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Die Tagung fand unter der Leitung der Herren:

J. GRAY (Urbana) und H. SCHUBERT (Düsseldorf)

statt.

TEILNEHMER:

Anghel, C.	BP 129 Kinshasa XI Rep. du Zaire, Faculsi de Math.-Mec., Bucarest, Rue Academ. 14, Roumanie
Acusista, Herman	Universited de Puerto Rico, Dept. de Matemàticas, Rio Piedras, Puerto Rico USA
Bainbridge, Stewart	Mathematics Dept. University of Ottawa, Ottawa, Ont., Canada
Blass, Andreas	Mathematics Dept., University of Michigan, Ann Arbor, Michigan 48104, USA
Börger, Reinhard	Mathematisches Institut der Universität Münster, 44 Münster, Roxeler Str. 69, Germany
Diers, Yves	U.E.R. de mathêmatiques pures et appliquées, Université des Sciences et Techniques de Lille, BP 36 59650 Villeneuve d'Ascq France
Duskin, J.	Department of Mathêmatics, University Suny, 4246 Ridge Lea Rd. Buffalo New York 14226 USA

Echivard-Michon, M.G.	Departement de Mathematiques, Universite de Dijon, 21000 Dijon, France
Ehrig, H.	Technische Universität Berlin, Fachbereich 20, 1 Berlin 10 Otto-Suhr-Allee 18-20, Germany
Eilenberg, S.	65 Roebuck House Stag Place London SW 1, England
Faehling, P.	Freie Universität Berlin, Mathematisches Institut II, 1 Berlin 33, Königin-Luise-Str. 24/26, Germany
Gray, John	Forschungsinstitut für Mathematik ETH Zürich, CH-8006 Zürich, Schweiz
Guitart, R.	Université de Paris VII, d'p. de math. Tom 45-55, 2 place Jussien, 75005 Paris, France
Harting, Roswitha	Mathematisches Institut der Universität 4 Düsseldorf 1, Universitätsstr. 1 Germany
Hoffmann, R.-E.	Mathematisches Institut der Universität 4 Düsseldorf 1, Universitätsstr.1 Germany
Hoff, G.	Depar ment de Mathemaiques, Centre Scientifique et Polytechnique Universite Paris-Nord, Place du 8 Mai 1945, 93206 Saint-Denis, France
Joyal, André	Université du Québec à Montreal, Case Postale 8888, Montreal, Canada
Kean, Orville	Department of Mathematics College of the Virgin Islands St. Thomas, V.I. 00801 USA
Kock, Anders	Matematisk Institut Aarhus Universitet, Ny Munkegade, DK 8000 Aarhus C, Dänemark
Kučera, L.	Charles University, 18600 Praque 8 Sokolovska' 83 Czechoslovakia
Lambeck, J.	Department of Mathematics McGill University, Montreal, Canada
Lavendhomme, René	Institut de Mathématique, 2, chemin du cyclotron, 1348 Louvain-la-Neuve Belgium
Lecouturier, Pierre	Departmentof Mathematics Suny 4246 Ridge Lea Road, Buffalo, N.Y. 14226, USA

Lindner, H.	Mathematisches Institut der Universität 4 Düsseldorf 1, Universitätsstr. 1 Germany
Linton, Fred, E.J.	Department of Mathematics Weseleyan University Middle Town, Conn 06457 USA
Mac Donald, John	Mathematics Department University of British Columbia, Vancouver, B.C. Canada
Masseron, Marcel	93 rue Jeanne d' Arc, 75013 Paris, France
Maurer, Ch.	II. Mathematisches Institut der FU Berlin, 1 Berlin 33, Königin-Luise-Str. 24/26, Germany
Mulvey, Christopher	Mathematics Division, University of Sussex, Falmer, Brighton, BNI 9QH England
Osius, G.	Fachsektion Mathematik, Universität Bremen, 28 Bremen, Achterstr. Germany
Pfender, M.	Fachbereich Mathematik TU Berlin, 1 Berlin 12, Str. des 17. Juni 135, Germany
Rattray, Basil	Department of Mathematics, McGill University, Montreal, Que., Canada
Reyes, Gonzalo	Université de Montreal, CP 6128, Department de Mathémati Montreal, Canada
Riguet, Jacques	UER Math. Logique Informatique Université René Descartes (Paris V) 12, rue Cujas 75005 Paris, France
Roberts, John E.	Centre de physique théorique C.N.R.S 31, chemin J. Aiguier, F-13 Marseille, Cedex 2, France
Rosicky, I.	Department of Mathematics UIEP, Ianackovo NAM. 2a, 66295 BRNO, Czechoslovakia
Street, Ross	School of Mathematics + Physiks Macquarie University, North Rydes, N.S.W. 2113, Australia

- 4 -

Swirszcz, T.	Institute of Mathematics, Polish Academy of Sciences, 00-656 Warsaw
Sydow, Walter	Mathematisches Institut der Universität Münster, 44 Münster, Roxelerstr. 64 Germany
Thode, Th.	Mathematisches Institut der Universität 4 Düsseldorf 1, Universitätsstr.1 Germany
Van de Wauw - De Kinder, G.	Department of Mathematics University Leuven, Celestynenlaan 61/12 B-3030 Heverlee, Belgium
Volger, Hugo	Mathematisches Institut der Universität 74 Tübingen, Auf der Morgenstelle 10 Germany
Wraith, G.C.	Department of Mathematics, University of Sussex, Falmer, Brighton, Sussex, England
Zöberlein, Volker	Mathematisches Institut der Universität 4 Düsseldorf 1, Universitätsstr. 1 Germany

Automata and Systems in a Hyperdoctrine

E.S. Bainbridge, University of Ottawa

The logic of the hyperdoctrine of set-valued functors provides a two-dimensional generalization of ordinary logic. This 2-logic provides an appropriate language for automata and system theory.

A transition function $\delta : Q \times X \to Q$ determines an action of the free monoid X^* generated by X, say $\delta^* : Q \times X^* \to Q$. A transition function equipped with a read-in function $\alpha : I \to Q$ and a read-out function $\beta : Q \to J$ constitutes a (Moore-type) automaton, and specifies a computation

$$I \times X^* \xrightarrow{\alpha \times X^*} Q \times X^* \xrightarrow{\delta^*} Q \xrightarrow{\beta} J$$

If T is the graph $\begin{tikzcd} u \arrow[r, "a"] & w \arrow[loop above, "x \in X"] \arrow[r, "b"] & v \end{tikzcd}$ and T^* is the free category generated by T, then an automaton is a functor $\Phi : T^* \to \mathrm{Sets}$ $\left(\begin{tikzcd} I \arrow[r, "\alpha"] & Q \arrow[loop above] \arrow[r, "\beta"] & J \end{tikzcd}\right)$. The computation is obtained as follows, where $\mathbf{1} \xrightarrow{u} T^* \xleftarrow{v} \mathbf{1}$, $u^* = - \circ u$, $v^* = - \circ v$, $\Sigma u \dashv u^*$, $v^* \dashv \Pi v$.

$$\frac{\Sigma u\ u^* \Phi \xrightarrow{\text{counit}} \Phi \xrightarrow{\text{unit}} \Pi v\ v^* \Phi}{v^*\ \Sigma u\ u^* \Phi \longrightarrow v^* \Phi}$$

Conversely, given $C : v^*\ \Sigma u\ I \to J$ we obtain $\Phi_C : T^* \to \mathrm{Sets}$ whose computation simulates C.

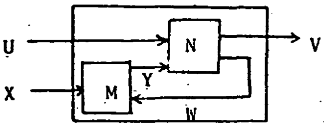
$$\frac{v^*\ \Sigma u\ I \xrightarrow{C} J}{\Sigma u\ I \to \Pi v\ J}$$

$$\frac{f}{\llcorner f} \qquad \begin{tikzcd} & \Sigma u\ I \arrow[r] \arrow[dl, dashed, "\Sigma u\ \llcorner f"'] \arrow[d, "f"] & \Pi v\ J & \\ \Sigma u\ u^* \Phi_C \arrow[r] & \Phi_C \arrow[ur, "g"] \arrow[r] & \Pi v\ v^* \Phi_C \arrow[u, dashed, "\Pi v\ g\lrcorner"'] & \end{tikzcd} \qquad \frac{g}{g\lrcorner}$$

Moreover, Φ_C is a sub-quotient of any other such Φ. With suitable other choices of $U \xrightarrow{u} W \xleftarrow{v} V$, $\Phi:W \to \mathrm{Sets}$, the above scheme gives the standard minimal realization theory for linear systems and algebra automata (including many-sorted theories arising from grammars), among others. Here we are using quantification along terms in the set-valued functor hyperdoctrine.

A transition function together with an <u>output function</u> $\lambda:Q\times X \to Y$ constitutes a <u>(Mealy-type) system</u> and specifies an <u>input-output behaviour</u> $\lambda^*: Q\times X^* \to Y^*$ defined below. The <u>state graph</u> G of $\delta:Q\times X \to Q$ is $Q \xleftarrow{\text{proj}} Q\times X \xrightarrow{\delta} Q$. The functions $Q\times X \xrightarrow{\text{proj}} X$, $Q\times X \xrightarrow{\lambda} Y$ extend to functors from the free category G^* generated by G; $X^* \xleftarrow{p_{\delta^*}} G^* \xrightarrow{\lambda^*} Y^*$. The connection with the hyperdoctrine structure is that p_{δ^*} is the discrete op-fibration assigned to $\delta^*:X^* \to \mathrm{Sets}$ by the comprehension schema. Moreover, Lawvere has observed that spans in <u>cat</u> with one projection a discrete op-fibration are the analogue in this hyperdoctrine of partial functions in the subobject hyperdoctrine of a topos. Such spans also model Thatcher's generalized[2] sequential machines, among other examples; so a <u>system</u> is a span $U \xleftarrow{p_\Phi} (1{\downarrow}\Phi) \xrightarrow{f} V$ for some $\Phi:U \to \mathrm{Sets}$.

To each system Φ,f assign its <u>characteristic profunctor</u> $M:V^{op}\times U \to \mathrm{Sets}$, $M(v,-) = \Sigma p_\Phi\, f^*V[v,-]$. Φ,f can be recovered up to isomorphism from M. The construction of M has system theoretic significance, as does the bifibration associated with M. Lawvere has observed that there is a classifier for such profunctors viewed as $U \to \mathrm{Sets}^{V^{op}}$, analogous to the partial function classifier of a topos. The characteristic profunctor of any interconnection of systems (parallel, cascade, feedback) is computed by a coend formula from the component profunctors, e.g. :

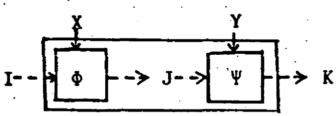


$M: Y^{op} \times W \times X \to Sets$, $N: W^{op} \times V^{op} \times U \times Y \to Sets$

$$\int^{yw} M(y,w,-) \times N(w,-,-,y) : V^{op} \times X \times U \to Sets$$

Moreover, the proof of this fact is system theoretic. An approach to the feedback control problem is suggested by the existence of a right adjoint to profunctor composition. Indeed, the logical formalism of the hyperdoctrine provides a deductive calculus for system homomorphism.

Finally, to a Moore automaton $\mathbb{1} \xrightarrow{u} T^* \xleftarrow{v} \mathbb{1}$, $\Phi: T^* \to Sets$, assign the cospan $I = (1 \downarrow u^*\Phi) \to (1 \downarrow \Phi) \leftarrow (1 \downarrow v^*\Phi) = J$ obtained by pulling back $p_\Phi: (1 \downarrow \Phi) \to T^*$ along u and v. Arbitrary interconnections (DO_OR_, DO_THEN_, DO_WHILE_) can be obtained as colimits of suitable diagrams, e.g.



$$\begin{tikzcd} I \arrow[dr] & & J \arrow[dl] \arrow[dr] & & K \arrow[dl] \\ & (1\downarrow\Phi) \arrow[dr, dashed] & & (1\downarrow\Psi) \arrow[dl, dashed] & \\ & & A & & \end{tikzcd}$$

The cospan $I \to A \leftarrow K$, while not a Moore automaton, is what you want. This provides an interconnection theory for automata (special case : programs) dual to the Goguen interconnection-by-limit theory for systems. Hardware and software are dual, e.g. looping is dual to feedback. A translation to a logical formalism like the preceding for systems seems possible using the higher order logic of the hyperdoctrine.

Categories of Games

Andreas Blass

We consider games in which two players (0 and 1) alternately select elements from some set; depending on the (usually infinite) sequence of their choices, one of the players wins and the other loses. In [1] we proposed that such games be preordered by the relation

(*) A $\leq$ B iff there is a strategy whereby player 1 can win B if he is shown how to win A,

and we studied the lattices ($\mathcal{S}$ and $\mathcal{W}$) that arise from two natural ways of making this proposal precise.

By thinking of the strategies mentioned in (*) as morphisms from A to B, we define a category $\underline{S}$ of games. The pre-ordered class associated to $\underline{S}$ is the lattice $\mathcal{S}$ of [1], but, as might be expected, the category $\underline{S}$ contains more information than $\mathcal{S}$. For example, $\underline{S}$ contains nontrivial retractions, so games can be equivalent in $\mathcal{S}$ without being isomorphic in $\underline{S}$. If we restrict attention to games in which every play has finite lenth, we find a nearly trivial sublattice of $\mathcal{S}$ but a rather interesting subcategory of $\underline{S}$; a quotient of this subcategory is an "initial category with arbitrary products and coproducts."

The category $\underline{S}$ has a (symmetric monoidal) closed structure. Its tensor product is not the cartesian product; the difference between the two products can be viewed, using ideas of Lorenzen [2], as reflecting the difference between the classical and intuitionistic meanings of "or."

We define a cotriple (R,ε,μ) on $\underline{S}$. The object part of R is the operator called R in [1], and the Kleisli category $\underline{W}$ of this cotriple is related to the lattice $\mathcal{W}$ of [1] as $\underline{S}$ is to $\mathcal{S}$. From a natural isomorphism $R(A \times B) \cong R(A) \otimes R(B)$ in $\underline{S}$ it follows that $\underline{W}$ is cartesian closed.

We re-examine Lorenzen's idea of defining the basic logical connectives in terms of games [2]. Lorenzen and others have produced such definitions capable of yielding either classical or intuitionistic logic, but the definitions are fairly complex. When the problem is attacked using the concepts that occur naturally in the study of S, two particularly simple approaches present themselves. One has an intuitionistic flavor but leads to nothing beyond lattice theory. The other comes very close to producing classical propositional logic. In fact, we can get exactly classical propositional logic by using the functor R; if we refrain from using R, we get a weaker logic which may be of some independent interest.

References

1. A. Blass, Degrees of indeterminacy of games, Fund. Math. 77(1972) pp. 151-166

2. P. Lorenzen. Ein dialogisches Konstruktivitätskriterium, in Infinitistic Methods, PWN, Warsaw, 1960

An interpretation theory for "triple" cohomology

J. Duskin

A categorical, semi-simplicical interpretation theory for "triple" cohomology is outlined which uses the groups $TORS^n_U\,|X;\pi|$ of connected components of certain categories of $K(\pi,n)$-torsors (higher dimensional analogs of locally trivial principal fibre bundles) to interpret the higher dimensional cohomology groups, i.e., $TORS^n_U\,|X;\pi| \xrightarrow{\sim} H^n_G(X;\pi)$, $n \geq 1$. Details of these results will appear in Memoirs A.M.S. and a detailed outline may be found in Proc. Nat. Acad. Sci. (USA) Vol 71 , No. 6 pp. 2554 - 2557 (June 1974).

Equational Traduction Of Set Theoretical Notions

by René Guitart

I. Involutive Monads.- The main problem is to find an "equational" context which exists in Sets and which would allows us to develop in an equational way the theory of the definition (of the types of structures).

