divisors in A. The "relative group" $U_n(A,S)$ is π_n of a certain category built out of the category of S-torsion modules provided by an hermitian form with values in A_S/A.
This theorem has many applications, some of which are known:

1.) If $\hat{A} = \lim A/_{SA}$ and $\hat{A}_S = S^{-1}A_S$ one has a Mayer-Vietoris exact sequence

$$\ldots \to L_{n+1}(F) \oplus L_{n+1}(\hat{A}) \to L_{n+1}(\hat{F}) \to L_n(A) \to L_n(F) \oplus L_n(\hat{A}) \to L_{n+1}(\hat{F}) \to \ldots \quad n \in \mathbb{Z}.$$

2.) If A is a Dedekind ring one has an exact sequence

$$0 \to SK_1(A) \to L_1(A) \to \mathbb{Z}/_2 \times \left(A^*/_{(A^*)^2}\right) \times \left(2 \text{ torsion } cl(A)\right) \to 0 \qquad \text{(Bass)}$$

3.) For any field k, put $W''(k) = \mathrm{Ker}(W/k) \xrightarrow{(\mathrm{rank},\mathrm{disc.})} \mathbb{Z}/_2 \times k^*/_{(k^*)^2}$

Then if A is a Dedekind ring such that $W''(A/p)$ finite for any maximal ideal p, one has $Sp(A)/_{[Sp(A),Sp(A)]} \simeq S\,K_1(A)$.

4.) If A is any Dedekind ring, F the field of fractions the homomorphism $L_2(A) \to L_2(F)$ is injective.

5.) Assume A and F as above. Then one has an exact sequence

$$L_{n+1}(A) \to L_{n+1}(F) \to \bigoplus_p U_n(A/p) \to L_n(A) \to L_n(F).$$

If A/p is finite of cardinality q, one has $U_{8k+1}(A/p) \approx \mathbb{Z}/_{(q^{4k+1}-1)\mathbb{Z}}$,

$U_{8k+2}(A/p) = U_{8k+3}(A/p) = U_{8k+4}(A/p) = 0$, $U_{8k+5}(A/p) = \mathbb{Z}/_{(q^{4k+3}-1)\mathbb{Z}}$,

$U_{8k+6}(A/p) = \mathbb{Z}/_2$, $\# U_{8k+7}(A/p) = 4$, $U_{8k} = \mathbb{Z}/_2$. (Friedlander-Quillen)

6.) Assume A and F as in 5.); then one has an exact sequence

$$0 \to W(A) \to W(F) \xrightarrow{\alpha} \bigoplus_p W(A/p)$$

where coker α is in the exact sequence.

$$0 \to Sp(A)/_{[p(A),Sp(A)]\cdot GL(A)} \to \mathrm{coker}\ \alpha \to cl(A)/_{cl(A)^2} \to 0.$$

Inparticular, if $W''(A/p)$ is finite, one has

$$0 \to S\,K_1(A)/_{2\,S\,K_1(A)} \to \mathrm{coker}\ \alpha \to cl(A)/_{cl(A)^2} \to 0$$

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A.RANICKI: Geometric L-theory

In §17G of "Surgery on compact manifolds" Wall suggests a reformulation of surgery obstruction theory in terms of quadratic forms on chain complexes. Misčenko (Izo. Akad. Nauk SSSR, 1971) carried out this programme for bilinear forms on chain complexes, describing the bilinear part of the surgery groups. It is possible to obtain the quadratic structure in this way as well. Given a ring with involution A let $L_n(A)$ be the bordism group of suitably defined n-dimensional "algebraic Poincaré complexes" over A. Then $L_n(\mathbb{Z}[\pi])$ is just the Wall surgery group $L_n(\pi)$ of a group π. Algebraic surgery shows that

$$L_n(A) \cong L_{n+4}(A).$$

However, the homogenous appearance of n as a dimension rather than as a residue mod 4 allows the definition of algebraic analogues of familiar geometric techniques (such as glueing manifolds together), justifying the title of the talk. Algebraic glueing can be used to establish a Mayer-Vietoris sequence in L-theory

