MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 19/1977

Kombinatorik

8.5. bis 14.5.1977

Die diesjährige Tagung über Kombinatorik stand unter der Leitung von D. Foata (Strasbourg) und K. Jacobs (Erlangen). Ein wesentlicher Teil der Veranstaltung war der Frage gewidmet, welche Rolle Techniken aus dem Bereich der speziellen Funktionen (hypergeometrische Reihen, orthogonale Polynome, q-Erweiterungen) in der Kombinatorik - insbesondere in der Theorie der Partitionen, bei Enumerationsproblemen, bei der Untersuchung homogener Räume - spielen. Außerdem wurden Existenz-, Anzahl- und Optimierungsfragen bei Problemen kombinatorisch-geometrischer Natur (im Zusammenhang mit kombinatorischen Designs, Verbänden, endlichen Vektorräumen, Hypergraphen, Permutationen) behandelt, sowie symmetrische Funktionen und zahlentheoretische Fragestellungen.

Die Vorträge und Diskussionen wurden ergänzt durch ein Tutorial über hypergeometrische Reihen (R. Askey). Eine Sitzung war der Darstellung von kurzen und eleganten Beweisen gewidmet; in einer weiteren Veranstaltung wurden ungelöste Probleme präsentiert und diskutiert.

Das Forschungsinstitut mit seiner anregenden Atmosphäre bot einen idealen äußeren Rahmen für die erfolgreiche Durchführung der Tagung; auch die vielfältigen Möglichkeiten der Entspannung wurden gerne genutzt. Die Teilnehmer der Tagung danken dem Direktor des Instituts, Herrn Prof. Dr. M. Barner, und seinen Mitarbeitern für die organisatorische Unterstützung bei der Vorbereitung und der Durchführung der Tagung.

Teilnehmer:

Ahlswede, R. (Bielefeld)
Andrews, G.E. (University Park)
Arditti, J.-C. (Paris)
Askey, R. (Madison)
Beth, Th. (Erlangen)
Cohen, D.I.A. (Boston)
Daykin, D.E. (Reading)
Désarmeniên, L. (Strasbourg)
Deuber, W. (Bielefeld)
Dumont, D. (Strasbourg)
Duboué, M. (Paris)
Foata, D. (Strasbourg)
Greene, C. (Amherst)
Guinand, A.P. (Peterborough)
Halder, H.-R. (München)
Heise, W. (München)
Hoare, M.R. (London)
Ismail, M.E.H. (Hamilton)
Jacobs, K. (Erlangen)
Jamison, R.E. (Erlangen)
Kerber, A. (Aachen)
Knödel, W. (Stuttgart)
Koornwinder, T. (Amsterdam)
Lascoux, A. (Paris)
Leeb, K. (Erlangen)
Lenz, H. (Berlin)
Macdonald, I.G. (London)
Schützenberger, M.-P. (Paris)
Sprinkhuizen-Kuyper, I.G. (Amsterdam)
Strehl, V. (Erlangen)
Viennot, G. (Paris)
Vijayan, K.S.(Erlangen)

Vortragsauszüge:

R. Ahlswede: A combinatorial puzzle

Let X be an n element set and let $\mathcal{E} = \{E_i : i = 1,\dots,N\}$ be a set of N subsets of X. Consider $g(\mathcal{E}) = \sum_{i=1}^{N-1} |\{j : j>i , E_j \cap E_i \neq \emptyset\}|$ and define $G(n,N) = \max_{\mathcal{E}:|\mathcal{E}|=N} g(\mathcal{E})$. Write N as

$N = \sum_{t=m+1}^{n} \binom{n}{t} + N_m$, $0 \leq N_m \leq \binom{n}{m}$. An "upper sphere" S_n^N is defined by

$S_n^N = \{E : E \subset X , |E| \geq m+1\} \cup \{E_1,\dots,E_{N_m}\}$, where $|E_i| = m$ for

$i = 1,2,\dots,N_m$.

Theorem: For all natural n, N with $1 \leq N \leq 2^n$:

$$G(n,N) = \max_{S_n^N} g(S_n^N)$$

The structure of the "boundary" of an optimal sphere is unknown.

G.E. Andrews: Partitions, q-Series and Computers

Given $Q(n_1,\dots,n_j)$ a positive definite integral quadratic form, a computer can be used to examine the a_n in the equation

$$\sum_{n_1,\dots,n_j \geq 0} \frac{q^{Q(n_1,\dots,n_j)}}{(q)_{n_1}\cdots(q)_{n_j}} = \prod_{n=1}^{\infty} (1-q^n)^{a_n}$$

where $(q)_n = (1-q)(1-q^2)\dots(1-q^n)$. If the a_n are periodic the resulting identity is often important in the theory of partitions. When $j=1$, $Q(n) = n^2$, then the above identity reduces to the first of the famous Rogers-Ramanujan identities with the a_n periodic modulo 5. Numerous other examples have now been discovered; perhaps the simplest new (I think) result is

$$\sum_{n,m\geq 0} \frac{q^{n^2-nm+m^2}}{(q)_n (q)_m} = \prod_{n=1}^{\infty} (1+q^n)(1+q^{2n-1})^2 .$$