Roughly speaking we have to solve the equation

"topos = finitely complete cat. + cartesian closed cat. + ?"

Let us begin with an abelian sup-monoid $A = (\underline{A}, \sup, k)$ that is to say a complete lattice $(\underline{A}, \sup)$ and an abelian monoid $(\underline{A}, k)$ (whose unit is denoted by e) where the law is a sup-lattice morphism (examples of this situation are distributive lattices, and, also, the set $[0,1]$ with its usual order and multiplication).

For a set X let A^X be denoted by FX, and let $i_X: X \to FX$ and $d_X: FX \to F^2X$ be defined by

$$i_X xx' = \begin{cases} e \text{ if } x = x' \\ 0 \text{ if } x \neq x' \end{cases} \quad \text{and } d_X pp' = \sup_{x \in X} k(px, p'x).$$

The system (F,i,d) is a "contravariant standard construction" (c.s.c) over Sets, i.e. satisfies the 4 equations given in ([1]) or ([2]). It is a theorem that over each category $\underline{C}$ there is a bijection between the set of c.s.c. over $\underline{C}$ and the set of involutive monads over $\underline{C}$ (we call involutive monad (i.m.) over $\underline{C}$ a pair (P,I) where P is a monad over $\underline{C}$ and I an involution on the Kleisli's category $\underline{KL}P$ of P).

A definition of a complemented i.m. (c.i.m.) is given in ([2]), and an equivalent one can be found in ([0]). In my talk at the Open House on Category Theory organised at the University of Sussex (July 74) I have given a lot of examples (on a topos, on the category of relations, on the "special" category of compact spaces, on the category of "quasi-topologies", etc...). As many people here were last week in Sussex, I would not repeat this list. Looking for some examples in Cat I have studies the notion of a machine (cf. my lecture at the Amiens'Meeting on July 73); actually this is studied now by R. Street (and exposed in his lecture here) in the context of 2-categories, which suggests how to extend the concept of involutive monad into the 2 dimensional case. Let us notice that, in Cat, the case $F = \underline{Sets}^{(-)}$ is not an example of an involutive monad only because of the question of size. However, the case $F = 2^{(-)}$ is an example of i.m. in Cat.

We denote by $\underline{U}_A$ the i.m. over Sets exhibited at the beginning of this paragraph. In the category of i.m. over Sets we have the simplicial object $\hat{\underline{U}}$

$$\hat{\underline{U}} : \underline{U}_1 \xleftarrow{\ \varepsilon\ } \underline{U}_2 \begin{array}{c} \xleftarrow{\ \alpha\ } \\ \xrightarrow{\ u\ } \\ \xleftarrow[\ \beta\ ]{} \end{array} \underline{U}_3 \xleftarrow{\ \kappa\ } \underline{U}_4 \longleftarrow \cdots$$

which in fact is a category, and comes from the involutive monad $2^{(-)}$ over Cat.

Everyone knows how to make use of $\underline{U}_1$ and $\underline{U}_2$ to work out some mathematical notions. But the question is: what comes after 1 and 2 ? A somewhat natural reaction would then make us think about using $\underline{U}_3$ and $\underline{U}_4$.

Notice that if we add to $\underline{U}_N$ the data of all the maps from $\underline{U}_N$ to itself coming from the maps ε, α, β, u, κ,... in the simplicial object $\hat{\underline{U}}$, we obtain a system reacher than an i.m., which we denote by $\vec{\underline{U}}_N$.

II. Structural Equations.- Let $\underline{C}$ be a category and $(F, i, d) = \underline{U}$ an i.m. over $\underline{C}$. An equation for $f_1, \ldots, f_n$ in $\underline{U}$ consists of an identity "A = B" where A and B are composites in $\underline{C}$ of morphisms of the form $F^m f_i$, $F^m i_F p_{f_i}$ or $F^m d_F p_{f_i}$. Clearly for every abelian sup-monoid A, an equation E can be interpreted as a formula written $J_A E(f_1, \ldots, f_n)$. Hence, every such equation defines a theory $T_A E$ whose models are (by definition) n-tuples $(f_1, \ldots, f_n)$ of maps verifying the identity "$A(f_1, \ldots, f_n) = B(f_1, \ldots, f_n)$" in $\underline{U}_A$.

Main problem: If T is a type of structure, could we find an equation $E_A T(f_1, \dots, f_n)$ whose solutions in $\underline{U}_A$ are exactly the models of T?

Such an equation, if it exists, will be called a "structural equation of the theory T in the context $\underline{U}_A$" (or simply a.s.e. of T in A).

Nota.- If A and A' are two abelian sup-monoids then, given a theory T, we can transform it into a new theory T' by "modifying the underlying logics" as follows: if T admits a structural equation $E_A T$ in A, then the interpretation $J_{A'}(E_A T)$ defines the new theory T'.

The following theories admit structural equations for A = 2: the theory of the void set, the theory of the set 2^n, theories of relations, of order relations, of congruences, of injections, of surjections, of complete atomic boolean algebras, of points (elements).

When we work out the notion of a structural equation in the context of c.i.m., we get structural equations in A = 2 for notions of filters, ultrafilters, compact spaces; we also get new equations for the notion of a point.

It is a fact that we cannot get structural equations in 2 for the notion of a reflexive relation and for that of a topological space. However, these notions admit s.e. for $\vec{\underline{U}}_3$.

So, it is natural to try to measure "complexity" of theories according to the invariant

$$\delta(T) = \inf \{n \;/\; T \text{ admits a structural equation for } \vec{\underline{U}}_n\} .$$

III. Related Functors.- Let $\underline{U}$ be an i.m. over a category $\underline{C}$. If $\underline{R}X$ is the set $\mathrm{Hom}_C(X,FX)$ the function $\underline{R}: \underline{C}_0 \to \underline{Sets}$ can be extended to a contravariant functor $\underline{R}^-$ from $\underline{C}$ to $\underline{Sets}$ and also to a covariant functor $\underline{R}^+$ from $\underline{C}$ to $\underline{Sets}$. In the same way the function $\underline{E}X = \{r \in \underline{R}X \;/\; E(r)\}$ where the equation $E(r) \equiv$ "$r = Fr.i_{FX}.r$" is a structural equation in 2 of the notion of an equivalence relation, gives rise to a contravariant functor $\underline{E}^-$ from $\underline{C}$ to $\underline{Sets}$.

If E is a structural equation of the notion of a point, and if $\underline{U}$ satisfies some equations related to E, we can define a functor $\underline{V}_E : \underline{C} \to \underline{\mathrm{Sets}}$, which assigns to each $X \in \underline{C}_0$ the set $\underline{V}_E X$ of "points" of X ("points" being solutions of E in $\underline{U}$).

More generally, we can obtain functors $\underline{V}^+_E$ or $\underline{V}^-_E$ for E element of a large class of equations (the functors $\underline{R}^+$, $\underline{R}^-$ and $\underline{E}^-$ are of this form $\underline{V}_E$ for some E).

Now, if we assume some supplementary properties to be satisfied by some $\underline{V}_E$ we obtain more precise theories than the theory of c.i.m., and these theories are of course a better approximation of the theory of topoi.

In order to find which supplementary properties are interesting, we can look at the $\underline{V}_E$ as candidates for "concrete functors", or for "deductive categories", or perhaps for functors defining "dogmas".

The method of structural equations may, of course, be used in different contexts than the one of involutive monads. We could start, for examples, with (in the $\underline{\text{Sets}}$'case) the functor $P(E^2)$ instead of the functor $P(E)$, and develop a parallel theory. We call "<u>typical system</u>" such a context (cf. ([5])).

I would like to conclude this talk by the following question:

Let $E(f_1, \ldots, f_n)$ be an equation satisfied in $\underline{U}_2$. Of course E is not necessarily satisfied in each involutive monad. But, if $\underline{T}$ is an elementary topos and $\underline{U}(\underline{T})$ the canonical involutive monad $\Omega^{(-)}$ over $\underline{T}$, is it true that E is satisfied in $\underline{U}(\underline{T})$?

([0]) Esq. Math. Paris VII, vol. 1 (June 70).

([1]) C.R.A.S. Paris, t 275 (July 72) p. 259.

([2]) C.R.A.S. Paris, t 277 (Nov. 73) p. 935.

([3]) C.R.A.S. vol. to appear (2 notes presented on the 1st and 8th of July 74).

([4]) Monades involutives complémentées, to appear in "Cahiers top. et géo. diff.".

([5]) Systèmes typiques (in preparation).

(E,M) - Universally Topological Functors

Rudolf-E. Hoffmann

A cone $(C,\lambda: C_\Sigma \to T)$ in a category $\underline{C}$ is said to be V-co-identifying (= "V-co-idt.") with respect to a functor $V: \underline{C} \to \underline{D}$, iff whenever $(V*\lambda)u_\Sigma = V*\eta$ for some cone $(X,\eta: X_\Sigma \to T)$ and some morphism $u: VX \to VC$, then there is a morphism $h: X \to C$ in $\underline{C}$ being unique with respect to the following properties

(1) $\qquad Vh = u$

(2) $\qquad \eta = \lambda h_\Sigma$

A functor $V: \underline{C} \to \underline{D}$ is said to be "topological" provided that (1) for every V-datum $(T; D,\psi)$, i.e. for every diagram $T: \Sigma \to \underline{C}$ of "discrete type" (where Σ is assumed to be $\underline{U}$-small and - moreover - to be discrete, i.e. a set; $\underline{U}$ = [fixed] universe) and every cone $(D,\psi: D_\Sigma \to VT)$ in $\underline{D}$ there is a V-co-identifying (= "V-co-idt.") lift $(C,\lambda;i)$, i.e. a V-co-idt. cone $(C,\lambda: C_\Sigma \to T)$ and an isomorphism $i: VC \to D$ in $\underline{D}$ with $\psi i_\Sigma = V*\lambda$, and (2) that V satisfies the "smallness condition" for functors, i.e. whenever $M \subseteq \mathrm{Ob}\,\underline{C}$ consists of nonisomorphic objects, which are taken by V into objects isomorphic to some $Y \in \mathrm{Ob}\,\underline{D}$, then M is $\underline{U}$-small.

[The relationship to O. Wyler's top categories is clarified as follows:

(a) $V: \underline{C} \to \underline{D}$ is (a projection from) a top category, iff (1) it is a topological functor and (2) it lifts isomorphisms uniquely;

(b) every topological functor is isomorphic to the composite of (at first) an equivalence and (then) a projection from a top category.]

(Some fundamental properties of these functors are to be found in the author's abstract for the Oberwolfach Kategorientagung 1972).

Topological functors abound: forgetful functors <u>Top</u> (topological spaces and continuous maps) $\to$ <u>Ens</u>, <u>Unif</u> (uniform spaces and uniformly continuous maps) $\to$ <u>Ens</u>, <u>Preord</u> (preordered sets and

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isotone maps) → Ens, ..., Top-Gr (topological groups and continuous homomorphisms → Gr, ...

Since one is also interested in epi-reflective subcategories of Top, etc.*) and their forgetful functors to Ens, H. Herrlich has introduced the concept of (E,M)-topological functor:

(a) Let E be a class of epimorphisms in D with Iso D $\subseteq$ E, which is compositive, and let M be a class of (not necessarily monic) cones in D indexed by U-small sets, such that Iso D $\subseteq$ M and composition of an M-morphism and an M-cone gives (whenever this is defined) an M-cone. If every cone in D indexed by some U-small set factors uniquely (up to ...) over an E-morphism and an M-cone, and if D is E-co-well-powered, then D is said to be an (E,M)-category.

(b) Let D be an (E,M)-category. V: C → D is said to be (E,M)-topological, iff

(1) every V-datum (T;D,ψ) for every discrete, U-small graph Σ with (D,ψ) $\in$ M has a V-co-idt. lift;

(2) V satisfies the smallness condition [H.Herrlich's definition drops assumptions on smallness and co-well-poweredness, but includes cones indexed by U-classes].

From an (E,M)-topological functor V: C → D H.Herrlich reconstructed a topological functor U: B → D and a full reflective embedding F: C → B, such that (1) UF = V and (2) for the unit η (of the adjunction given by F) holds $U\eta_B \in$ E for every B $\in$ Ob B (i.e. "C is E_U-reflective in B" via F):

Objects of B are pairs (e: D → VC,C) with e $\in$ E and C $\in$ Ob C, morphisms from (e,C) to (e': D' → VC',C') are pairs (f: D → D', g: C → C') with Vg e = e' f [of course, one has to make the hom-sets disjoint to one another]; F is given by C $\longmapsto$ $(1_{VC}, C)$.
We have discovered, that this construction has a nice universal property: whenever T: X → D is topological, and K: C → X with

*) E.g. T_o-, T_1-,T_2- spaces, regular spaces, completely regular spaces (and continuous maps), but also posets (and isotone maps)

$V \cong TK$ takes those V-co-idt. cones (indexed by $\underline{U}$-sets) into T-co-idt. cones, which are taken by V into M-cones, then there is a unique (up to ...) functor $H: \underline{B} \to \underline{X}$ with $HF \cong K$, $U \cong TH$ and taking U-co-idt. cones into T-co-idt. cones [in order to make the statement correct, one needs coherence conditions for the above isomorphisms]. Consequently, V "determines" U and F. More important is, of course, the observation, that U "determines" $\underline{C}$ (upto...): The objects of $\underline{C}$ are characterized (up to...) by the fact that they are (U,M)-separated in $\underline{B}$; here (U,M)-separated means that

(a) every set-indexed U-co-idt. cone in $\underline{B}$ with domain C is taken by U into an M-cone;

(b) every U-co-idt. morphism in $\underline{B}$ with domain C is taken by U into an M-morphism;

(c) every U-co-idt. morphism f in $\underline{B}$ with domain C, such that $Uf \in E$, is an isomorphism;

(a), (b), (c) are pairwise equivalent, provided that U is an (arbitrary) topological functor. Functors U obtained by the above "universal" construction from an (E,M)-topological functor are called "(E,M)-universally topological functors".

<u>Examples</u> (with the usual forgetful functors to <u>Ens</u> and, resp, <u>Gr</u>):
<u>T_o-Top</u> reconstructs <u>Top</u>,

<u>Sep Unif</u> (separated uniform spaces) reconstructs <u>Unif</u> ,
<u>Poset</u> reconstructs <u>Preord</u>,...

<u>T_o-Top-Gr</u> reconstructs <u>Top-Gr</u>,...

where E = {surjections}, M = {"point separating" families of maps}. It turns out that many of the Ens-valued topological functors are (E,M)-universally topological and that "separatedness" (the name here is due to Brümmer) gives a natural analogue of T_o in the case of <u>Top</u> (which often coincides with" Hausdorffsch").

There are Ens-valued topological functors which are not of the above described type; however, for the class of those Ens-valued topological functors $T: \underline{X} \to$ Ens, which are represented by a terminal object, there is an approximation theorem:

There is a largest bi-reflective subcategory $\underline{B}$ of $\underline{X}$, such that $T|\underline{B} = U: \underline{B} \to$ Ens is (E,M)-universally topological. The (U,M)-separated objects of $\underline{B}$ are exactly the non-cogenerators of $\underline{X}$, which can be described as follows:

Let 2^c denote a T-co-discrete object of $\underline{X}$ with card $T(2^c) = 2$ ($X \in$ Ob $\underline{X}$ is said to be T-co-discrete, iff for every $\underline{X}$-object Y and every morphism $f: TY \to TX$ there is a unique morphism $g: Y \to X$ with $Tg = f$): $C \in$ Ob $\underline{X}$ is a non-cogenerator, iff every morphism $2^c \to C$ is taken by T into a constant map. - $\underline{B}$ is the bi-reflective hull of the class of non-cogenerators in $\underline{X}$.