$$\ldots \to L_n(A) \xrightarrow{\binom{\beta}{\gamma}} L_n(B) \oplus L_n(C) \xrightarrow{(\beta' - \gamma')} L_n(A') \xrightarrow{\partial} L_{n-1}(A) \to \ldots$$

for a commutative square of rings with involution

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ \gamma\downarrow & & \downarrow\beta' \\ C & \xrightarrow[\gamma']{} & A' \end{array}$$

which is cartesian and such that

either: β' (& hence γ) is onto (excision)

or: $B = A_S$, $C = \hat{A}$, $A' = \hat{A}_S$ for some multiplicative subset $S \subseteq A$ of non-zero-divisors (Localization & completion).

M.R.STEIN: Estimates for the order of $K_2(\mathbb{Z}G)$

Let G be a finite abelian group and let a prime $p \mid |G|$. Write $G = H \times \pi$, π cyclic of order p^n, $n \geq 1$ with generator σ. The K-theory exact sequence

$$K_2(\mathbb{Z}G) \to K_2(\mathbb{Z}G/p\mathbb{Z}G) \to SK_1(\mathbb{Z}G, p\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \to 1$$

may be used to estimate the p-part of the order of $K_2(\mathbb{Z}G)$. The crucial estimate is that of $K_2(\mathbb{Z}G/p\mathbb{Z}G)$ which uses theorems of Bloch (p odd) and van der Kallen ($p = 2$).

It is easily seen that $\mathbb{Z}G/p\mathbb{Z}G \approx (F_p[H])(T)/(T^{p^n})$. Supposing for simplicity that H is also a p-group and p is odd. the order of the p-part of $K_2(\mathbb{Z}G/p\mathbb{Z}G)$ may be computed from the exact sequences

$$1 \to \Phi_i \to K_2(R[T]/(T^i)) \to K_2(R[T]/(T^{i-1})) \to 1$$

with $R = \mathbb{F}_p[H]$.

Since ord $p(\Phi_i)$ is known by Bloch's work when $i \not\equiv 1 \mod p$. The answer depends on ord $p(\Omega'_R)$ and the number of elements of H of order dividing p^ℓ, $\ell \geq 1$. If G itself is an elementary abelian p-group of rank m, the estimate obtained is

$$\text{ord } p(K_2(\mathbb{Z}G)) \geq (m-1)(p^m-1) - \binom{p+m-1}{p}$$

using the results of Alpein-Dennis-Stein on $|SK_1(\mathbb{Z}G)|$. In particular $WL_2(G) \neq 1$ if $m \geq 2$.

D.QUILLEN: Finite generation of K-groups for rings of S-integers

Bass has conjectured that the groups $K_i A$ are finitely generated if A is a regular (commutative) ring finitely generated over $\mathbb{Z}$.

Theorem: Bass's conjecture is true if Krull dim $A \leq 1$.

For the proof one must consider three cases:

i.) A = finite field (here the K-groups are finite except for K_0,

ii.) A = ring of integers in a number field,

iii.) A = coordinate ring of an complete non-singular curve minus one point defined over $\mathbb{F}_q$.

The proof of ii.) appears in the Seattle Proceedings on Alg. K-theory. The same method is used to reduce iii.) to the following

Theorem: A as in iii.), let P be a finitely gen. projective A-module, let $I(F \otimes_A P)$ be the Steinberg module of the vector space $F \otimes_A P$ (F=quotient field of A). Then the group $H_i(\mathrm{Aut}(P), I(F \otimes_A P))$ is finite for $i > 0$ and finitely generated for $i = 0$.

On Homology of General Linear Groups.