J.-C. Arditti: Combinatorial Compositions

Let $\mathcal{H}_1 = (E_1+e, \mathcal{F}_1)$, $\mathcal{H}_2 = (E_2, \mathcal{F}_2)$ be two hypergraphs with $E_1+e = E_1+\{e\}$; $E_1+e \cap E_2 = \emptyset$. ($\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is a subset of the power set of E_1+e (resp. E_2)). After Cunningham (Ph.D. thesis Waterloo) one defines $\mathcal{H} = (\mathcal{H}_1;(\mathcal{H}_2,e))$ to be the hypergraph $\mathcal{H} = (E = E_1+E_2, \mathcal{F})$. "+" is the disjoint union and

$\mathcal{F} = \{F_1 \mid F_1 \in \mathcal{F}_1, e \notin F_1\} \cup \{F_1 \cup F_2 \setminus e \mid F_2 \in \mathcal{F}_2 ; e \in F_1 \in \mathcal{F}_1\}$.

If $\mathcal{H}$ is given and can be written $\mathcal{H} = (\mathcal{H}_1;(\mathcal{H}_2,e))$ where $E = E_1+E_2$ is a true partition then $(\mathcal{H}_1;(\mathcal{H}_2,e))$ is a factorization of $\mathcal{H}$, and $\mathcal{H}$ is called factorizable. A committee of $\mathcal{H}$ is a subset $A \subset E$ such that $F_1, F_2 \in \mathcal{F}$, $F_1 \cap A \neq \emptyset \neq F_2 \cap A \Longrightarrow (F_1 \cap A) \cup (F_2 \cap \bar{A}) \in \mathcal{F}$.

Theorem: (Cunningham) $\mathcal{H}$ factorizable $\longleftrightarrow$ $\mathcal{H}$ has a proper committee.

One can perform the decompositions of $\mathcal{H}$ by finding a committee.

Theorem: The committees of $\mathcal{H}$ are "homogeneous subsets" of the graph $\mathcal{G}(\mathcal{H}) = (E,v)$ where $xy \in V \longleftrightarrow \exists F \in \mathcal{F}$ s.t. $x,y \in \mathcal{F}$.

These homogeneous subsets are easy to construct.

R. Askey: Orthogonal polynomials and combinatorics

A survey of orthogonal polynomials will be given. This will include the classical continuous orthogonal polynomials, the classical discrete orthogonal polynomials, and a new set of orthogonal polynomials found this year by J. Wilson.

This new set includes all of the other classical polynomials as limiting cases, sometimes in many ways. It also includes the 6-j symbols or Racah coefficients, which have been used by physicists in studying angular momentum, and many of the very interesting sets of polynomials found by Pollaczek. There are also basic hypergeometric function extensions fo all these polynomials. Various combinatorial applications will be mentioned.

T. Beth: Partitions of the niveaux of the power set into Steiner systems

Let X be finite set of v elements. Let k be an integer, $1\leqslant k\leqslant v$. A survey of the problem to partition $\binom{X}{k}$ into Steiner systems S(t,k,v) is given. The few known results are presented and compared. It is shown that the trick of using O-sum-systems in additive groups cannot be generalized further.

D.I.A. Cohen: Sieves for the Rogers-Ramanujan-Schur identities

From the Jacoby Product one finds that the number of ways in which n can be written as the sum of parts congruent to ± 1 (mod 5) is given by the formula

$$A_n = \sum_{-\infty}^{\infty} (-1)^m \, p\left(n - \frac{m(5m+1)}{2}\right) = p(n) - p(n-2) - p(n-3) + p(n-9) + \dots$$

By inclusion - exclusion the number of ways in which n can be written

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as the sum of distinct non-consecutive parts is

$B_n = \sum_{i=0}^{\infty} c_i\, p(n-i)$ where $c_i = \sum_{\pi(i)} \mu(\pi)$ and μ is a specific Möbius function, defined by a certain lattice. The goal is to show $A_n = B_n$ by combinatorial means.

D.E. Daykin: Lattice inequalities

(1) Let $\mathbb{R}_+$ be the reals ≥ 0 . Let D be a distributive lattice; for example D_m is subsets of $\{1,2,\dots,m\}$ ordered by inclusion. Let $\mu : D \longrightarrow \mathbb{R}_+$ satisfy $\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$ for all $x,y \in D$. For $A \subset D$ put $\mu(A) = \Sigma(x \in A)\mu(x)$. For $A,B \subset D$ put $J = \{x \vee y : x \in A , y \in B\}$ and $M = \{x \wedge y : x \in A , y \in B\}$.

Theorem: If $A,B \subset D$ there is an injection $A \times B \longrightarrow J \times M$, and $\mu(A)\mu(B) \leq \mu(J)\mu(M)$.

(2) A map $f : D \longrightarrow \mathbb{R}_+$ is an up-function if $x,y \in D$ and $x \leq y$ implies $f(x) \leq f(y)$. Write (μf) for $\Sigma(x \in D)\mu(x)f(x)$ and (μ) for $\Sigma(x \in D)\mu(x)$.