Finally, we want to mention some related results:

1) The correspondence between "separation axioms": T_o, T_1, T_2, $T_3 \wedge T_o$, complete regularity $\wedge$ T_o, etc. and "regularity axioms": Top itself, "R_o", "R_1", "R_2" (i.e. "T_3 without T_o"), "complete regularity without T_o", etc., which was described by A.S. Davis, can be nicely translated into the context of (E,M)-universally topological functors.

2) The relationship between normed linear spaces and quasi-normed linear spaces can be analysed by separatedness and co-separatedness with respect to the obvious topological functor to linear spaces [with (surjective-joint injective)-factorization].

3) Applying the above approximation theorem one can show the following result on (extremal epi)-reflective subcategories of top categories over Ens: Let $T: \underline{X} \to$ Ens be a topological functor, which is represented by a terminal object. An (extremal epi)-reflective subcategory $\underline{Y}$ of $\underline{X}$ is closed under coproducts (as e.g. T_o, T_1, T_2 is) iff card $TY \geq 2$ for some $Y \in$ Ob $\underline{Y}$.

Cohomologie non-abélienne et homotopie
André Joyal

Il s'agit de comparer des notions et résultats de la Cohomologie non-abélienne de Giraud avec les méthodes homotopiques. Si E est un topos (élémentaire) on construit succesivement un objet catégorie Δ (utilisant N) et son complété pour l' adjonction de limites à droite finies $\hat{\Delta}$: cette dernière catégorie interne est celle des "ensembles" simpliciaux de dimension finie. On définit sur $\hat{\Delta}$ la "classe" Σ des extensions anodines et on définit la catégorie homotopique interne finie Ho_o comme étant $\Sigma^{-1}\hat{\Delta}/\sim$ ou $\sim$ est la relation d'équivalence homotopique. On considère ensuite la catégorie localement représentable Ind (Ho_o) obtenue par complétion inductive filtrante interne de Ho_o. Le foncteur canonique $\hat{\Delta} \xrightarrow{U} Ho_o$ possède un prolongement canonique $E^{\Delta opp} = \mathrm{Ind}(\hat{\Delta}) \xrightarrow{U'} \mathrm{Ind}(Ho_o)$.

<u>Définition:</u> Un morphisme d'object simplicial $(F \xrightarrow{f} G)$ $E^{\Delta opp}$ est une équivalence faible si $U'(F) \xrightarrow{U'(f)} U'(G)$ est un isomorphisme. On peut alors définir la catégorie homotopique Ho(E) comme la catégorie de fractions $\Gamma^{-1}E^{\Delta opp}$ ou Γ est la classe des équivalences faibles. <u>Définition</u> Un complexe de Kan H est <u>saturé</u> si le foncteur $E^{\Delta opp}(-,H):(E^{\Delta opp})^{opp} \to Set^{\Delta opp}$ transforme les équivalences faibles en équivalence homotopique ordinaire.

<u>Proposition</u>. Si H est un complexe quelconque, alors il existe un complexe de Kan saturé $\tilde{H}$ et une équivalence faible $H \to \tilde{H}$.

Cette proposition est valide sous l'hypothèse que E est un topos de Grothendieck. Si on considère les complexes d' Eilenberg-MacLane ordinaires $K[\pi,n]$ alors les groupes de cohomologie $H^n(X,\pi)$ sont donnés par.

$\pi_o(\mathrm{Hom}_E(X,\widetilde{K[\pi,n]})) = H^n(X,\pi)$. Si π est un groupoide alors π est un champ un sens de Giraud si et seulement si $K[\pi,1]$ est saturé. De plus on a $\widetilde{K[\pi,1]} = K[\tilde{\pi},1]$ ou $\tilde{\pi}$ est le champ associé à π. Ceci montre que la théorie précédente peut englober la théorie de Giraud en dimension 1. Si on considère des 2-groupoides π il semble que la théorie de Giraud en dimension 2 puisse se ramener à la considération des complexes $\widetilde{K[\pi,2]}$.

An application of Categoy Theory to Model Theory

Orville Kean

Let L be an elementary one sorted language with finitary operations and relations, I be a theory with language L, and T be a class of formulas in L. By $C_{I,T}$ we shall mean the category whose objects are the models for I and whose morphisms are the maps between the models with preserve the formulas in T. Let:

$\overline{T}$ = the set of all formulas in L preserved by all the maps in $C_{I,T}$

If $A(\overline{X}) \in \overline{T}$, then there is a set-valued functor:

$$\widetilde{A}: C_{I,T} \to \text{Sets}$$

such that $\widetilde{A}(M) = \{\vec{a} \in M^n \mid M \models A(\bar{a})$ for every $M \in \text{Ob}\,(C_{I,T})$.
For representable $\widetilde{A}$, there is a model M_A and an n-tuple $\vec{e} \in M_A^n$ such that $M_A \models A(\vec{e})$; and given any $M \in \text{Ob}(C_{I,T})$ and n-tuple $\vec{b} \in M^n$ satisfying $M \models A(b)$, there is a unique map $M_A \to M$ with $\vec{e} \longmapsto \vec{b}$. Such an M_A is said to represent the formula $A(\bar{x})$. A familiar example of this is that given any topos $\mathbf{E}$ and object X in $\mathbf{E}$, the topos $\mathbf{E}_{/X}$ is the topos which represents the formula $(\delta_o(x) = 1) \wedge (\delta_1(x) = X)$.

Henceforth we shall let T be the set of atomic formulas in L and we shall omit T in the notation for the category of models. If C_Y is complete and admits the standard construction for products and equalizers, ie:

(1) $\quad \widetilde{A}\,(\prod_{\alpha<\beta} M_\alpha) = \prod_{\alpha<\beta} (\widetilde{A}(M_\alpha))$

(2) $\quad \widetilde{A}(\text{Eq}(x,y)) = \text{Eq}(\widetilde{A}(x,y))$

for every atomic formula $A(\bar{x})$ in L; then we have the following characterisations.

Proposition 1: If C_Y admits the standard construction for products and equalizers, then there is a finitary one sorted Gabriel-Ulmer theory $\mathbb{T}$ such that $C_Y \simeq$ Fin.Cont $(\mathbb{T}$, Sets).

A finitary one sorted Gabriel-Ulmer theory theory $\mathbb{T}$ is a small finitely complete category $\mathbb{T}$ with a distinguished object G such that for every $X \in Ob\ (\mathbb{T})$ there is a natural number n and a monic $X \rightarrowtail G^n$ in $\mathbb{T}$.

Proposition 2: A category is a finitary one-sorted Gabriel-Ulmer theory iff there is a simple Hom theory I such that $C_Y \cong F$ in. Cont. $(\mathbb{T}$, Sets).

An elemtary theory is a simple Hom theory if it logically equivalent to a theory with axioms of the form:

1. $A_1(\vec{x})$
2. $A_2(\vec{x}) \to B_1(\vec{x})$
3. $A_3(\vec{x}) \quad \exists! y\ B(\vec{x},\vec{y})$

where A_i and B_j are conjunctions of atomic formulas.

Theorem: An elementary theory I admits the standard construction for products and equalizers iff:

(1) I has an extension by definition I' which is a simple Hom theory.

(2) If $A(\vec{x})$ is an atomic formula in L(I') which is not in L(I), then the defining formula for $A(\vec{x})$ in I' is of the form $A(\vec{x}) \longleftrightarrow \exists y\ B(\vec{x},\vec{y})$ where $B(\vec{x},\vec{y})$ is a conjunction of atomic formulas in L(I).

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Linear algebra and projective geometry in the Zariski topos

Anders Kock

What status may be given to synthetic geometry inside the framework of abstract or generic algebraic geometry? Put differently, we seek a framework in which certain geometric "theorems" which admit different geometric interpretations when a change of rings is performed, are actual theorems about something, and not just "theorem-schemes". (An example of such a theorem-scheme is the one whose version over the reals says that the altitudes of a spherical triangle intersect in one point, and which for the ring of dual numbers gives a theorem about (non-plane) hexagons in space with all angles right; this is a application of Study's transfer principle.)

A "generic" theorem, which specializes to the two mentioned theorems (and whose proof is the same), exists: it is a theorem about the the universal ring R in the Zariski topos $\mathfrak{Z}$. The Zariski topos $\mathfrak{Z}$ is the category of sheaves on $\mathcal{R}^{op}$, where $\mathcal{R}$ is the category of finitely presented commutative rings, equipped with an easily described Grothendieck topology. The forgetful functor

$$\mathcal{R} = (\mathcal{R}^{op})^{op} \rightarrow \text{Sets}$$

is a commutative ring object R in $\mathfrak{Z}$, and it is a local ring object, and universal as such, by an important observation of Hakim. Being a local ring object is a property which is preserved by left exact left adjoint functors between toposes. However, R has also some properties which are not preserved by such functors: it is a field object in the sense that it for each natural number n satisfies the statement

$$(*) \qquad \neg \left(\bigwedge_{i=1}^{n} x_i = 0 \right) \Longrightarrow \bigvee_{i=1}^{n} (x_i \text{ is invertible})$$

and also

$$(**) \qquad \neg\, (1 = 0).$$

(The meaning of objects in a topos satisfying 1^{st} order statements is by now well known. One explanation may be found in [2].)

Being a field object in the sense (*) and (**) turns out to be precisely what is needed in order to make standard linear algebra work (remembering that deductions in 1^{st} order logic in an elementary topos have to be intuitionistically valid). In particular, we can prove that under the assumptions (*) and (**), we have

<u>Theorem</u>. For each $m \times n$ matrix A, the row Rank of A is $\geq$ r if and only if the determinant Rank is $\geq$ r.

(Row-Rank being defined in terms of linear independence of the rows of the matrix; determinant-Rank being defined in terms of invertibility of $r \times r$ sub-determinants. In particular, we have as a Corollary Row-Rank = Column-Rank).

Essentially because standard linear algebra works, we can develop projective geometry over a ring object R in an elementary topos, provided R satisfies (*) and (**). This means that we can construct a first order structure, "the projective plane" in the given topos: an object of "points" and an object of "lines", and an "incidence" relation between these two objects, such that, for instance, the following statements are satisfied:

"Given two points which are not equal; then there is a unique line containing them".

" Given two lines which are not equal; then they intersect in a unique point."

- as well as, for instance, Pappus' theorem about plane hexagons with vertices lying on two lines, and other theorems from synthetik projective geometry.

To "transfer" these theorems to $\mathcal{S}$ (the category of sets) equipped with the ring of reals, or with the ring of dual numbers over the reals, one has to use the universal property of $\mathfrak{Z}$, R noted by Hakim. But first, one has to change the theorems to be transferred into transferable form, i.e. to a form which is preserved by left exact left adjoints, in particular, the form should be negation-free. So I do not believe one get geometric theorems which one could not get by using Study's transfer principle in its purely heuristic form.

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Full Embeddings of Categories

Luděk Kučera

I. Set-theoretical assumptions and full embeddings of categories

A category is called binding if any concretisable category can be fully embedded into it. In 1969 author and Z. Hedrlín proved [3] that the category of graphs and compatible mappings is binding. According to papers of Věra Trnková, Z. Hedrlín, J.Lambek, E. Mendelsohn, J.Nešetřil, A.Pultr, J.Sichler and others there are many binding categories "from the life", e.g. the categories of semigroups, commutative groupoids, various categories of algebraic and topological type.

However, there is one set-theoretical difficulty in this full embedding theory. If we want to prove bindability of above categories we have to assume the next axiom (M):

(M) There is a cardinal number m such that any two-valued m-additive measure is trivial.

As it was shown by author and A.Pultr in 1971 ([4]), the role of (M) is essential: The axiom (M) is even equivalent to the existence of a full embedding of Set^{op} into the category of graphs (or into any category of universal algebras).

Therefore we shall call a category to be binding if above described full embeddings do exist under (M). If the existence of those embeddings can be proved in Gödel-Bernays set theory without any set theoretical assumptions then the category is called universal.

The difference between binding and universal categories is not purely set-theoretical. E.g. if F: $Set^{op} \to Gra$ (Gra being the category of graphs) is a full embedding then the cardinality of a set of vertices of F(1) is at least the cardinal number m from the axiom (M) (i.e. at least $\aleph_0$ if there is no measurable cardinal); on the other hand, changing Gra by a universal category, F(X) can be an object of finite size for every finite set X.

The first result to be referred to is that adding a locally compact T_2-topology to many categories of universal algebras we obtain universal categories. Especially, though the category of semigroups is binding only, we have the next theorem:

Theorem: The category of locally compact semigroups and their continuous homomorphisms is universal.

II. Categories with O-morphisms

Any category, which is fully embeddable into the category of semigroups with unity, is evidently a concretisable category with O-morphisms, Using the technique based on ideas of [1], we can prove the next theorem:

Theorem: Assuming (M), any concretisable category with O-morphisms can be fully embedded into the category of semigroups with unity. Any concretisable category with O-morphisms can be fully embeded into the category of locally compact semigroups with unity.

III. Full embeddings of non-concrete categories.

No non-concrete category can be (fully) embedded into concrete one. That is the reason for restricting to concrerisable categories in the definition of both binding and universal categories.

Now, simple examples of categories, into which every (even non-concrete) categories can be embedded, will be given. The main lemma for the construction of them it is proved in [2] and says that every category is a "homotopy-like" factorization of a concrete one.

Definition: Let K be a concrete category. Define a category $\bar{K}$ as follows: objects of $\bar{K}$ are triples (o,e_1,e_2), where o is an object of K, e_1,e_2 are equivalences on the underlying set of o, morphisms have a form $(o,e_1,e_2) \xrightarrow{f} (\acute{o},\acute{e}_1,\acute{e}_2)$, where $f: o \to \acute{o}$ is a morphism of K such that $x\ e_i\ y$ implies $f(x)\ \acute{e}_i\ f(y)$, $i=1,2$.

Let E be the smallest congruence on $\overline{K}$ such that

(i) if $f,g:(o,e_1,e_2) \longrightarrow (\acute{o},\acute{e}_1,\acute{e}_2)$ are morphisms of $\overline{K}$ such that

either $x\ e_1\ y$ implies $f(x)\ \acute{e}_1\ g(y)$

or $x\ e_2\ y$ implies $f(x)\ \acute{e}_2\ g(y)$

then $f\ E\ g$.

(It can be proved that E is the smallest equivalence s.t. (i) holds).

Denote the factorized category $\overline{K}/E$ by $\tilde{\overline{K}}$.

Theorem: Let K be a concrete category. If either K is binding and (M) holds or K is universal then every category can be fully embedded into $\tilde{\overline{K}}$.

The theorem together with the list of binding and universal categories, yields categories universal w.r.t. full embedding of arbitrary category.

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From Types to Sets

J. Lambek

Dogmas are related to Lawvere's hyperdoctrines and Volger's logical categories, as well as to the languages of Bénabou, Coste and Fourman. They are categories with finite products with a specified object Ω which admits arbitrary exponents, Moreover, Ω is a Heyting algebra object and the canonical morphismm $\Omega \to \Omega^A$ has a right adjoint $\forall_A$ and a left adjoint $\exists_A$. Finally, one postulates extensionality. The point of a dogma is that it permits set abstraction (as a special case of λ-conversion): given any "propositional function" $\varphi(x): 1 \to \Omega$ in the indeterminate $x: 1 \to A$, there exist a unique morphism $f: A \to \Omega$ (not depending on x) such that $fx = \varphi(x)$, its "name" $1 \to \Omega^A$ is written $\{x \in A \mid \varphi(x)\}$. Moreover, all sentences of set theory involving constants from the dogma A appear as "propositions" $1 \to \Omega$ in A. Each dogma canonically generates a topos: its objects are "sets" $f: 1 \to \Omega^A$ and its morphisms are relations (ρ, g, f) between sets f and $g: 1 \to \Omega^B$ which happen to be universally defined and single valued. (Take $\rho \leq f \times g = \{\langle x, y\rangle \in A \times B \mid x \in f \wedge y \in g\}$, where $x \in f$ is the evaluation of f at x in the polynomial dogma $A[x]$.) This result is due to Volger; in fact, dogmas are his closed logical categories. It allows one to regard topos as forming a reflective subcategory of dogmas.