Theorem: Let A be any ring. Then for $0 < r \leq \infty$

$$\varinjlim_n H_*\left(\begin{pmatrix} I & * \\ 0 & GL_n(A) \end{pmatrix}, \mathbb{Z}\right) \cong \varinjlim_n H_*(GL_n(A), \mathbb{Z})$$

Application: $H_i(GL(\mathbb{F}_p), \mathbb{Z}) = 0!$ for $i > 0$

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J.L.LODAY: Multiplicative Structure in Algebraic K-theory

Let A and A' be rings with unit. The tensor product $\otimes$: $GL_n(A) \times GL_m(A') \to GL_{nm}(A\otimes A')$ can be extended (with some care) to GL(A) and then modified to give a continuous map $BGL(A)^+ \wedge BGL(A^2)^+ \longrightarrow BGL(A\otimes A')^+$. ($BGL(A)^+$ = Quillens space, $\pi_n(BGL(A)^+) = K_nA$). We define so $K_nA \times K_p(A') \xrightarrow{\star} K_{n+p}(A\otimes A')$. The application has all the properies we expect for a product: naturality, bilinearity, associativity and (graded-)comm. if A = A' = commut. ring. Moreover

- $\star$ coincides with Milnor's product in case n = 1, p = 1 (cf. Introd. to Alg. K.th.)
- if {t} is the class of $t \in GL_1(\mathbb{Z}[t,t^{-1}])$ in $K_1(\mathbb{Z}[t,t^{-1}])$ the product by {t} identifies $K_n(A)$ with a direct summand of $K_{n+1}(A[t,t^{-1}])$.

(this result was conjectured by Gersten and necessary for Karoubi's theorem on periodicity in Hermitian K-theory). In the case n = 2, p = 1, we construct an explicit homomorphism $H_2(E(A);\mathbb{Z}) \times H_1(GL(A);\mathbb{Z}) \to H_3(St(A);\mathbb{Z})$ which coincides with the product $\star$ after the identifications with $K_i(A)$ (i = 1,2,3).

Example: Let $\alpha,\beta,\gamma \in GL(A)$ define $\{\alpha\}, \{\beta\}, \{\gamma\} \in K_1A$, A comm.

Put $D_\alpha = (\alpha\otimes 1\otimes 1) \oplus (\alpha^{-1}\otimes 1\otimes 1) \oplus (1\otimes 1\otimes 1) \oplus (1\otimes 1\otimes 1)$

$D'_\beta = (1\otimes \beta\otimes 1) \oplus (1\otimes 1\otimes 1) \oplus (1\otimes \beta^{-1}\otimes 1) \oplus (1\otimes 1\otimes 1)$

$D''_\gamma = (1\otimes 1\otimes \gamma) \oplus (1\otimes 1\otimes 1) \oplus (1\otimes 1\otimes 1) \oplus (1\otimes 1\otimes \gamma^{-1})$

We have $D_\alpha, D'_\beta, D''_\gamma \in E(A)$. We lift them in St(A), say $\tilde{D}_\alpha, \tilde{D}'_\beta, \tilde{D}''_\gamma$.

The elements $\tilde{\tilde{D}}_\alpha = \tilde{D}_\alpha \oplus 1 \oplus \tilde{D}_\alpha \oplus \tilde{D}_\alpha^{-1}$

$\tilde{\tilde{D}}'_\beta = \tilde{D}'_\beta \oplus \tilde{D}'^{-1}_\beta \oplus 1 \oplus \tilde{D}'_\beta$

$\tilde{\tilde{D}}''_\gamma = \tilde{D}''_\gamma \oplus \tilde{D}''_\gamma \oplus \tilde{D}''^{-1}_\gamma \oplus 1$

commute in St(A) and define thus a homomorphism of groups $\mathbb{Z}^3 \xrightarrow{\mu} St(A)$. The product $\{\alpha\}\star\{\beta\}\star\{\gamma\} \in K_3A \approx H_3(St\,A;\mathbb{Z})$ is the image of the foudam class of the torus in $H_3(\mathbb{Z}^3;\mathbb{Z}) = H_3(S^1\times S^1\times S^1;\mathbb{Z})$ by $\mu_*: H_3(\mathbb{Z}^3;\mathbb{Z}) \to H_3(St A;\mathbb{Z})$.

