MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 28/1976

ON BOL AND MOUFANG LOOPS

June 26 to July 3, 1976

The conference on "Bol and Moufang Loops" was organized in honor of Professor Emeritus Ruth Moufang, Frankfurt and Professor Emeritus Gerrit Bol, Freiburg i.Br., and was chaired by Professor Hala O. Pflugfelder, Temple University, Philadelphia. The meeting was attended by 13 Mathematicians including the guest of honor Professor Bol. Unfortunately Professor Moufang was not able to attend.

Papers presented at the meeting dealt with geometrical, algebraical and topological problems concerning Bol and Moufang loops and also with the application of loops to differential equations.

The "Bol and Moufang Loops" conference was held concurrently with a conference on "Near-rings and Near-fields" under the chairmanship of Professor Betsch, Tübingen, a circumstance which offered a welcome opportunity for discussions and exchange between the members of both groups. It was possible to coordinate the scheduales of both meetings so as to enable all participants to attend paper presentations of mutual interest. This cooperation proved to be very stimulating and fruitful, and it was suggested that a meeting of both groups should be convened in 1978 in Scotland. Professor Meldrum of Edinburgh was asked to make the necessary arrangements.

Participants

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Artzy, R., Haifa

Bewersdorf, E., Duisburg

Bohun-Chudnyniv, V. Baltimore

Bol, G., Freiburg

Chein, O., Philadelphia

Gerber,P.D., Yorktown Heights, N.Y.

Karzel, H., Munchen

Pflugfelder, H., Philadelphia

Robinson, D.A., Atlanta

Sharma, B.L., Ile-Ife

Strambach, K., Erlangen

Abstracts

Andre, J. Some topics on Linear algebra over near-fields.

We generalize the well-known linear algebra over skewfield on nearfields. There are essential differences between left and right vector spaces over nearfields. Some structure theorems and applications on geometry will be stated.

Arnold, H.J. Geometrically equivalent vectorial loops.

A vectorial gruppoid $(\mathcal{G}, +, \mathfrak{I}, o)$: $\mathcal{G} = \{a, b, c, \ldots\}$

$$\left\{\begin{array}{l}\mathcal{G}\times\mathcal{G} \to \mathcal{G} \\ a, b \to a+b \in \mathcal{G}\end{array}\right\}, \left\{\begin{array}{l}\mathcal{G} \to \mathcal{P}(\mathcal{G}) \\ a \mapsto a\mathfrak{I} \subset \mathcal{G}\end{array}\right\}; \ o \in \mathcal{G},$$

satisfying the following axioms

G_1 $\bigwedge\limits_a o + a = a + o = a \wedge o\mathfrak{I} = \{o\} \subset a\mathfrak{I}$;

G_2 $\bigwedge\limits_a \bigwedge\limits_c \bigvee\limits_b a + b = c$; G_3 $\bigwedge\limits_a \bigwedge\limits_b b \in a\mathfrak{I} \Leftrightarrow b\mathfrak{I} \subset a\mathfrak{I}$;

G_4 $\bigwedge\limits_a \bigwedge\limits_b \bigwedge\limits_c b \in c\mathfrak{I} \Rightarrow (a+b)+c\mathfrak{I} = a + c\mathfrak{I}$

defines an affine line-geometry, and each affine line-geometry (see: Arnold, Die Geometrie der Ringe im Rahmen allgemeiner affiner Struckturen, Hamburger Mathematische Einzelschriften, NF, Heft 4) can be defined by a suitable vectorial gruppoid, which is not uniquely determined (in the general case). In the special case of weak-translation-set-transitivity (WST) the corresponding vectorial groupoids are loops satisfying $\bigwedge\limits_a \bigwedge\limits_b \bigwedge\limits_c (a+b)+c\mathfrak{I} = a + (b + c\mathfrak{I})$, each being a subloop. In addition to this each $a\mathfrak{I}$ is a normal subloop in the case TST (Translation-set transitivity). It is an open problem whether TST $\succ$ TGT holds or not.

Artzy, R. Some geometric aspects of loops.

An isostrophism ϕ is a map taking a loop L onto a loop L^{ϕ} over the same set. Isostrophisms are obtained by permuting the pencils of the three-net belonging to L. The isostrophisms form a group which is a homomorph of the infinite dihedral group. Indentification of L and

L^{ϕ} yields a loop law, and many well known laws are thus obtained, their mutual relations can be predicted, and the behavior of the loop nuclei can be explained. Combination of isostrophisms and isotopies yields theorems about the connection between the existence of collineations in a projective plane and the validity of a law in the multiplicative loop of the coordinatizing ternary ring the classification of finite loops.

Bewersdorf, E. Vectorial loops which are geometrically equivalent.

Let $G(t)$ be a loop and γ: G $U(G)$ a mapping from G into the lattice of subloops of G, obeying (1) $0\gamma = \{0\}$ (2) $a\varepsilon b\gamma$ $a\gamma c\ b\gamma$ (3) $(a+b)+c\gamma = a+(b+c\gamma)$ (4) $a+b\gamma = b\gamma + a$ for all $a,b,c, \varepsilon G$. There $(G(t),\gamma,0)$ is called a normal vectorial loop. If the full associative law holds in G, $(G(t),\gamma,0)$ is called a normal vectorial group. $(G(t),\gamma,0)$ is called geometrically equivalent to $(G'(t'),\gamma',0')$, if the ocrresponding geometries are isomorphic (see Arnold). The following loops are geometrically equivalent to groups for every γ obeying (1) - (4):

a) The hamiltonian, di-ass. loops with $(a,b,c) = 1$ a,b,c group

b) The commutative, finitely generated Moufang-loops, centrally nilpotent of class 2.

c) The loops G, where $K = \{U<G \mid a+U = U+a,\ (a+b) + U = a+(b+U);\ U \neq \{0\}\} \neq \{0\}$ and where G/K is a group.