Cohomologie non-abélienne à coéfficients dans une 2-catégorie *)

R. Lavendhomme (Louvain - la - Neuve)

1. Catégories de 1- et de 2-cocycles.

Soit $\underline{X}$ une catégorie et $\underline{\underline{A}}$ une 2-catégorie.

Un 1-cocycle de $\underline{X}$ à coéfficients dans $\underline{\underline{A}}$ est un foncteur de $\underline{X}$ dans la catégorie $\underline{\underline{A}}_{o2}$ des 2-flèches de $\underline{\underline{A}}$ munie du produit de Godement. Plus généralement, on definit la catégorie des 1-cocycles de $\underline{X}$ à valeurs dans $\underline{\underline{A}}$ par

$$\underline{Z}^1(\underline{X},\underline{\underline{A}}) = \underline{Cat}\ (\underline{X},\underline{\underline{A}}_{o2}).$$

La source et le but d'un 1-cocycle sont des foncteurs de $\underline{X}$ dans la catégorie $\underline{\underline{A}}_{o1}$ des 1-flèches de $\underline{\underline{A}}$. Si $H: \underline{X} \to \underline{\underline{A}}_{o1}$ est un foncteur, on désigne par $Z^1_H\ (\underline{X},\underline{\underline{A}})$ l'ensemble des 1-cocycles de $\underline{X}$ à coéfficients dans $\underline{\underline{A}}$ de but H.

Un 2-cocycle de $\underline{X}$ à coéfficients dans $\underline{\underline{A}}$ est un triple (γ,Q,c) ou

a) γ est une application de l' ensemble $\underline{X}_o$ des objets de $\underline{X}$ dans l'ensemble $\underline{\underline{A}}_o$ des objets de $\underline{\underline{A}}$.

b) Q est une application de l'ensemble $\underline{X}_1$ des flèches de $\underline{X}$ dans l'ensemble A_1 des 1-flèches de $\underline{\underline{A}}$, qui à $x: x_1 \to x_2$ de $\underline{X}$ associe $Q_x: \gamma(x_1) \to \gamma(x_2)$

c) c est une application de l'ensemble $\underline{X}_1 \underset{\underline{X}_o}{\times} \underline{X}_1$

des couples de flèches composables de $\underline{X}$ dans l'ensemble $\underline{\underline{A}}_2$ des 2-flèches de $\underline{\underline{A}}$ qui associe à $x_o \xrightarrow{x} x_1 \xrightarrow{y} x_2$ la 2-flèche $c_{y,x}: Q_{yx} \Longrightarrow Q_y Q_x$.

Les données doivent satisaire aux conditions suivantes.

α) les conditions de normalisation: Si x $\underline{X}_o$, $Q_{1x}=1_{\gamma(x)}$ et si x on y est un identité de $\underline{X}$, $c_{y,x}$ est une identité.

β) la condition d' associativité. Si zyx est défini dans $\underline{X}$, le diagramme suivant est commutatif;

$$\begin{tikzcd} Q_{zyx} \arrow[r, Rightarrow, "c_{z,yx}"'] \arrow[d, Rightarrow, "c_{zy,x}"'] & Q_z Q_{yx} \arrow[d, Rightarrow, "Q_z * c_{y,x}"] \\ Q_{zy} Q_x \arrow[r, Rightarrow, "c_{z,y} * Q_x"'] & Q_z Q_y Q_x \end{tikzcd}$$

Soient (γ,Q,c) et (γ',Q',c') deux 2-cocycles avec $\gamma=\gamma'$. Un morphisme de (γ,Q,c) vers (γ,Q',c') consiste en la donnée pour chaque x de $\underline{X}$ d'une 2-flèche φ_x: $Q_x \Longrightarrow Q'_x$ telle que: si x est neutre, φ_x est neutre; si yx est défini on a

$$(\varphi_y * \varphi_x) \circ c_{y,x} = c'_{y,x} \circ \varphi_{yx} .$$

On obtient donc une catégorie de 2-cocycles notée $\underline{Z}^2(\underline{X},\underline{\underline{A}})$. Son ensemble d'objets est noté $Z^2(\underline{X},A)$. Si on identifie deux objets isomorphes, on obtient une catégorie de 2-cohomologie, notée $\underline{H}^2(\underline{X},\underline{\underline{A}})$ dont l'ensemble d'object est notée $\underline{H}^2(\underline{X},\underline{\underline{A}})$. On peut aussi considérer un ensemble de 2-cohomologie plus mince formé de l'ensemble des composantes connexes de $\underline{Z}^2(\underline{X},\underline{\underline{A}})$: $\pi_o\underline{Z}^2(\underline{X},\underline{\underline{A}})$.

On dira qu'un 2-cocycle (γ,Q,c) est neutre si $c_{y,x}$ est une identité quels que soint x et y. On a alors une bijection entre l'ensemble des 2-cocycles neutres de $\underline{X}$ à coéfficients dans $\underline{\underline{A}}$ et l'ensemble des foncteurs de $\underline{X}$ dans $\underline{\underline{A}}_{01}$. Plus généralement un 2-cocycle (γ,Q,c) est neutralisable par un foncteur F: $\underline{X} \to \underline{\underline{A}}_{01}$ s' il existe un morphisme α de (γ,Q,c) vers le 2-cocycle neutre associé à F. On a alors $\alpha(yx) = (\alpha(y) * \alpha(x)) \circ c_{y,x}$. On dit que α est une neutralisation de (γ,Q,c). Si on travaillait dans $\pi_o\underline{Z}^2(\underline{X},\underline{\underline{A}})$, il n'y aurait pas lien de distinguer entre 2-cocycle neutre et neutralisable.

2.-Suites exactes.

Une suite $\underline{\underline{C}} \xrightarrow{I} \underline{\underline{A}} \xrightarrow{P} \underline{\underline{B}}$ de 2-catégories et de 2-foncteurs est une suite exacte courte de 2-catégories si

a) $\underline{\underline{C}}_o = \underline{\underline{A}}_o = \underline{\underline{B}}_o$

b) $\underline{\underline{C}}_{01} = \underline{\underline{A}}_{01}$ et le foncteur $P_1 : \underline{\underline{A}}_{01} \to \underline{\underline{B}}_{01}$ est plein et surjectif.

c) La suite de catégories $\underline{\underline{C}}_{12} \xrightarrow{I_2} \underline{\underline{A}}_{12} \xrightarrow{P_2} \underline{\underline{B}}_{12}$ est une suite exacte courte oppréfibrée, c'est-à-dire que pour tout morphisme $\varphi : F_o \Longrightarrow F_1$ de $\underline{\underline{B}}_{12}$ et pour tout objet G_1 de $\underline{\underline{A}}_{12}$ au-dessus de F_1, il existe un morphisme opcartésien ξ au-dessus de φ de but G_1. [ξ est op-cartésien signifie que pour tout morphisme η de projection φ et de but G_1, il existe un unique morphisme η' de projection 1_{F_o} tel que $\eta \circ \eta' = \xi$].

Soit $\underline{\underline{C}} \xrightarrow{I} \underline{\underline{A}} \xrightarrow{P} \underline{\underline{B}}$ une suite exacte courte de 2-categories. Soit $G : \underline{X} \to \underline{\underline{A}}_{01} = \underline{\underline{C}}_{01}$ un foncteur et posons $H = P_1 \circ G : \underline{X} \to \underline{\underline{B}}_{01}$. On définit une application

$$\delta : Z^1_H(\underline{X},\underline{\underline{B}}) \to H^2(\underline{X},\underline{\underline{C}})$$

de la manière suivante. Soit $\beta \in Z^1_H(\underline{X},\underline{\underline{B}})$ et soit k un opclivage normalisé de l'oppréfibration $\underline{\underline{A}}_{12} \to \underline{\underline{B}}_{12}$. On définit (γ, Q^k, c^k) par $\gamma(x) = G(x) = H(x)$; $Q^k(x)$ est la source du 2-morphisme opcartésien de l'opclivage k de projection $\beta(x)$ et de but $G(x)$; enfin comme $\alpha^k(yx)$ et $\alpha^k(y) * \alpha^k(x)$ ont même projection, on a une factorication $c^k_{y,x} : Q^k_{yx} \Longrightarrow Q^k_y\, Q^k_x$. Il est trivial que (γ, Q^k, c^k) est une 2-cocycle et qu'un changement d'op-clivage k en k' détermine un isomorphisme entre les 2-cocycles correspondants. On a donc bien défini l'application δ.

Théorème:

La suite $Z^1_G(\underline{X},\underline{\underline{C}}) \to Z^1_G(\underline{X},\underline{\underline{A}}) \to Z^1_H(\underline{X},\underline{\underline{B}}) \xrightarrow{\delta} H^2(\underline{X},\underline{\underline{C}}) \to H^2(\underline{X},\underline{\underline{A}}) \to H^2(\underline{X},\underline{\underline{B}})$ est G-exacte au sens suivant:

1) en $Z^1_G(\underline{X},\underline{\underline{A}})$: un 1-cocycle dans $\underline{\underline{A}}$ provient d'un 1-cocycle dans

$\underline{\underline{C}}$ si et seulement si son image est neutre dans $\underline{\underline{B}}$.

2) il en est de même en $H^2(\underline{X},\underline{\underline{A}})$ pour les classes de 2-cohomologie

3) un élément de $H^2(\underline{X},\underline{\underline{C}})$ provient d'un 1-cocycle de $Z^1_H(\underline{X},\underline{\underline{B}})$ ssi son image dans $H^2(\underline{X},\underline{\underline{A}})$ est neutralisable par G, le morphisme de neutralisation étant opcartésien.

4) En $Z^1_H(\underline{X},\underline{\underline{B}})$ on a:

a) un 1-cocycle β de $Z^1_H(\underline{X},\underline{\underline{B}})$ provient d'un 1-cocycle opcartésien ssi $\delta(\beta)$ est neutre.

b) β provient d'un 1-cocycle ssi la neutralisation par G de $\delta(\beta)$ dans $\underline{\underline{A}}$ se factorise par une neutralisation dans $\underline{\underline{C}}$. (Notons que si on avait pris la cohomologie mince $\pi_o \underline{Z}^2(\underline{X},-)$ la formulation serait plus simple mais moins fine)

3.- Examples.

a) On obtient un exemple trivial en associant à chaque groupe abélian G la 2-categorie $\underline{\underline{G}}$ avec $\underline{\underline{G}}_o = 1$, $\underline{\underline{G}}_1 = 1$, $\underline{\underline{G}}_2 = A$. On retrouve alors la cohomologie à coëfficients dans un groupe abelien.

b) Si on se limite à des 2-groupoïdes à un seul objet qui correspondent à des groupes croisés, on retrouve la théorie de P. Dedecker. Pour des 2-groupoides plus généraux on a la théorie de I. Valdenama *) et I.C. Donai *).

c) A une suite cofibrée de catégories $\underline{C} \to \underline{A} \to \underline{B}$ on peut associer une suite exacte de 2-catégories $\underline{\underline{C}} \to \underline{\underline{A}} \to \underline{\underline{B}}$ opérant sur $\underline{C}$, $\underline{A}$, $\underline{B}$. La 2-cohomologie s' interprète alors en termes d' extensions de $\underline{X}$ par $(\underline{C}, \underline{\underline{C}})$.

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*) Das (handgeschriebene) Manuskript war stellenweise schwer lesbar. Über einzelne Schreibungen, insbesondere von Eigennamen, ließ sich keine Sicherheit gewinnen. (R.-E.Hoffmann)

Relative functorial semantics, III. Triples vs. theories.

F.E. J. Linton

1. The construction of Kleisli associates with each triple $\mathbb{T} = (T, \eta, \mu)$ on a category $\underline{A}$ a category $Kl(\mathbb{T})$ (cf. [1] and [3]), having the same objects as $\underline{A}$, and a functor $f^{\mathbb{T}} : \underline{A} \to Kl(\mathbb{T})$, working as the identity on the objects and having a right adjoint $u^{\mathbb{T}}$ that, on objects, works like T, the adjunction isomorphisms being the <u>identity</u> maps

$$Kl(\mathbb{T})(f^{\mathbb{T}}A,\ B) = Kl(\mathbb{T})(A,\ B) \underset{\text{def}}{=\!=} \underline{A}(A,\ TB) = \underline{A}(A,\ u^{\mathbb{T}}B).$$

2. This note records a simple and informative conceptual argument for the complete identification (announced in [2] and arduously established in [3]) of the Eilenberg-Moore category $\underline{A}^{\mathbb{T}}$ of algebras over $\mathbb{T}$ (cf. [0]) with the category of Lawvere-style algebras over the Kleisli category $Kl(\mathbb{T})$. It will be recalled that the former is equipped with a canonical "underlying $\underline{A}$-object" functor $U^{\mathbb{T}} : \underline{A}^{\mathbb{T}} \to \underline{A}$, and that the latter is, by definition, any $\underline{A}$-valued functor serving as a pullback of the diagram

$$\text{(1)} \qquad \begin{tikzcd} & \underline{S}^{Kl(\mathbb{T})} \arrow[d, "\underline{S}^{f^{\mathbb{T}}}"] \\ \underline{A} \arrow[r, "Y"'] & \underline{S}^{\underline{A}} \end{tikzcd}$$

in which Y is the Yoneda embedding and <u>the functor category notation is used to indicate categories of contravariant functors.</u>

3. To see that $U^{\mathbb{T}}$ serves as pullback of (1), use is first made of the Yoneda Lemma and the category $(\underline{S}^{\underline{A}})_{\check{\mathbb{T}}}$ of Eilenberg-Moore coalgebras over the "composition with the ingredients of $\mathbb{T}$" cotriple $\check{\mathbb{T}} = (\check{T}, \check{\eta}, \check{\mu})$ on the (contravariant functor) category $\underline{S}^{\underline{A}}$. Here

$$\check{T}(X) = X \circ T, \quad \check{\eta}_X = X \circ \eta, \quad \check{\mu}_X = X \circ \mu.$$

Each Eilenberg-Moore $\mathbb{T}$-algebra $\mathbb{B} = (B, \beta)$ becomes [3] a coalgebra

$\check{Y}(B) = (YB, \check{\beta}) = (\underline{A}(-,B),\ \underline{A}(-,B) \xrightarrow{T} \underline{A}(T-,TB) \xrightarrow{\beta} \underline{A}(T-,B))$ over the cotriple $\check{\mathbb{T}}$. In this way, there arises a functor

$$\check{Y} : \underline{A}^{\mathbb{T}} \longrightarrow (\underline{S}^{\underline{A}})_{\check{\mathbb{T}}}$$

lifting Y over $U^{\mathbb{T}} : \underline{A}^{\mathbb{T}} \to \underline{A}$ and $U_{\check{\mathbb{T}}} : (\underline{S}^{\underline{A}})_{\check{\mathbb{T}}} \to \underline{S}^{\underline{A}}$. The Yoneda Lemma now indicates: first, that a $\check{\mathbb{T}}$-coalgebra structure on a represented functor YB "is" nothing more than a $\mathbb{T}$-algebra structure on B; next, that $\check{Y}$ is fully faithful; and last, that $\check{Y}$ makes $U^{\mathbb{T}} : \underline{A}^{\mathbb{T}} \to \underline{A}$ the pullback of the diagram

(2)
$$\begin{array}{ccc} & & (\underline{S}^{\underline{A}})_{\check{\mathbb{T}}} \\ & & \downarrow U_{\check{\mathbb{T}}} \\ \underline{A} & \xrightarrow[Y]{} & \underline{S}^{\underline{A}} \end{array}$$

4. For $U^{\mathbb{T}}$ to be the pullback of diagram (1), therefore, it would be nice if $\underline{S}^{KL(\mathbb{T})}$ and $(\underline{S}^{\underline{A}})_{\check{\mathbb{T}}}$ were isomorphic as categories over $\underline{S}^{\underline{A}}$. It is nice: they are. The adjointness betweem $f^{\mathbb{T}}$ and $u^{\mathbb{T}}$, with adjunction triple $\mathbb{T}$ on $\underline{A}$, provides an adjunction making $\underline{S}^{u^{\mathbb{T}}}$ right adjoint to $\underline{S}^{f^{\mathbb{T}}}$, with adjunction cotriple $\check{\mathbb{T}}$ on $\underline{S}^{\underline{A}}$.
Moreover, $\underline{S}^{f^{\mathbb{T}}}$ is easily seen to create equalizers of $\underline{S}^{f^{\mathbb{T}}}$-split pairs, $f^{\mathbb{T}}$ being a bijection on the object classes. Thus, Beck's Theorem (cf. [3]), in its cotriple version, completes this proof and ends the argument. Of course, Beck's Theorem could have been applied dimctly to the pullbacķ of (1), but checking its hypotheses would have been more tedious, and the isomorphism of this paragraph would have escaped notice.