H.BEHR: (Further) Variations on Milnor's computation of $K_2 \mathbb{Z}$

Theorem: Let G be a Chevalleygroup, simply connected and of simple type with root system Φ; denote by $St(\Phi, \mathbb{Z})$ the Steinberg-group with respect to Φ. One has the following exact sequence.

$$1 \longrightarrow L(\Phi, \mathbb{Z}) \longrightarrow St(\Phi, \mathbb{Z}) \longrightarrow G(\mathbb{Z}) \longrightarrow 1$$

and
$$L(\Phi, \mathbb{Z}) = \begin{cases} \mathbb{Z}, \text{ if } \Phi \text{ is of type } A_1 \text{ or } C_e \\ \mathbb{Z}/2\mathbb{Z} \text{ otherwise.} \end{cases}$$

A sketch of a proof for this theorem was given, which consists of two parts: the first one is an elementary proof for the rank 2-groups, the second one a reduction process to the rank 2-case.

1) The case Φ of type A_2 was done by Milnor (and classically by Nielsen) in Milnor's Introduction to algebraic K-theory.
For the symplectic group $Sp_4(\mathbb{Z})$, one uses the operation of this group on $\mathbb{Z}^4$ and has to define a partial ordering on $\mathbb{Z}^4$, which reflects the special properties of Sp_4, and which is more conveniant than the usual norm, which was used in SL_3.
The problem for $G_2(\mathbb{Z})$ has been settled by Hurrelbrink and Rehmann, who use the same basic idea.

2) The reduction to rank 2-groups can be described in the algebraic case as follows: G (k) (k a field) is the amalgamated product of its rank 2-parabolic standard subgroups P_2 (due to Tits). Associate with this product a connected and simply connected simplicial complex K and let G (A) (A a ring, $A \subsetneq k$) operate on K. If A is a principal ideal domain, there exists a simple"fundamental complex" for the operation of G (A) and one gets: G (A) is the amalgamated product of the groups P_2 (A). From this one deduces easily a finite presentation for the groups G (A), which determines also $L(\Phi, \mathbb{Z})$.

R.K.DENNIS: K_2 of Local Rings

Let R be a commutative local ring. It has been conjectured for some time that the groups $K_2(n,R)$, $n \geq 3$, and K_2R are all isomorphic. Dennis and Stein have now shown this to be the case. This theorem is proved in a manner analogous to that in which the corresponding theorem for fields was proved by Matsumoto. A number of technical difficulties arise; their solution involves the proof of a theorem of independent interest: Namely, a simpler presentation for the Steinberg group of an arbitrary ring is given. As in the case of Matsumoto's Theorem, a set of defining relations for K_2R are simultaneously derived (R a commutative local ring).

J.B.WAGONER: Continuous Algebraic K-Theory for Local Rings and Fields

Let E be a local field (complete with finite residue field) with $\mathcal{O}$ its ring of integers and $\mathfrak{p}$ its maximal ideal. Then E and $\mathcal{O}$ are locally compact, totally disconnected and one would like to determine that part of K_iE (and $K_i\mathcal{O}$) coming from continuous invariants of the p-adic group $SL(\ell+1,E)$. Using the affine BN pair structure of $SL(\ell+1,E)$ we define topological groups $K_i^{top}(E)$ and $K_i^{top}(E)$ and $K_i^{top}(\mathcal{O})$ so that there is a natural commutative diagram

$$\begin{array}{ccc} K_i(\mathcal{O}) & \rightarrow & K_i(E) \\ \downarrow & & \downarrow \\ K_i^{top} & \rightarrow & K_i^{top}(E) \end{array}$$

Theorem A: $K_2^{top}(E) = \mu(E)$ and $K_2^{top}(\mathcal{O}) = \mu(E)_p$ where $\mu(E)$ is the group of roots of unity in E and $\mu(E)_p$ is p-primary part ($p = \text{char}\ \mathcal{O}/\mathfrak{p}$).