Gerber, P.D. Some Applications of loops to Quadratic Differential equations.

If $x \in R^n$ and $Q(*)$ is an n dimensional commutative algebra, the equation $\dot{x} = x*x$ is a quadratic system. When n = 1, such equations are solvable by a linearization: $x = \dot{F}(u)u$, $\dot{u} = Bu+Cu+d$. We show that for n > 1, the linearization exist iff $x*x = F_u(0)\{x\}x$ and the composition $xy = F^{-1}(y)x + y$ defines a loop with the left inverse property. The infintessimal equations are derived and integrated in two cases: group and quadratic. We show finally that locally the loop is an algebra satisfying a 4th degree identity, and that the two special cases correspond respectively to right symmetric algebras and to left alternative algebras.

Chein, O. Moufang Loops of Small Order.

In [Chein, O., Moufang loops of small order I., Trans Amer. Math. Soc. 18(1974), pp. 31-51] all Moufang loops of order ≤ 31 were found and their properties were investigated. The reason that 31 was chosen as a cut-off point is that the main tool of the paper is a technique for constructing nonassociative Moufang loops of order 2n as extensions of nonabelian groups of order n and that the groups of order sixteen are more complex than those of lower order.

As a next step, we consider Moufang loops of order ≤ 63. (Again this stopping point is selected to avoid the necessity of considering groups of order 32.) More general constructive techniques are needed to find the loops in question, and lengthy combinatorial analyses are required to show that all loops have been found.

The present paper reports on these techniques and the results obtained. It is found that there are 155 nonassociative Moufang loops of order ≤ 63. All of these are solvable, satisfy Lagrang's theorem, have Sylow subloops and are isomorphic to all their isotopes.

Robinson, D.A. Concerning Small Bol Loops and Related Issues.

Recall that a loop (G,.) is a Bol loop provided that the identity $(xy\cdot z)y = x(yz\cdot y)$ holds for all $x,y,z \in G$. Clearly the Moufang loops are precisely those Bol loops which are di-associative. In this lecture it is shown that any Bol loop which is not a group must have at least eight distinct elements and that (up to isomorphism) there exist exactly six Bol loops of order eight which are not groups. In addition to some *ad hoc* constructions, a general "group-theoretic" construction of Bol loops is examined in detail and various open questions are posed.

Sharma, B.L. Left Loops which satisfy the left Bol Identity.

It is our purpose in this paper to initiate a study of the algebraic properties of a left loop Q(.) satisfying the identical relation

(1) $y(z.y\ x) = (y.z\ y)x$

for all $x,y,z \in Q$, It is shown that (1) implies right division in Q(.). By introducing a new operation o in Q, the connection between the left loop Q(.) and Bol loop Q(o) is established. Further we show that the role of neclei in the left loop theory is not the same as that in the loop theory. We conclude the paper by describing situations in which the left loop Q(.) is Moufang.

Pflugfelder, H. Self-adjoint subgroups of Bol loops.

For a Moufang loop L a self-adjoint subgroup is defined as a subloop H satisfying the identity $(H,H,L)=1$. The concept can be generalized for Bol loops where distinction must be made between right, left and middle self-adjoint subgroups H_ρ, H_λ, H_μ. Let $\mathfrak{M}_S^R$, ($\mathfrak{M}_S^L$) be a group generated by right (left) multiplications by elements of a set S. The following relationships exist between self-adjoint subgroups and their multiplication groups: (a) $\mathfrak{M}_{H_\rho}^R$ and H_ρ are isomorphic, (b) $\mathfrak{M}_{H_\lambda}^L$ and H_λ are anti-isomorphic, (c) in $\mathfrak{M}_H$ left and right multiplications commute, (d) if H is right, left and middle self-adjoint, then $\mathfrak{M}_H = \{L_a R_a\}$, and if H is of order k, then $\mathfrak{M}_H$ is of order at most k^2.

Strambach, K. Mehrfach scharf transitive Moufanf-Loops.

Operiert auf einem local compakten Raum M eine mehrfach scharf transitive Liesche Moufang-Loop L, so lässt sich auf M eine Addition + und eine Multiplication $\cdot$ so einführen, dass M die klassische Divisionsalgebra $\mathcal{O}$ der Oktaven wird, und $L=L(h)$ aus den Abbildungen $\{x \mapsto ax+b, a=0, b \in \mathcal{O}\}$ besteht; das Produkt $\beta\alpha$ der Elemente $\alpha: x \mapsto ax+b$ und $\beta: x \to cx+d$ wird in $L(h)$ gemässe der Regel $\beta\alpha: x \mapsto (ca)x + [c(bh)]h^{-1}+d$ festgesetzt, wobei $h \neq 0$ eine beliebige Oktave ist. Zwei Moufang-Loops $L(h_1)$ und $L(h_2)$ sind genau dann isomorph (abstrakt und als Transformationsloops, wenn $h_1 h_2$ reell ist.

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