5. P.S.: The reader whom our notation (and references) successfully misled into assuming, comfortably, that $\underline{S}$ referred to his favorite category of sets and functions is hereby invited to choose an arbitrary multilinear category $\underline{S}$ and to place himself in the cosmos of $\underline{S}$-categories, where, bearing in mind that, even though Kl($\mathbb{T}$) remains (cf. [5]) an $\underline{S}$-category when $\mathbb{T}$ is an $\underline{S}$-triple on the $\underline{S}$-category $\underline{A}$, the constructions of $\underline{A}^{\mathbb{T}}$, the functor categories,

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the pullback of (1), and ($\underline{S}^{\underline{A}}$)$_{\mathbb{T}}$ may force him to enter the larger cosmos (cf. [7]) of pro- $\underline{S}$ -categories (so that Street's suggested procedure [8] isn't readily applied), he may nevertheless assure himself, using [4] and [6] for the Yoneda Lamma and Beck's Theorem, that the argument here presented remains entirely valid.

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Author's permanent adress:

Department of Mathematics
Wesleyan University
Middletown, Conn.
06457 U.S.A.

Categorical Shape Theory

John L. MacDonald

Let $K:P \to T$ be a functor. The shape category S_K of K is defined by $|S_K| = |T|$ and $S_K(X,Y) = \mathrm{Nat}\ (T(Y,K-),\ T(X,K-))$. There is an obvious canonical functor $D:T \to S_K$. Furthermore using Yoneda we have $S_K \cong S_L$ for $L = DK$. The Kan extension F_K of $F:P \to C$ along K factors as $F_K = F_L D$ where F_L is the Kan extension of F along $L = DK$. If K is the inclusion of a full subcategory then (a) L is codense (i.e. $L_L = 1_{S_K}$), (b) $S_K(X,Y) \cong T(X,Y)$ if $Y \in |P|$ and (c) $|S_K| = |T|$. Furthermore S_K is determined up to isomorphism by (a), (b) and (c).

For many functors the shape morphisms $S_K(X,Y)$ or the coshape morphisms $\hat{S}_K(X,Y) = \mathrm{Nat}\ (T(K-,X),\ T(K-,Y))$ can be described more explicitly. For example if $K:P \to T$ is the inclusion of a full coreflective subcategory with right adjoint $R:T \to P$, then $\hat{S}_K(X,Y) \cong P(RX,RY)$. The shape morphisms $S_K(X,Y)$ can be described as $\varinjlim T(X,KY_\alpha)$ under the more general condition than being reflective that $(Y \downarrow Y_\alpha)$ is a cofinal subcategory of $Y \downarrow P$ for some inverse system $\{Y_\alpha\}$.

The shape theories of Borsuk and Mardesic - Segal for compact spaces each lead to shape categories which are isomorphic to the restriction of S_K to compakt spaces for $K:P \to T$ the inclusion, T the homotopy category of topological spaces and P the full subcategory of spaces of the homotopy type of a polyhedron. We mention that P is coreflective and although not reflective does satisfy a cofinality condition of the type described in the preceding paragraph.

We haven seen that the Kan extension is shape invariant, i.e. it factors through S_K. Under what circumstances are shape invariant extensions of functors simply Kan extensions? We examine this question in the topological context of Mardesic-Segal by using their results to construct a shape invariant extension $\tilde{F}:C \to A$ of any functor $F:ANR \to A$ where C is the category of compact spaces, ANR is the full subcategory of compact absolute neighborhood retracts and A is any category in which $\varprojlim$ exists. Using a cofinality condition it can be shown that this shape invariant extension $\tilde{F}:C \to A$ is the Kan extension in much the same way that Dold shows that the Cech and Kan extensions are equivalent. We note that many of the preceding results are due to Hilton or Levin.

C^*-algebras in a topos

Christopher Mulvey.

Any C^*-algebra A (with identity) admits a compact representation in a ring A_X in the category of sheaves on the maximal ideal space X of the centre Z(A). The representation is classified by the quotient map

$$\text{Prim } A \to X$$

which intersects each primitive ideal of A with the centre of A. Under the representation the centre Z(A) becomes isomorphic to the sheaf of continuous complex functions on X, since the induced representation is the Gelfand representation of the commutative C^*-algebra Z(A).

Approaching the representation from the viewpoint using the language of toposes one feels that the representation should yield a C^*-algebra in the category Top(X) of sheaves on X. The effect of the representation would then have been to have converted the C^*-algebra into one in Top(X) having centre the complex numbers. The problem arising is that of defining the concept of a C^*-algebra, or more generally a normed algebra, in a topos.

Generalising the usual definition of a normed algebra to the case of an algebra over the ring $\mathbb{C}$ of complex numbers in a topos $\mathbb{E}$ one might be tempted to require the existence of a map

$$B \xrightarrow{\|\cdot\|} \mathbb{R}$$

from an algebra B to the (Dedekind) reals in $\mathbb{E}$ satisfying the axioms

i) $\|a\| \geq 0$ and $\|a\| = 0 \iff a = 0$

ii) $\|a+b\| \leq \|a\| + \|b\|$

iii) $\|1\| = 1$

iv) $\|\alpha a\| = |\alpha| \; \|a\|$

v) $\|ab\| \leq \|a\| \cdot \|b\|$

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for all $a, b \in B$ and $\alpha \in \mathbb{C}$ where $|\cdot| : \mathbb{C} \to \mathbb{R}$ is the modulus map.

In the case of the algebra A_X obtained by representing a C^*-algebra over the maximal ideal space of its centre, the canonical candidate for a norm unfortunately assigns to a section a real function which although upper semi-continuous is not in general continuous. The algebra A_X would therefore fail to be a C^*-algebra through not admitting a map to $\mathbb{R}$ satisfying the required conditions.

The difficulty may be resolved by remarking that the presence of a norm is needed essentially to describe a topology of a particular kind. Indeed the definition of a normed algebra is equivalent for the case of sets to the existence of a map

$$\mathbb{R}^+ \xrightarrow{N} \Omega^B ,$$

from the object of positive reals to Ω^B satisfying

i) $(\forall_{r>0} \quad a \in N(r)) \iff a = 0 \quad$ and $\quad r>s \implies N(r) \supset N(s)$

ii) $a \in N(r) \wedge b \in N(s) \implies a + b \in N(r + s)$

iii) $1 \in N(r) \iff r > 1$

iv) $a \in N(r) \iff \alpha a \quad N(|\alpha|. r)$

v) $a \in N(r) \wedge b \in N(s) \implies ab \in N(rs)$

to which one ought to add the additional axiom

vi) $\exists_{r>0} \quad a \in N(r)$

where r, s range over the positive reals and $\alpha \in \mathbb{C}$ is such that $|\alpha| > 0$.

According to this definition it may be verified that the algebra obtained from a C^*-algebra A is indeed a normed algebra in the category of sheaves on the maximal ideal space of $Z(A)$. It is complete with respect to this norm and admits an involution $*$ which satisfies the condition

$$a^*a \in N(r^2) \iff a \in N(r)$$

for it to be a C^*-algebra internally. Further the norm and involution induced on the centre are the canonical ones.

Finally, remark that in the case of the topos of sets, a normed algebra defined in the above way admits a map

$$B \xrightarrow{\|\cdot\|} \mathbb{R}$$

defined by the formula

$$\| a \| = \inf \{ r \in \mathbb{R}^{+} \mid a \in N(r) \}$$

which yields an element of $\mathbb{R}$ since in this case $\mathbb{R}$ is inf-complete. In a general topos $\mathbb{R}$ may fail to be inf-complete, in which case the formula determines an element of the inf-completation of $\mathbb{R}$, which in the case of Top(X) is of course exactly the sheaf of upper semi-continuous real functions on X.

Topos theory $\subset$ many-sorted intuitionstic set theory. *)

Gerhard Osius

Elementary topoi serve as a generalization of "the" category of sets, and our aim is to investigate to which extent topos theory actually "is" set theory. We will work within the theory ET of elementary topoi.

To discover that topos theory ET is contained in some kind of set theory we first introduce the set theoretical (or internal) language L(SET) which goes back to W. Mitchell. L(SET) is a many-sorted firstorder language whose terms x (called: elements) have objects A of the topos (i.e. terms of ET) as types. We write "$x\varepsilon A$" instead of "x is of type A". The terms and their types are simultaneously given by

- there are countable many variables of each type
- $\mathring{0}\varepsilon 1$ is a constant (1 is the terminal object)
- any map $A \xrightarrow{f} B$ induces a unary operation: $x\varepsilon A \longmapsto fx\varepsilon B$.
- there is an ordered pair operator: $x\varepsilon A,\ y\varepsilon B \longmapsto \langle x,y\rangle \ \varepsilon\ A\times B$.

The only primitive predicate of L(SET) is equality "=" which may hold only beetween terms of the same type. The formulas are formed from the atomic ones using the connectives $\neg$, $\wedge$, $\vee$, $\Longrightarrow$ and quantifiers $\exists x$ (or $\exists x\varepsilon A$ if $x\varepsilon A$), $\forall x$.

The language L(SET) admits an internal interpretation in topos theory ET in the following sense. For any formula φ resp. term $t\varepsilon B$ of L(SET) with free variables among $x_1\varepsilon A_1, .., x_n\varepsilon A_n$ one can define

a subobject $\{\langle x_1,..x_n\rangle \mid \varphi\} : A_1\times..\times A_n \longrightarrow \Omega$

resp. a map $\{\langle x_1,..x_n\rangle \mapsto t\} : A_1\times..\times A_n \longrightarrow B$

By induction on the length of resp. t.

*) This is an abstract of the author's paper "Logical and set theoretical tools in elementary topoi" (to appear in Springer Lecture Notes).

A formula φ having exactly the free variables $y_1,..y_n$ is called internally valid, denoted $\models \varphi$, iff $\{<y_1,..y_n> \mid \varphi\}$ factors through $1 \xrightarrow{true} \Omega$.

The language L(SET) together with internal validity as a notion of truth will be called the set theory SET defined over topos theory ET. The axioms and rules of intuitionistic logic hold in SET (i.e. are internally valid), however the modus ponens

$$\models \varphi \quad \text{and} \models (\varphi \Longrightarrow \psi) \text{ imply } \models \psi$$

requires the additional assumption that all free variables of φ occur free in ψ .

The atomic formula x=y with $x,y \varepsilon A$ admits a realization

$$\{<x,y> \mid x=y\} \quad : \quad A \times A \longrightarrow \Omega$$

and since Ω is a complete Heyting-algebra, SET may be viewed as an intuitionistic many-sorted Heyting-valued theory. In fact, SET is a set theory since for $x \varepsilon A$, $Y \varepsilon PA$ a membership predicate "$\in$" can be defined

$$x \in Y \ :\Longleftrightarrow \ (PA \times A \xrightarrow{ev} \Omega) <Y,x> = (1 \xrightarrow{true} \Omega)(\mathring{0}) .$$

For a subobject $A \xrightarrow{M} \Omega$ we put $\mathring{M} := (1 \xrightarrow{\overline{M}} PA)(\mathring{0})$ and write simply $x \in M$ instead of $x \in \mathring{M}$.

With the above definition of membership the following axioms of (many-sorted) set theory hold in SET: Extensionality, Empty Set, Sinletons, Binary and Arbitrary Unions, Powersets, and Separations-scheme.

To explain and establish the title of this note it remains to show that all considerations in topos theory ET might as well be carried out within the set theory SET. This will be done by internally characterizing all fundamental notions of ET within SET.

First, the maps $A \xrightarrow{f} B$ are (via their graphs) in 1-1-correspondence with functional relations $A \times B \xrightarrow{R} \Omega$, i.e. $\models \forall x \varepsilon A \ \exists ! y \varepsilon B <x,y> \in R$, such that the composition of maps corresponds to relational composition. Second, equality of maps can be characterized internally by

$$A \xrightarrow{f} B = A \xrightarrow{g} B \qquad \text{iff } \models \forall x \varepsilon A \; fx = gx \quad .$$

Hence the category structure is characterized and for the internal characterization of the remaining topos structure of ET we only give some examples:

1. A is a terminal object iff $\models \exists ! x \varepsilon A \; x=x$.

2. A commutative square

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

is a pullback

iff $\models \forall x \varepsilon A \; \forall y \varepsilon B \; (fx = gy \Longrightarrow \exists ! u \varepsilon D \; (hu = x \wedge ku = y))$.

3. $C \xrightarrow{f} PA$ is the exponential adjoint of $C \times A \xrightarrow{R} \Omega$

iff $\models \forall x \varepsilon C \; \forall y \varepsilon A \; (y \in fx \Longleftrightarrow \langle x,y \rangle \in R)$.

4. $1 \xrightarrow{o} \mathbb{N} \xrightarrow{s} \mathbb{N}$ is a natural number object iff the axioms of Peano are internally valid, i.e.

$$\models \quad \neg \exists n \varepsilon \mathbb{N} \quad o = sn$$

$$\models \quad \forall m,n \varepsilon \mathbb{N} \quad (sm = sn \Longrightarrow m=n)$$

$$\models \quad \forall X \varepsilon P\mathbb{N} \quad (o \in X \wedge \forall n \varepsilon \mathbb{N} \; (n \in X \Longrightarrow sn \in X) \Longrightarrow \forall n \varepsilon \mathbb{N} \; n \in X)$$

Having thus characterized all primitive notions of ET within SET, any statement α of topos theory ET may be translated into a set theoretical statement $\alpha*$ of SET such that α holds in ET if and only if $\alpha*$ holds in SET. However it should be pointed out that $\models \alpha*$ (i.e. $\alpha*$ holds in SET) is by definition of internal validity a statement of ET, so that we are not really leaving topos theory ET but rather look at it from a different point of view, the set theoretical one.

This set theoretical method of establishing results in topos theory is extremly useful since it reduces the arguments and constructions to their set theoretical nature (which we use in heuristic ideas anyway).

On different logical tools in elementary topoi

Gerhard Osius

In addition to our abstract "Topos theory $\subset$ Many-sorted intuitionistic set theory" (which is presupposed here) let us describe another natural interpretation of the set-theoretical language L(SET) of elementary topoi. This interpretation, called Kripke-Joyal-Semantics, goes essentially back to A. Joyal and has been used extensivły in topoi by A. Kock and Ch. Mikkelsen.