The construction is by the nerves of coverings of $SL(\ell+1,E)$ by certain families of open subgroups.

Theorem B: $K_i^{top}(\mathcal{O}) = \varprojlim_n K_i^{top}(\mathcal{O}/\mathfrak{p}^n)$.

Whatever the correct definition is Quillen has conjectured that it should satisfy

Conj. (Quillen) Let char E = 0, $[E,\mathbb{Q}_p] = d$.

(a) $K_i^{top}(E)/\text{Torsion} = \begin{cases} 0, & i = 2n \\ Z_p^d, & i = 2n-1 \end{cases}$

(b) Torsion $K_i^{top}(E) = Z/w_i(E)\cdot Z$, $i = 2n, 2n-1$

where $w_i(E)$ is the largest m such that $\text{Gal}(E(\mu_m)/E)$ has exponent dividing i.

The motivation for (a) is the Lazard-Wegner-Casselman Theorem that $H_c(SL(E);\mathbb{Q}p)$ = Exterior algebra over $\mathbb{Q}p$ with d generators in each dimension 1,3,5,7,...: (b) is the analogue of the Lichtenbaum conjectures.

S.MAUMARY: Categorical L-theory

The L-theory as defined by Karoubi is related to K-theory by the forgetful map $F_{\pm}: BO_{\pm}(A)^+ \longrightarrow BGL(A)^+$ and the hyperbolic map $H_{\pm}: BGl(A)^+ \longrightarrow BO_{\pm}(A)^+$. The periodicity "theorem" says that the homotopy fibre $V_{\pm}$ of $F_{\pm}$ is the loop space on the homotopy fibre $U_{\mp}$ of $H_{\mp}$, up to π_0. In general, this theorem is unsettled, but it is known by Karoubi in special cases.

Theorem: One has a categroy $Q_{\pm}(A)$ such that

i.) $\Omega(Q_{\pm}A)$ has the homotopy type of $L_0(A)_{\mp} \times BO_{\mp}(A)^+$

ii.) there is a forgetful functor $Q_{\pm}(A) \to Q(A)$ (= Quillen category of projective modules) which deloops the above map F.

As a result, $V_{\mp} = \Omega W_{\pm}$, when $W_{\pm}$ is the homotopy fibre of the lifting $\hat{Q}_{\pm}(A) \to \hat{Q}(A)$ to universal coverings. There is a canonical map $U_{\pm} \to W_{\pm}$ which should be a homotopy equivalence, at least up to a covering.

H.BASS: Russian Progress on Serre's Problem

Let $A = k[t_1,\dots,t_n]$, k a field, and let P be a projective A-module of rank r. Serre's problem asks if P is free.. Recent progress has been made by M. Roitman, M.P. Murthy & J. Towber, and R.G. Swan, but the most far reaching results are due to A. Suslin (Leningrad) and L. Vaseršteín (Moscow). They prove P is free in the following cases:

a.) $r > 1 + \frac{n}{2}$ or $1 + \frac{n-1}{2}$ if k is finite; b.) $n \leqslant 3$; c.) $n = 4$ and $\mathrm{char}(k) \neq 2$; d.) $n = 5$, $\mathrm{char}(k) \neq 2$, and k finite. The main ingredients of the proofs are as follows. Let A be a commutative ring and denote by $Un_r(A)$ the set of unimodular elements in A^r. Let SR(A) denote the least r such that, given $a = (a_1,\dots,a_{r+1}) \in Un_{r+1}(A), \exists\ a_i' = a_i + b_i a_{r+1}$ $(1 \leqslant i \leqslant r)$ s.t. $(a_1',\dots,a_r') \in Un_r(A)$.

Then stably free A-modules of rank $\geqslant SR(A)$ are free.

Theorem 1: a.) If A is noetherian of dim d, $SR(A) \leqslant d + 1$.

b.) (Vaseršteín) If A is affine over a finite field $SR(A) \leqslant \max(z,d)$.