Starting with the interpretation, maps $X \to A$ in an elementary topos are called "elements of A at the stage (or: time, place) X". With respect to a fixed stage X we interprete the primitive operations of L(SET): - $X \to 1$ is the interpretation of the constant $\delta \varepsilon 1$

- For $A \xrightarrow{f} B$: $f(X \xrightarrow{a} A) := X \xrightarrow{a} A \xrightarrow{f} B$
- For $X \xrightarrow{a} A, X \xrightarrow{b} B$: $\langle a,b \rangle := X \xrightarrow{(a,b)} A{\times}B$.

Now let $\varphi(x_1,..x_n)$ be a formula of L(SET) with free variables among $x_i \varepsilon A_i$ and let $X \xrightarrow{a_i} A_i$ be elements at stage X (i=1,..n). We define by induction on the length of formulas what it means that " $\varphi(a_1,..a_n)$ holds (at stage X)", written "$\models_X \varphi(a_1,..a_n)$":

(0) $\models_X \quad X \xrightarrow{a} A = X \xrightarrow{a'} A$ iff $a=a'$.

(1) $\models_X \neg\varphi(a_1,..a_n)$ iff
For all $Y \xrightarrow{t} X$: $\models_Y \varphi(a_1t,..a_nt)$ implies $Y \simeq 0$.

(2) $\models_X \varphi(a_1,..a_n) \wedge \psi(a_1,..a_n)$ iff
$\models_X \varphi(a_1,..a_n)$ and $\models_X \psi(a_1,..a_n)$.

(3) $\models_X \varphi(a_1,..a_n) \vee \psi(a_1,..a_n)$ iff
There exist jointly epic maps $Y \xrightarrow{t} X$, $Z \xrightarrow{s} X$ such that $\models_Y \varphi(a_1t,..a_nt)$ and $\models_Z \psi(a_1s,..a_ns)$.

(4) $\models_X \varphi(a_1,..a_n) \Longrightarrow \psi(a_1,..a_n)$ iff

For all $Y \xrightarrow{t} X$: $\models_Y \varphi(a_1t,..a_nt)$ implies $\models_Y \psi(a_1t,..a_nt)$

(5) $\models_X (\forall y \varepsilon B)\ \varphi(y,a_1,..a_n)$ iff

For all $Y \xrightarrow{t} X$, $Y \xrightarrow{b} B$: $\models_Y \varphi(b,a_1t,..a_nt)$.

(6) $\models_X (\exists y \varepsilon B)\ \varphi(y,a_1,..a_n)$ iff

There exist $Y \xrightarrow{t} \!\!> X$ epic, $Y \xrightarrow{b} B$ such that $\models_Y \varphi(b,a_1t,..a_nt)$.

For an intuitive understanding of this definition, the maps $Y \xrightarrow{t} X$ (and $Z \xrightarrow{s} X$) should be viewed as a passage from the "later" stage Y (and Z) to the "present" stage (time) X. Thus (5) can be read "$(\forall y \varepsilon B)\ \varphi(y,a_1,..a_n)$ holds at stage X iff for all passages $Y \xrightarrow{t} X$ from later stages Y to X, $\varphi(b,a_1t,..a_nt)$ holds at Y for all $Y \xrightarrow{b} B$".

The important connection between this Kripke-Joyal-Sematics and the internal interpretation of the language L(SET) is given by the well-known (among specialists)

Metatheorem: $\models_X \varphi(a_1,..a_n)$ if and only if

$$X \xrightarrow{\langle a_1,..a_n\rangle} A_1\times..\times A_n \xrightarrow{\{\langle x_1,..x_n\rangle \mid \varphi(x_1,..x_n)\}} \Omega = \mathrm{true}_X .$$

Concerning the definable predicate $(-)\in M$ for a subobject $A \xrightarrow{M} \Omega$ which is the characteristic map of $B \rightarrowtail^{m} A$, we get in particular

for $X \xrightarrow{a} A$: $\models_X a \in M$ iff a factors through m .

According to the metatheorem Kripke-Joyal-Semantics and the internal interpretation of the language L(SET) provide "equivalent" logical tools in elemtary topoi and since each method has some advantage over the other both should be used (in some situations one may be more appropriate than the other).

Let us finally observe how Kripke-Joyal-Semantics can be simplyfied if the topos is generated by a class $\mathbb{G}$ of objects which is closed under subobjects. Then we can restrict the stages X, Y, Z.. above (i.e. the domains of elements) to objects in the class $\mathbb{G}$, and all previous results hold unchanged if we only change (6) into

$(6)_{\mathbb{G}} \models_X (\exists y \varepsilon B) \quad \varphi(y, a_1, \ldots a_n)$ iff

There exists a subclass $\mathbb{G}' \subset \mathbb{G}$, a <u>jointly</u> <u>epic</u> family $(Y \xrightarrow{t_Y} X)_{Y \in \mathbb{G}'}$ and a family of elements $(Y \xrightarrow{b_Y} B)_{Y \in \mathbb{G}'}$ such that for all $Y \in G'$: $\models_Y \varphi(b_Y, a_1 t_Y, \ldots a_n t_Y)$.

The important example is of course $\mathbb{G}$ = {open objects} for well-opened topoi. We note, that in this particular case $(6)_{\mathbb{G}}$ can again be replaced by the original (6) <u>if</u> in addition "<u>Support splits</u>". One further example is $\mathbb{G} = \{0,1\}$ for well-pointed topoi. In both examples the definitions (0)-(5), $(6)_{\mathbb{G}}$ can be simplified because of the particular nature of $\mathbb{G}$.

Model-theoretic methods in the theory of topoi

Gonzalo E. Reyes.

This paper (written in collaboration with Michael Makkai) makes more explicit and further develops the connections between coherent topos (in the sense of SGA4, Exp. VI) and certain first order theories which we call coherent. These latter are defined to be sets of formal expressions of the form $\varphi \Longrightarrow \psi$, where φ, ψ are formulas of a (many-sorted) language obtained (from the atomic ones) by using $\wedge$ (and), $\vee$ (or), $\exists$ (there is), $\uparrow$(true), $\downarrow$(false), $\approx$ (equal) as logical connectives. The logic of this language can be considered as the "geometric" (1 st order) logic of topoi inasmuch as the concepts expressible in it are preserved by (inverse image of) geometric morphisms.

We define the <u>category</u> of <u>models</u> of such a theory T in any pretopos $\mathcal{P}$ (in particular in a topos), $\mathrm{Mod}_{\mathcal{P}}(T)$ and we obtain:

0) (Existence of classifying topos)
If T is coherent, there is a coherent topos $\mathcal{E}(T)$ and a model M of T in $\mathcal{E}(T)$ such that the functor induced by this model $\hat{M}$: $\mathrm{Top}(\mathfrak{X}, \mathcal{E}(T))^{op} \to \mathrm{Mod}_{\mathfrak{X}}(T)$ is an equivalence, for every topos $\mathfrak{X}$. (Here Coh($\mathcal{E}$) is the pretopos of coherent objects of $\mathcal{E}$. The topos $\mathcal{E}(T)$ is called the <u>classifying topos for T</u>.

1) If $\mathcal{E}$ is coherent, then there is a coherent theory T such that $\mathcal{E}(T) \xrightarrow{\sim} \mathcal{E}$, i.e., any coherent topos is the classifying topos of some coherent theory.

2) ("Points are enough for classifying").
Let T be a coherent theory, let $\mathcal{E}$ be a coherent topos and let M be a model of T in Coh($\mathcal{E}$) which induces an equivalence

$$\hat{M} : \mathrm{Points}\ (\mathcal{E})^{op} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{Sets}}\ (T)$$

There $\mathcal{E}$ is the classifying topos for T.

A new proof that the Zariski topos classifies local rings is quickly obtained, as well as a description of the coherent theory of the être étale topos.

In order to prove the existence of points in (coherent) topoi, the first 2 results assure us that we need to construct models only for the coherent theory associated. We set up a formal system (whose details will appear elsewehre) and prove a completeness theorem. We combine this theorem with the method of diagrams of Tarski-Robinson to obtain some new and old results in a uniform manner:

2) above is thus obtained.

Delignes theorem (i.e., every coherent topos has a surjective boolean point $Sh(2^X) \xrightarrow{p} \mathcal{E}$ for some set X, where surjective means p^* faithful).

A coherent topos $\mathcal{E}$ has a surjective point iff for all coherent objects A,X,B,Y such that $A \rightarrowtail X$, $B \rightarrowtail Y$, if $A \times Y \vee X \times B = X \times Y$, then $A = X$ or $B = Y$ (This gives a characterization of classifying topos which are cotripleable over Set).

Existence of enlargements in the sense of Robinson.

Joyal's Theorem (unpublished, 2 years ago).

Let $\mathcal{H}$ be a pretopos with $\forall$. Then there is a small $\mathbb{P} \to \mathrm{Mod}(|\mathcal{H}|)$ where $\mathrm{Mod}(|\mathcal{H}|)$ is the category of functors from $\mathcal{H}$ into Set preserving the pretopos structure such that $\mathcal{H} \xrightarrow{ev} \mathrm{Set}^{\mathbb{P}}$ is a conservative functor preserving the pretopos structure and $\forall$.

There is an infinitary generallization of the completenes theorem (changing Set to Sh(B), i.e., a category of sheaves for a complete B.A. with the canonical topology).

As a corollary, we obtain:

Barr's Theorem.

Every topos has a surjective boolean point.

Let $\mathcal{H}$ be a pretopos with $\forall$. Then there is a complete Heyting algebra $\mathbb{H}$ and a conservative functor $M: \mathcal{H} \to Sh(\mathbb{H})$ preserving the pretopos structure, $\forall$, all (possible infinite) stable $\vee$ and all infs which exist in $\mathcal{H}$.

A Categorial Problem in Group Duality.

J. E. Roberts.

The classical Tannaka duality theorem for compact groups tells us that a compact group G can be recovered from the symmetric monoidal category $\mathcal{U}(G)$ of finite-dimensional continuous unitary representations (G-modules) as the group of monoidal natural unitary transformations of the forgetful functor of $\mathcal{U}(G)$ into the category of Hilbert spaces.

It is possible to construct categories which apparently have the same abstract structure as $\mathcal{U}(G)$ but without recourse to a group G. This raises the quenstion of whether one can improve on this classical duality theorem by characterizing symmetric monoidal categories of the form $\mathcal{U}(G)$ without referring explicitly to a forgetful functor into the category of Hilbert spaces.

The construction arose during the course of investigations into the superselection structure of elementary particle physics [1] and for this reason cannot be described adequately here. However some idea of the construction and the results can be gained from the following simplified mathematical setting.

Let M be a von Neumann algebra. Consider the category EndM whose objects are endomorphisms of M, i.e. normal identity-preserving *-homorphisms of M into M. The arrows of EndM are defined by $\mathrm{Hom}(\rho,\rho') = \{t \in M : t\rho(x) = \rho'(x)t,\ x \in M\}$. EndM has a lot of structure; $\mathrm{Hom}(\rho,\rho')$ inherits algebraic structure from M and EndM becomes a strict monoidal category if we define

$$(\rho\otimes\rho')(x) = \rho\rho'(x),\quad x \in M$$

$$s\otimes t = s\sigma(t) = \sigma'(t)s,\quad s \in \mathrm{Hom}(\sigma,\sigma'),\ t \in \mathrm{Hom}(\rho,\rho').$$

The identity object is the identity automorphism ι of M. The monoidal structure is best understood by considering M as a category with a single object and endomorphisms as endofunctors of M.

We are interested in certain full monoidal subcategories $\mathcal{J}$ of EndM which allow a (coherent) symmetry ε with $\varepsilon(\rho,\rho')^* = \varepsilon(\rho',\rho)$. The coherence of ε associates with any object ρ of $\mathcal{J}$ a representaion $\varepsilon_\rho^{(n)}$ of the permutation group $\mathbb{P}_n$. The main line of attack is to analyse the chain of representations $\varepsilon_\rho^{(n)}$, n=1,2... One condition which allows such an analysis is to suppose that $s \otimes t = 0$ implies $s = 0$ or $t = 0$; this might be useful in quite different contexts. More complete results follow by supposing that ρ has a left inverse φ. This is a positive linear mapping of M into M such that $\varphi(1) = 1$ and $\varphi(\rho(x)y) = x\,\varphi(y)$, $x,y \in M$. What is important is the way φ acts on the arrows of $\mathcal{J}$; we have $\varphi(\mathrm{Hom}(\rho\rho_1,\rho\rho_2)) \subset \mathrm{Hom}(\rho_1,\rho_2)$. Mappings of this nature arise in symmetric monoidal categories whenever there is a $\bar{\rho}$ such that the operations of tensoring by ρ and $\bar{\rho}$ are adjoint functors. We may then compute a class function of positive type

$$\varphi^n \varepsilon_\rho^{(n)} : \mathbb{P}_n \to \mathrm{Hom}(\iota,\iota).$$

It is multiplicative on disjoint cycles and takes the value $\varphi(\varphi(\varepsilon(\rho,\rho)^k)$ on a k+1 -cycle. Suppose now that M is a factor, i.e. $\mathrm{Hom}(\iota,\iota) = \mathbb{C}1_\iota$ and that ρ is irreducible, i.e. $\mathrm{Hom}(\rho,\rho) = \mathbb{C}1_\rho$. Then $\varphi(\varepsilon(\rho,\rho)) = \lambda 1_\rho$ where $\lambda \in \{0\} \cup \{\pm d^{-1}$: d integer}. λ determines the irreducible representations contained in the chain $\varepsilon_\rho^{(n)}$, n = 1,2,... and $|\lambda|^{-1}$ is called the dimension of ρ, $d(\rho)$.

In our context the full monoidal subcategory $\mathcal{J}_f$ of $\mathcal{J}$ generated by the finite dimensional irreducibles has a structure like the monoidal category generated by the continuous unitary irreducible representations of a compact group. The dimension function can be extended to $\mathcal{J}_f$ so that

$$d(\rho\rho') = d(\rho)d(\rho'), \qquad d(\rho\oplus\rho') = d(\rho)+d(\rho').$$

Every object has a decomposition as a direct sum of irreducibles. The sign of λ is important in the physical context (it gives the difference between Bose and Fermi statistics) but has little relevance for the structure of $\mathcal{J}_f$. In fact it is possible and convenient to adjust the symmetry so that $\lambda > 0$.

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Objects ρ with $d(\rho) = 1$ are automorphisms and possess an inverse ρ^{-1} in $\mathcal{J}_f$. As a consequence every object ρ has an adjoint $\bar{\rho}$ so that the functors of tensoring by ρ and $\bar{\rho}$ are adjoints. Once the symmetry has been adjusted as above, $\bar{\rho}$ may be constructed by following the group theoretical recipe. One takes subobjects ρ' and γ of ρ^{d-1} and ρ^d respectively, $d = d(\rho)$, which correspond to total antisymmetrization. One computes that $d(\gamma) = 1$ so that γ has an inverse and then shows that $\rho'\gamma^{-1}$ is an adjoint for ρ.

This completes the description of the basic structure of $\mathcal{J}_f$ although there is much further structure of a derivative nature. For example we have a bitrace on the arrows of $\mathcal{J}_f$ and an anti-unitary involutory functor on $\mathcal{J}_f$ commuting with *, the Hermitian conjugation, and mapping objects into their adjoints [2].

To date is has been possible to prove that $\mathcal{J}_f$ is associated with a compact group G only in the case where G is Abelian. One way of tackling the problem would be to show that $\mathcal{J}_f$ allows a monoidal embedding into the category of Hilbert spaces. This reduces to showing that a certain 3-cocycle in a non-Abelian cohomology theory is a 3-coboundary.

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Extensions of full embeddings and binding categories

J. Rosický

Given categories M, C, A and full and faithful functors $K: M \to C$, $T: M \to A$, it is often important to know whether there is a full and faithful functor $S: C \to A$ with $SK = T$. When K is dense, then a good candidate for S is a pointwise left Kan extension $\mathrm{Lan}_K(T)$ of T along K and in some papers it was (implicitely) used for this purpose. Under some restrictive suppositions it can be shown that $\mathrm{Lan}_K(T)$ is full and faithful whenever a full and faithful functor S with $SK = T$ exists. This result has applications for proving that there is no such S. In the general case the role of $\mathrm{Lan}_K(T)$ plays a new functor $L_K^*(T)$ which can be pointwise defined by a suitable colimit construction using the transfinite induction.

<u>Theorem 1</u>: Let M be small, A cocomplete and co-well-powered. Then $L_K^*(T)$ exists and $L_K^*(T)K = T$. If K is dense and cogenerating, then $L_K^*(T)$ is full and faithful whenever a full and faithful extension S exists.

Moreover, let F_M be a full subcategory of the functor category A^M consisting of full and faithful functors T having a full and faithful extension S such that the family $\{h: Tn \to T'm / n \in M\}$ is jointly epi for any $T, T' \in F_M$ and $m \in M$, F the full subcategory of the functor category A^C consisting of all full and faithful functors S with $SK \in F_M$ and $A^K: F \to F_M$ the functor given by the composing with K on the right. Then $L_K^*: F_M \to F$ is a functor left adjoint to A^K.

If the existence of S is replaced by the codenseness of K, then the first part of Theorem 1 remains true. Another result ensuring $L_K^*(T)$ to be full and faithful, which is convenient for the following application to binding categories, can be proved by means of more elaborate arguments originated from |5|.

A category is binding if any full category of algebras can be fully embedded into it. Any small category can be fully embedded into a binding category (see [1]) and the same holds for any concrete category under the following assumption (M): There is a cardinal a such that every ultrafilter closed under intersections of a elements is trival. It was proved by Hedrlín and Kučera and communicated in [2]. In [5] it was found a three-object category M full embeddability of which into an equational class A of unary algebras make A to be binding. This testing category M was taken as a full subcategory of a suitable binding category C of graphs and $Lan_K(T)$ yields a full embedding $C \to A$. Using the functor $L^*_K(T)$ the following result can be obtained.

Theorem 2: Let b a regular infinite cardinal. Then there is a three-object category M_b such that an equational class A of algebras having less than b-ary operations is binding iff M_b can be fully embedded into it.

Under non(M) there is a non-binding monadic category containing any small category as a full subcategory.

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Double categories as a 2-topos

Ross Street

Our purpose is to find some axioms for the elementary theory of the 2-category CAT of all categories from which a large part of category theory can be developed in a natural way, and yet weak enough to be satisfied by a topos, the 2-category of 2-categories, and other hyperdoctrines. Size considerations should appear in the development of the theory in an elementary categorical way and should not be imposed from outside the 2-category by an elaborate meta-set-theory.

Let K denote a 2-category with finite 2-limits. Then each object A is an object of objects for a category object

$$\mathrm{Arr}\ A \underset{d_1}{\overset{d_o}{\rightrightarrows}} A$$

in K, where $K(B,\mathrm{Arr}\ A) = K(B,A)^{\mathbf{2}}$; so that Arr A is the cotensor of $\mathbf{2}$ with A in K. This allows us to define the 2-category $[B,K]$ of internal functors from B to K as the 2-category of algebras for the 2-monad on K/B obtained by pulling back along $\mathrm{Arr}\ B \xrightarrow{d_o} B$ and using $\mathrm{Arr}\ B \xrightarrow{d_1} B$ to get an arrow into B.

An Object Ω in K is called an ideal classifier when there is an internal functor $\Omega \xrightarrow{T} K$ satisfying the following two axioms:

I C 1. the functor $J: K/\Omega \to K$ obtained from T has a right 2-adjoint;

I C 2. for each object B, the functor $K(B,\Omega) \to [B,K]$ obtained by "composing" with T is fully faithful. The internal functors in the image of the functor in I C 2. are called B-ideals. If $\Gamma X \xrightarrow{\delta_x} \Omega$ is the value of the right 2-adjoint of the 2-functor of I C 1 at X then ΓX represents partial maps defined in terms of ideals in the appropriate way. An object U is called small when $U \to 1$ is an ideal. Exponentiation to small powers exists.

Moreover, $-\times B: K \to [B,K]$ has a right 2-adjoint and takes the ideal classifier in K to an ideal classifier in $[B,K]$.

A 2-topos is a finitely 2-complete 2-category K with an ideal classifier Ω satisfying:

2T 1. each identity arrow is an ideal;

2T 2. a composite of ideals is an ideal;

2T 3. ideals are closed under cotensor with $\mathbf{2}$ in $[B,K]$.

Perhaps we should also suppose K 2-cartesian closed (not just exponentiation to small powers, but to all powers). In a 2-topos the theory of cocomplete objects works well and Ω is internally cocomplete modelling internally the sub-2-category of K consisting of the small objects.

A topos provides an example of a 2-topos by extending to the 2-category of ordered objects therein; the ideal classifier is the subobject classifier. Also the 2-category CAT is the motivating example of a 2-topos with Ω = Cat, the category of small categories.

The main example presented in our lecture is the 2-category DBL of double categories; that is, category objects in CAT. Here Ω = 2-Cat, the 2-category of small 2-categories appropriately regarded as a double category. There are many ways in which a 2-category can be regarded as a double category. Making use of -his observation and the fact that DBL is a 2-topos, we are able to turn much 2-category theory into formal category theory.

It would be good to internalize this example and, we conjecture that if K is a 2-topos then so is Cat(K) with a natural ideal classifier.

Monadic Functors and Convexity

Tadeusz Swirszcz

The well-known Theorem of Linton can be strengthened as follows:

<u>Theorem.</u> Let U: $\mathcal{B} \to \mathfrak{A}$ be a functor having a left adjoint, let $\mathfrak{A}$ have kernel pairs of retractions and let $\mathcal{B}$ have kernel pairs and coequalizers. If

(i) for each morphism f in $\mathcal{B}$, f is a coequalizer iff Uf is a coequalizer,

(ii) for each parallel pair (f,g) in $\mathcal{B}$, (f,g) is a kernel pair iff (Uf,Ug) is a kernel pair,

then the canonical comparison functor $\emptyset$: $\mathcal{B} \to \mathfrak{A}^{\mathbb{T}}$ is an equivalence of categories. ($\mathfrak{A}^{\mathbb{T}}$ is the Eilenberg-Moore category of the monad $\mathbb{T}$ determined by the functor U and its left adjoint.)

The assumption that each epimorphism in $\mathfrak{A}$ is a retraction in $\mathfrak{A}$ is superfluous.

The above Theorem is proving very useful in functional analysis. For example, using this Theorem we can prove that the forgetful functor

U: Compconv $\to$ Comp

is monadic. Compconv is the category of compact convex sets and continuous affine maps. Comp is the category of compact spaces and continuous maps.

The functor

$\mathcal{L}$: Comp $\to$ Compconv

left adjoint of U, is defined as follows: given a compact space X, $\mathcal{L}$(X) is the set of all probability measures on X, convex and compact with the *-weak topology. If f: X $\to$ Y is a continuous map, $\mathcal{L}$f: $\mathcal{L}$(X) $\to$ $\mathcal{L}$(Y) is defined by $\mathcal{L}f(\mu)(B) = \mu(f^{-1}(B))$ for μ in $\mathcal{L}$(X) and a borel subset B of Y.

The monadicity of the functor U gives the following axiomatic characterization of the centroid of a probability measure on a compact space:

<u>Theorem</u>. Let X be a compact space and let $\gamma : \mathcal{L}(X) \to X$ be a continuous map satisfying the following conditions:

(1) $\gamma(\delta_x^X) = x$ for each x in X, where δ_x^X is the Dirac measure at x,

(2) if $\lambda, \lambda_1, \lambda_2$ are elements of $\mathcal{L}(X)$ and $\gamma(\lambda_1) = \gamma(\lambda_2)$

then $\gamma[(1-t)\lambda_1+t\lambda] = \gamma[(1-t)\lambda_2+t\lambda]$ for $0 \leq t \leq 1$.

Define the convex combinations of elements of X as

$$\sum_{i=1}^{n} a_i x_i = \gamma\left(\sum_{i=1}^{n} a_i \delta_{x_i}^X\right).$$

Then X becomes a compact convex set such that $\gamma(\lambda)$ is the centroid of λ for each λ in $\mathcal{L}(X)$.

On the other hand, the category Conv of convex sets and affine maps is not monadic over the category of sets, in particular the forgetful functor U: Conv $\to$ Ens ist not monadic.

Let X be a set and let $(ⓢ : X\times X \to X)_{0 < s < 1}$ be a family of binary operations satisfying the following axioms:

(A) $x ⓢ x = x$

(B) $x ⓢ y = y \,(1-s)\, x$

(C) $(x ⓢ y) ⓣ z = x \,(s+t-st)\, \left(y \left(\frac{t}{s+t-st}\right) z\right)$

(D) $x_1 ⓢ y = x_2 ⓢ y \Longrightarrow x_1 = x_2$

for all x, x_1, x_2, y, z in X, $0 < s < 1$, $0 < t < 1$.

Define the convex combinations of elements of X as $(1-t)x+ty = x ⓣ y$ for x,y in X, $0 < t < 1$. Then X becomes a convex set.

Thus a convex set can be regarded as an abstract algebra. The axioms (A)-(C) are of an equational type, where as the axiom (D) is not. Since Conv is not monadic over Ens, there is no system of axioms of an equational type defining a convex structure on a set.

The Eilenberg-Moore category $Ens^{\mathbb{T}}$ of the monad $\mathbb{T}$ determined by the forgetful functor U: Conv → Ens and its left adjoint is the smallest category of equationally defined algebras over Ens containing the category Conv. The category $Ens^{\mathbb{T}}$ will be denoted by Sconv and its objects will be called semi-convex sets. It turns out that the pair $(X,(ⓢ)_{0<s<1})$ is a semi-convex set iff the family $(s)_{0<s<1}$ of binary operations in X satisfies the axioms (A)-(C).

It can be proved that Conv is a full and reflective subcategory of Sconv.

The semi-convex sets can be also described as follows:

Let K be a convex set and let φ: K → S be a surjection satisfying the following condition:

if $\varphi(x_1) = \varphi(x_2)$, $x \in K$, $0 \leq t \leq 1$, then $\varphi[(1-t)x_1+tx] = \varphi[(1-t)x_2+tx]$

Define the "convex combinations" of elements of a set S as

$$\sum_{i=1}^{n} a_i s_i = \varphi\Big(\sum_{i=1}^{n} a_i x_i\Big)$$

where $\varphi(x_i) = s_i$ for i=1,...,n. Then S becomes a semi-convex set.

On the other hand, each semi-convex set can be obtained in a such a way.

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Ultrafilters, ultrapowers and finiteness in a topos.

Hugo Volger

Alan Day raised the following question: What is the correct definition of an ultrafilter and an ultrapower in an elemtary topos in the sense of Lawvere and Tierney? To be more precise, we are looking for a generalization of the set-theoretic ultrapower construction which is internal, i.e. which can be described within the topos.

Thus an internal filter on an object X in a topos $\underline{E}$ should be a subobject of Ω^X with appropriate closure properties, which will be given as preservation properties of the characteristic function u: $\Omega^X \to \Omega$. In particular, an ultrafilter will be a Heyting algebra morphism from Ω^X to Ω. On the other hand, an external filter on X is a filter on $\underline{E}(1,\Omega^X)$, the set of subobjects of X.

Consequently, the construction of the ultrapower A^X/U should use the internal power A^X rather than the external power $A^{\underline{E}(1,X)}$ which might not even exist without further assumptions on external limits. However, there are several ways of defining the ultrapower. Usually the ultrapower A^X/U of a set A with respect to an ultrafilter U on the set X is defined as the quotient of A^X, obtained by identifying two functions if they agree on a subset in the filter. Okhuma observed in Ultrapowers in categories (Yokohama Math.J.14(1966), 17-37) that the ultrapower may be viewed as the filtered colimit of the partial powers A^Y with Y in U. This can be rephrased as follows. The ultrapower is defined as the quotient of $A^X|_U$, the set of partial functions with domain in U, obtained by identifying two functions if they agree on a subset in the filter. Therefore we will distinguish in an arbitrary topos between A^X/U and $A^X//U$, where the latter is the filtered colimit of the partial powers. However, we have not found yet an example to show that A^X/U and $A^X//U$ can be different.

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It will be shown that a filter U on an object X is an ultrafilter iff Ω is isomorphic to $\Omega^X//U$ ersp. Ω^X/U. Combining the preservation properties of the ultrapower functor we will prove that in every topos with the internal axiom of choice the ultrapower functor $(-)^X//U$ is a first order functor, i.e. it is left exact and preserves the propositional operations and the existential and universal quantification. This is an appropriate generalization of the basic result on ultrapowers of sets which states that the diagonal morphism from A to $A^X//U$ is an elementary embedding. It should be remarked that in the set-theoretic case also the axiom of choice has to be used. Therefore, $A^X//U$ will be regarded as the correct generalization of the set-theoretic ultrapower construction.

Any property which characterizes finite sets in the category of sets can be used to define a concept of finiteness in an arbitrary topos. We are interested in the following two variants which depend on ultrafilters. An object will be called ultrafinite iff it is isomorphic to all its ultrapowers. It will be called principally finite iff every ultrafilter on it is principal. Ultrafinite objects in categories with external ultrapowers in the sense of Okhuma have been studied by Day and Higgs (A finiteness condition in categories with ultrapowers, manuscript Lakehead Univ., Thunder Bay, 1973). We will prove that in a topos the class of ultrafinite resp. principally finite objects contains Ω and is closed under finite limits. However, we dot not know whether these classes are closed under the power set operation $\Omega^{(-)}$. As a side result, we obtain the description of the subtopos generated by a single object, which permits to generali[ze] W. Mitchell's results on free boolean topoi in the J. of Pure and Applied Math. 3(1973) to the non-boolean case.

Algebraic Theories in Topoi

G.C. Wraith

An object A of the object classifier $E[U]$ of a topos E may be identified either with i) a map of E-topoi $\tilde{A}: E[U] \to E[U]$ or ii) functors $A \otimes -: F \to F$ for every E-topos F, commuting with inverse image parts of maps of E-topoi. In consequence, $E[U]$ obtains a monoidal structure $(E[U], \otimes, U)$. We identify the category of monoids for $(E[U], \otimes, U)$ with the category of finitary algebraic theories in E. P. Johnstone has shown that if A is such a finitary algebraic theory in E, then the category of algebra for the monad $\tilde{A}^*$ on $E[U]$ is equivalent to the category of internal functors $F_A \to E$, where F_A is the internal category of finitely free A-algebras in E.

For any object T of $E[U]$ we show how to construct the free monoid on T in $(E[U], \otimes, U)$, and we use this construction to show that given a diagram of E-topoi

$$E \xrightarrow{X_O} E[U] \xrightarrow{T} E[U]$$

there exists a unique

$$E/_N \xrightarrow{X} E[U]$$

up to natural isomorphism, making the diagram

$$\begin{tikzcd}
 & E/_N \arrow[r, "s"] \arrow[dd, "X"] & E/_N \arrow[dd, "X"] \\
E \arrow[ur, "Q"] \arrow[dr, "X_O"'] & & \\
 & E[U] \arrow[r, "T"] & E[U]
\end{tikzcd}$$

commute.

If $\mathbb{E}$ is the fundamental locally internal category of E (see J. Penon 'Catégories localement internes') then an algebraic theory on E may be defined to be a map $\mathbb{E} \to \mathbb{E}$ of locally internal categories with a monad structure. Roughly speaking, this means that an algebraic theory on E is given by a strong monad on E/X for each X in E, commuting with pullback. We deduce that an algebraic theory is finitary if and only if this monad can be extended to all E-topoi, not just those of the form E/X.