Theorem 2: (Suslin) Let B be comm noeth of dim d. Put $A_n = B[t_1,\dots,t_n]$. For $n \geqslant 1$, $E_{r+1}(A_n)$ acts transitively on $Un_{r+1}(A_n)$ for

$$r \geqslant 1 + \max\left(d, \,^{SR(A_{n-1})}/2\right)$$

Theorem 3: (Vaseršteín) For any comm. ring A, $\exists$ a natural map

$$\varphi: SL_3(A)\backslash Un_3(A) \longrightarrow W(A) = \mathrm{Ker}(KSp_0(A) \rightarrow K_0(A)).$$

If $E_{r+1}(A)$ acts transitively on $Un_{r+1}(A)$ for $r \geqslant 3$, φ is bijective

Theorem 4: (Karoubi) If $\frac{1}{2} \in A$ then $W(A) \xrightarrow{\approx} W(A[t])$.

Proofs of these results, with their apllications to Serre's problem, can be found in the Seminaire Bourbaki of June, 1974.

A.BAK: Strong approximation and Mayer-Vietoris sequences in algebraic K-theory

Recall the classical strong approximation theorem for the special linear group SL_n.

Theorem: Let R be a Dedekind ring, F the field of fractions of R, $\hat{R}$ the finite adèle ring of R, and $\hat{F}$ the finite adèle ring of F. Then $SL_n(K)$ is dense in $SL_n(\hat{K})$, $n = 1,\dots,\infty$.

We used this result to motivate the following result. Call a fibred square of topological rings

$$\begin{array}{ccc} A & \overset{\beta}{\hookrightarrow} & B \\ \alpha\downarrow & & \downarrow g \\ \hat{A} & \underset{f}{\hookrightarrow} & \hat{B} \end{array}$$

an approximation square of rings if (i) the image α and g are dense; (ii) β and f are open; (iii) the topology on A is determined by the inverse image of the topology on $\hat{A}$; (iv) A and $\hat{A}$ have bases I(A) and I($\hat{A}$) of neighborhoods of zero consisting of 2-sided ideals such that if $\mathfrak{g} \in I(A)$ and $\mathfrak{g}' \in I(\hat{A})$ then there is a $t \in \mathfrak{g} \cap$ center A such that $t^i\mathfrak{g}$ and $t^i\mathfrak{g}'$ are open for all $i > 0$. We topologize St(A) by letting $S(A) = \{\ker(St(A) \to St(A/\mathfrak{g}))/\mathfrak{g} \in I(A)\}$ be a basis of neighborhoods of 1. Topologize St($\hat{A}$) similarly by S($\hat{A}$). Topologize St(B) and St($\hat{B}$) by letting the images of S(A) and S($\hat{A}$) be neighborhoods of 1. It is clear that St(A) and St($\hat{A}$) are topological groups, but one has to prove that St(B) and St($\hat{B}$) are topological groups.

Theorem (Strong approximation for St) If

$$\begin{array}{ccc} St(A) & \to & St(B) \\ \downarrow & & \downarrow \\ St(\hat{A}) & \to & St(\hat{B}) \end{array}$$

is the square of Steinberg groups associated to an approximation square of rings then St(B) is dense in St($\hat{B}$), St($\hat{B}$) = "St($\hat{A}$)" "St(B)", and St(A) maps onto the fibred product (pullback) of St($\hat{A}$) and St(B) over St($\hat{B}$).

Corollary for all $i \leq 2$ there is an exact Mayer-Vietoris sequence

$$\dots \xrightarrow{\partial} K_i(A) \longrightarrow K_i(\hat{A}) \oplus K_i(B) \longrightarrow K_i(\hat{B}) \xrightarrow{\partial} \dots$$

We have also a desrete anology. Here, no conditions are put on β and t, α and g are assumed surjective, and we assume β(ker(St(A) → St($\hat{A}$))is normal in St(B). The analogy of the theorem and corollary are true. There are also anologous theorems for quadratic modules.