Low dimensional cohomology of topoi

G.C. Wraith

If $\underline{E}$ is a Grothendieck topos, and $Ab(\underline{E})$ denotes the category of abelian groups in $\underline{E}$, then $Ab(\underline{E})$ is an abelian category with enough injectives (cf. work of Van Osdol). The cohomology functors $H^*(\underline{E},-): Ab(\underline{E}) \to Ab$ are defined to be the right derived functors of $Hom_{\underline{E}}(1,-): Ab(\underline{E}) \to Ab$. We have

$$H^0(\underline{E},A) = Hom_{\underline{E}}(1,A) = Top_{\underline{E}}(\underline{E}.\underline{E}/A),$$

so A may be taken to be any object here. The well known interpretation of $H^1(\underline{E},A)$ in terms of torsors allows us to interpret this for any group object A; we sketch this interpretation briefly:-

We identify a group object G in $\underline{E}$ with category object $(G \rightrightarrows 1)$. An internal functor $G \to \underline{E}$ is then just a right G-object (X,ξ) where the action $X \times G \xrightarrow{\xi} X$ satisfies the usual laws. Such a functor is flat iff (X,ξ) is a right G-torsor, i.e

i) $X \to 1$ is epic,

ii) $X \times G \xrightarrow{\langle P_1,\xi\rangle} X \times X$ is iso.

Denote by $TORS_{\underline{E}}(G)$ the full subcategory of $\underline{E}^{G^0}$ of left G-torsors. The following results are well known:-

1. $TORS_{\underline{E}}(G) \simeq Top_{\underline{E}}(\underline{E},\underline{E}^G)$.

2. $TORS_{\underline{E}}(G)$ is a groupoid,

3. $G \longmapsto TORS_{\underline{E}}(G)$ is a product preserving functor $Gp(\underline{E}) \to Cat$.

For any category C we denote the class of connected components of C by $\pi_0 C$. As a corollary of 3. we have that if $A \in Ab(\underline{E})$ then $\pi_0 TORS_{\underline{E}}(A)$ has a natural abelian group structure. We call a G-torsor trivial if it is isomorphic as a G-object to G itself with action given by multiplication.

4. A torsor is trivial iff it has an element.

<u>Theorem.</u> For $A \in \mathrm{Ab}(\underline{E})$, $H^1(\underline{E},A) \simeq \pi_0 \mathrm{TORS}(A)$ is a natural isomorphism.

The proof proceeds in two steps; (i) for any exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathrm{Ab}(\underline{E})$ one constructes an exact sequence

$$0 \to \mathrm{Hom}_{\underline{E}}(1,A) \to \ldots \to \mathrm{Hom}_{\underline{E}}(1,C) \xrightarrow{\delta} \pi_0 \mathrm{TORS}_{\underline{E}}(A) \to \ldots \to \pi_0 \mathrm{TORS}(C).$$

The connecting map δ is given as follows: given $1 \xrightarrow{c} C$, form the pullback

$$\begin{array}{ccc} P & \twoheadrightarrow & 1 \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & C \end{array} .$$

One shows that $P \times A \rightarrowtail B \times B \to B$ factors through $P \rightarrowtail B$ making P an A-object. Then one shows that P is an A-torsor, whose class we define to be $\delta(c)$. Simple diagram chasing arguments establish the exactness of the sequence.

(ii) One shows that the functor $A \longrightarrow \pi_0 \mathrm{TORS}_{\underline{E}}(A)$ is effaceable, i.e. for any $\alpha \in \pi_0 \mathrm{TORS}_{\underline{E}}(A)$ there is an injective homomorphism $A \rightarrowtail A'$ such that $\alpha \longrightarrow 0$ under the induced map

$$\pi_0 \mathrm{TORS}_{\underline{E}}(A) \to \pi_0 \mathrm{TORS}_{\underline{E}}(A').$$

This may readily be proved by taking $A' = A^T$ where T is an A-torsor representing α. It is an immediate consequence that $\pi_0 \mathrm{TORS}_{\underline{E}}(A) = \{0\}$ for any injective A. Standard comparison theorems of homological algebra now give the result.

The purpose of this talk is to suggest an analogous interpretation for $H^2(\underline{E},A)$. The ideas were suggested by a (very) partial understanding of some of Girauds' 'Cohomologie non-abêlienne'. Unfortunately I have not yet been able to give the effaceability part of the proof.

Diaconescu's theorem has a consequence that the functor

$$\mathrm{Cat}(\underline{E}) \to \mathrm{Top}_{\underline{E}} : \underline{C} \longmapsto \underline{E}^{\underline{C}}$$

preserves products. Hence if $A \in Ab(\underline{E})$, $\underline{E}^A$ is an abelian group in $Top_{\underline{E}}$, and we may consider $\underline{E}$-Topoi with $\underline{E}^A$-action and $\underline{E}^A$-equivariant maps. We denote this category by $Top_{\underline{E}}^A$. To give an $\underline{E}$-topos $\underline{I}$ an $\underline{E}^A$-action amounts to choosing for each object of $\underline{I}$ an A-action in such a way that all maps of $\underline{I}$ are A-equivariant. Multiplication $A \times A \to A$ makes $\underline{E}^A$ into an object of $Top_{\underline{E}}^A$ in a canonical way. We say that two objects $\underline{I}_1$, $\underline{I}_2$ of $Top_{\underline{E}}^A$ are locally equivariantly isomorphic if there exists an epic $K \to 1$ in $\underline{E}$ and an $\underline{E}^A{}_{/K}$-equivariant isomorphism $\underline{I}_1/K \longrightarrow \underline{I}_2/K$ of $\underline{E}/K$ - topoi. Call an object $\underline{I}$ of $Top_{\underline{E}}^A$ an <u>A-extension</u> of $\underline{E}$ if it is locally equivariantly isomorphic to $\underline{E}^A$; we denote the full subcategory of $Top_{\underline{E}}^A$ of A-extensions by $EXT_{\underline{E}}(A)$. We call an A-extension of $\underline{E}$ trivial if it is isomorphic to $\underline{E}^A$. The following results are analogous to those for torsors.

1. $EXT_{\underline{E}}(A)$ is a groupoid.

2. $A \longmapsto EXT_{\underline{E}}(A)$ is a product preserving functor.

3. An A-extension $\underline{I}$ is trivial iff it has an element, i.e a map of $\underline{E}$-topoi $\underline{E} \to \underline{I}$.

4. For any geometric morphism $\underline{E}' \xrightarrow{f} \underline{E}$, if $\underline{I}$ is an A-extension of $\underline{E}$, then $\underline{E}' \times_{\underline{E}} \underline{I}$ is a $f^*(A)$-extension of $\underline{E}'$.

<u>Conjecture</u>. There is a natural isomorphism $H^2(\underline{E},A) \simeq \pi_0 EXT_{\underline{E}}(A)$.

If $0 \to A \to B \to C \to 0$ is exact in $Ab(\underline{E})$, we define the connecting map $\delta H^1(\underline{E},C) \to \pi_0 EXT_{\underline{E}}(A)$ as follows: represent $\gamma \in H^1(\underline{E},C)$ by a map of $\underline{E}$-topoi $\underline{E} \xrightarrow{\gamma} \underline{E}^C$ and from the pullback

$$\begin{array}{ccc} \underline{I} & \longrightarrow & \underline{E}^B \\ \downarrow & & \downarrow \\ \underline{E} & \xrightarrow{\gamma} & \underline{E}^C \end{array}.$$

Since any two C-torsors are locally isomorphic, and we have a pullback diagram

$$\begin{tikzcd} \underline{E}^A \arrow[r] \arrow[d, dashed] & \underline{E}^B \arrow[d, dashed] \\ \underline{E} \arrow[r, "0"] & \underline{E}^C \end{tikzcd}$$

we get that $\underline{I}$ is an A-extension of $\underline{E}$, whose class denote by $\delta(\gamma)$.

The establishment of the exactness of the appropriate sequence proceeds in much the same way as for torsors, except that we have pullback diagrans of $\underline{E}$-topoi rather than of objects in $\underline{E}$. Assuming the conjecture, we get the following:

Corollary. Let $\underline{E}' \xrightarrow{f} \underline{E}$ be a geometric morphism, and let $\alpha \in H^2(\underline{E},A)$ be represented by an A-extension $\underline{I} \xrightarrow{p} \underline{E}$. Then f factors through p iff $\alpha \longmapsto 0$ under $H^2(\underline{E},A) \to H^2(\underline{E}',f^*(A))$

Proof.

$$\begin{tikzcd} \underline{I} \times_{\underline{E}} \underline{E}' \arrow[r] \arrow[d, shift left] & \underline{I} \arrow[d, "p"] \\ \underline{E}' \arrow[r, "f"] \arrow[u, dashed, shift left, "s"] & \underline{E} \end{tikzcd} \qquad \alpha \longrightarrow 0 \iff s \text{ in diagram}$$

This generalizes the situation well-known in the cohomology of groups, whereby elements of $H^2(G,A)$ classify extensions

$$1 \to A \to F \to G \to 1,$$

and $F \to G$ is universl for homomorphisms into G annihilating α, the element of $H^2(G,A)$ represented by the extension.

Problem: Does there exist a universal A-extension over an $\underline{E}$-topos $\mathbb{K}(A,2)$, so that $H^2(\underline{E},A) \simeq \pi_0 \mathrm{Top}_{\underline{E}}(\underline{E},\mathbb{K}(A,2))$?

Doctrines on 2-categories

V. Zöberlein

This is a summary of my doctoral thesis, which has been presented to a small audience already a year ago by F. Ulmer. 2-categories are denoted by $\underline{\underline{X}}$, its objects by $\underline{X}$, 1-morphisms by $\underline{X} \xrightarrow{F} \underline{Y}$ and 2-morphisms by $F \xrightarrow{\varphi} F'$ (just think of the 2-category $\underline{\underline{CAT}}$ of categories).

A doctrine (= 2-triple up to isomorphism) $\mathfrak{D} = [\mathfrak{D}, \mathfrak{E}, \mathfrak{M}, a, b, c]$ on $\underline{\underline{X}}$ consists of a 2-functor $\underline{\underline{X}} \xrightarrow{\mathfrak{D}} \underline{\underline{X}}$, of 2-transformations (natural up to isomorphism) $1_{\underline{\underline{X}}} \xrightarrow{\mathfrak{E}} \mathfrak{D} \xleftarrow{\mathfrak{M}} \mathfrak{D}\cdot\mathfrak{D}$ and of translations (= modifications) a, b, c, satisfying four nonobvious coherence-conditions, such that the usual triple-laws hold up to the isomorphisms a, b, c. In an obvious manner one defines $\mathfrak{D}$-algebras (up to isomorphism), 1-homomorphisms (u. t. i.) and 2-homomorphisms (six nonobvious coherence-conditions). One has an Eilenberg-Moore- and a Kleisli-decomposition of $\mathfrak{D}$.

A doctrine $\mathfrak{D}$ is called quasi-idempotent (lax-idempotent) iff in the diagramm of 2-transformations

$$\begin{tikzcd} \mathfrak{D} \arrow[r, shift left=2, "\mathfrak{D}\cdot\mathfrak{E}"] \arrow[r, shift right=2, "\mathfrak{E}\cdot\mathfrak{D}"'] & \mathfrak{D}\cdot\mathfrak{D} \arrow[l, "\mathfrak{M}"'] \end{tikzcd}$$

$\mathfrak{M}$ is adjoint to $\mathfrak{E}\cdot\mathfrak{D}$ (that is iff $\mathfrak{M}$ is coadjoint to $\mathfrak{D}\cdot\mathfrak{E}$). There is a dual notion of coquasi-idempotent doctrines.

For a doctrine $\mathfrak{D}$ there are equivalent:

a) $\mathfrak{D}$ is coquasi-idempotent.

b) For every $\mathfrak{D}$-algebra $\underline{\underline{X}} = [\underline{X}, M, \alpha, \gamma]$ the multiplication M is coadjoint to the unit $\mathfrak{E}_{\underline{X}}$.

c) For every $\mathfrak{D}$-algebra $\underline{\underline{Y}} = [\underline{Y}, M, \alpha, \gamma]$ and for every 1-morphism $\underline{X} \xrightarrow{F} \underline{Y}$ the 1-morphism $M\,\mathfrak{D}F$

$$\begin{tikzcd} \underline{X} \arrow[rr, "F"] \arrow[d, "\mathfrak{E}_{\underline{X}}"'] & & \underline{Y} \\ \mathfrak{D}\underline{X} \arrow[r, "\mathfrak{D}F"] & \mathfrak{D}\underline{Y} \arrow[ur, "M"'] & \end{tikzcd}$$

is "the" Kan-coextension of F along $E_{\underline{X}}$.

d) There is a translation $\mathfrak{D}\cdot\mathfrak{E} \xrightarrow{f} \mathfrak{E}\cdot\mathfrak{D}$, satisfying two coherence-conditions.

There is an almost coherence-free presentation of coquasi-idempotent doctrines, their algebras and homomorphisms by "bases". A base consists of $\mathfrak{D}$, $\mathfrak{E}$ like in a doctrine and of a family of coreflections $\mathfrak{D}\underline{X} \xleftarrow{M_{\underline{X}}} \mathfrak{D}(\mathfrak{D}\underline{X})$ to $\mathfrak{D}\underline{X} \xrightarrow{E_{\mathfrak{D}\underline{X}}} \mathfrak{D}(\mathfrak{D}\underline{X})$, $\underline{X} \in \underline{\underline{X}}$ with some (coherence-free) properties. These data are equivalent to a coquasi-idempotent doctrine. An object $\underline{X}$ admits a $\mathfrak{D}$-algebra-structure iff there is a coreflection $\underline{X} \xleftarrow{M} \mathfrak{D}\underline{X}$ to $\underline{X} \xrightarrow{E_{\underline{X}}} \mathfrak{D}\underline{X}$. A 1-morphism is a $\mathfrak{D}$-algebra-homomorphism iff the usual diagram commutes u. t. i. (no coherence). In this way one eliminates 16 of 19 coherence-conditions.

Simple examples are the coquasi-idempotent doctrines of coproducts, whose algebras and 1-homomorphisms are just coproduct-complete categories and coproduct-continuous functors. These doctrines are defined on $\underline{\underline{CAT}}$ respectively on the 2-category of preadditive categories. There is an idempotent doctrine on $\underline{\underline{CAT}}$, whose algebras are just categories with enough split equalizers.

More complicated is the general colimit-doctrine on $\underline{\underline{CAT}}$, whose algebras and 1-homomorphisms are just $\mathfrak{J}$-cocomplete categories and $\mathfrak{J}$-cocontinuous functors, where $\mathfrak{J}$ is a given class of (small) indexcategories. In order to get a strict 2-functor $\mathfrak{D}$ one has to look at the category $\mathfrak{J}/\underline{X}$ of indexcategories over $\underline{X} \in \underline{\underline{CAT}}$. The final (= confinal) functors between indexcategories over $\underline{X}$ form a calculus Σ of left-fractions and $\mathfrak{D}\underline{X} = \Sigma^{-1}(\mathfrak{J}/\underline{X})$ is the corresponding category of fractions. $\mathfrak{D}\underline{X}$ is equivalent (not isomorphic) to the Gabriel-Ulmer-completion of $\underline{X}$ under $\mathfrak{J}$-colimits. Because the canonical functor $P: \mathfrak{J}/\underline{X} \longrightarrow \mathfrak{D}\underline{X}$ in general has no adjoint (P is only a partially coadjoint of some functor in a higher universe), one has to work with "locally adjoints" of P defined on small full subcategories of $\mathfrak{D}\underline{X}$.