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M.KNEBUSCH: Real closures of commutative rings

We consider pairs (A,σ) consisting of a connected commutative ring A with 1 and a "signature" σ of A, i.e. a ring-homomorphism from the Wittring W(A) of symmetic inner product spaces to $\mathbb{Z}$. If A is a field then the signatures correspond uniquely to the orderings of A (Harrison, Leicht-Lorenz). Thus for fields our theory will be identical with Artin-Schreier's Theory of real closures.

There is an evident notation of morphism $\varphi: (A,\sigma) \to (B,\tau)$. φ is called a covering of (A,σ), if $\varphi: A \hookrightarrow B$ is a covering in the sense of galois theory. A pair (R,g) is real closed, if (R,g) does not admit coverings expect isomorphisms. Let covering $(A,\sigma) \to (R,g)$ of (A,σ) by a real closed pair is called a real closure of (A,σ). By Zorn's Lemma every (A,σ) has a least one real closure (R,g).

Theorem 1: Any two real closures of (A,σ) are isomorphic over (A,σ).

Theorem 2: ("fundamental theorem of algebra"). If (R,g) is a real closure of (A,σ), then the degree $[\tilde{A}:R]$ of the universal covering $\tilde{A}$ of A over R is ≤ 2. If there exists a prime number p which is a unit in A, then $[\tilde{A}:R] = 2$. If 2 is a unit then $\tilde{A} = R[\sqrt{-1}]$.

Theorem 3: If A is semi-local, then g is the unique signature of R, and $W(R) = \mathbb{Z} \oplus \mathrm{Nil}W(R)$. The Wittring of the hermitian inner product spaces over $(\tilde{A},J)$ with J the involution of $\tilde{A}/R$ coincices with $\mathbb{Z}$. If 2 is a unit, then $W(R) = \mathbb{Z}$, but otherwise this must not hold true.

Remark: If A is the affine ring of non singular real affine curre, then also $W(A) = \mathbb{Z}$.

Theorem 4: If A is semi-local, then $\sigma \mapsto J$ (see Th. 3) gives a 1-1-correspondence between the signatures of A and the conjugacy classes of elements of order 2 of the Galois group G. (Probably G contains no other elements of finite order.)

J.CLAVENS: Odd quadratic forms

Let A be a ring with anti-involution α: $I = \{x+\alpha(x) \mid x \in A\}$ Q(A) is the set $A \times A/I$ together with

$$(x,[\zeta]) + (y,[\eta]) = (x+y,[\zeta+\eta+\alpha(x)y]) \quad i(\zeta) = (0,[\zeta])$$

$$\Pi(x,[\zeta]) = \zeta + \alpha(\zeta) - \alpha(x)x \qquad (x,[\zeta])^a = (xa,[\alpha(a)\zeta a])$$

Then we can consider the Grothendieck group MA of pairs λ, μ: λ nonsingular α-symmetric sesquilinear form on a module P

$$\mu: P \to QA \text{ such that} \qquad \mu(xa) = \mu(x)^a$$
$$\mu(x+y) = \mu x + \mu y + i\lambda(x,y) \quad \pi\mu(x) = \lambda(x,x).$$

For example there is an exact sequence $0 \to L_oA \to MA \to QA$.

W.SCHARLAU: On Subspaces of inner product spaces

Let K be a field of char $\neq 2$, and I a finite partially ordered set with involution $\perp: i \mapsto i^{\perp}$. An I space is a tupel $(V, b, V_i \in I)$ where b is a non-singular form on V and the V_i are subspaces such that $i \leq j$ implies $V_i \subset V_j$ and $V_{i^{\perp}} = V_i^{\perp}$. Some examples and results to the following questions were discussed:

1.) For which $(I, ^{\perp})$ do there exist only finitely many indecomposable I-spaces (up to isomorphisms and multiplication of b by scalars).

2.) How can one classify I-spaces, in particular for which I?

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