Theorem FKG: If f,g are up-functions then $(\mu f)(\mu g) \leq (\mu)(\mu fg)$. Generalizations of Chebychev's inequality are like this result.

(3) A map $\pi : D \longrightarrow D$ is a polarity if $\pi(\pi x) = x$ and $x \leq y$ implies $\pi y \leq \pi x$. It is easy to find the polarities of D_1 and D_2 .

Theorem: If D has a polarity it can be embedded in a product of copies of D_1 and D_2 with the obvious polarity.

Theorem: Let $\alpha = \alpha(x_1,\dots,x_n) : D \to D$ be a function defined by joins $\vee$ meets $\wedge$ and a polarity. If $A \subset D$ there is an injection $A \to \{\alpha(x_1,\dots,x_n) : x_i \in A\}$.

M. Duboué, M.P. Schützenberger: Une application des fonctions de Schur en théorie des nombres

Given a polynomial equation $p(x) = 0$ with roots $\alpha_1, \alpha_2, \ldots, \alpha_d$, one shows that the function in t, $\sum_{0 \leq n} (\Delta_{n+1}/\Delta_1) \cdot t_n$ is rational where Δ_{n+1} denote for each n the product of the $\frac{d \cdot d-1}{2}$ differences $\alpha_j^{n+1} - \alpha_i^{n+1}$. This derives from an application of the theory of Schur symmetric functions. When $p \in \mathbb{Z}[x]$ the coefficients (Δ_{n+1}/Δ_1) have amusing arithmetic properties which generalize those well known for the 2nd-degree recurrence sequences.

D. Dumont: La génération de Seidel des nombres de Genocchi

On résume l'exposé que donna Seidel en 1877. Il définit, à partir d'une suite initiale de rationnels $a^o = (a_n^o)$, la matrice $(a_i^j)_{i,j \geq 0}$ définie par les récurrences $a_i^j = a_i^{j-1} + a_{i+1}^{j-1}$. Seidel s'interesse à la correspondance qui à a^o, de fonction génératrice exponentielle $f(t)$, fait correspondre $a_o = (a_o^j)$, de fonction géneratrice $e^t \cdot f(t)$. Les suites de Bernoulli, et de Genocchi, apparaissent comme des suites "presque fixes" dans cette correspondance.
Seidel déduit une suite-double triangulaire d'entiers naturels qui engendre les nombres de Genocchi sur la diagonale.
On donne de cette suite-double une interprétation combinatoire, et on la relie à celles déjà connues des nombres de Genocchi.

D. Foata: A combinatorial proof of Mehler's identity on Hermite polynomials

The Hermite polynomials $H_n(x)$ $(n \geqslant 1)$ may be defined by means of their exponential generating function

$$1 + \sum_{n \geqslant 1} \frac{u^n}{n!} H_n(x) = \exp(2xu - u^2) .$$

The bilinear extension

$$1 + \sum_{n \geqslant 1} \frac{u^n}{n!} H_n(x) H_n(y) = (1-4u^2)^{-1/2} \exp\left[\frac{4xyu - 4(x^2 + y^2)u^2}{1-4u^2}\right]$$

is known as Mehler's identity.

A combinatorial proof of this identity is obtained that makes use of the partitional complex algebraic set-up introduced by Schützenberger and the author, together with the simple observation that $H_n(x)$ is the generating polynomial for the number of fixed points over the set of involutions of the interval $\{1,2,...,n\}$.

A.P. Guinand: Landau's problem. Is there an arithmetic relation between the prime numbers and zeros of the Riemann zeta-function?

In his "Handbuch der Lehre von der Verteilung der Primzahlen", and elsewhere, Prof. Landau conjectured that there must be some arithmetic connection between the prime numbers and the complex zeros of the Riemann zeta-function. At the International Congress in 1912 at Cambridge he referred to this as "einen geheimnisvollen unbekannten Zusammenhang". Since then it has become clear that the connection can be regarded as a Fourier transformation relating the sequence of logarithms of powers of primes to the sequence of zeros of $\zeta(1/2 + it)$. Whether or not the connection can be expressed in a more explicit "arithmetic" form still remains obscure. Some numerical evidence supporting such a possibility will be presented.

H. R. Halder: On refining partitions

If $M=\bigcup M_i$ (resp. $n=\Sigma n_i$) and $M_i=\bigcup M_{ij}$ (resp. $n_i=\Sigma n_{ij}$) are partitions, we say that the partition $M=\bigcup M_{ij}$ (resp. $n=\Sigma n_{ij}$) is finer than the partition $M=\bigcup M_i$ (resp. $n=\Sigma n_i$). Let $P(M)$ be the set of all partitions of the set M with $|M| = n\in\mathbb{N}$. The symmetric group S_M operates in a natural way on $P(M)$ and on the set $C(M)$ of all maximal chains of $P(M)$. Let $h(n)$ be the number of orbits of S_M on $C(M)$.

Theorem 1: $h(n)$ is the Euler-number E_{n-1}.

Let $f(n)$ be the number of maximal chains in the set $P(n)$ of all partitions of $n\in\mathbb{N}$.

Theorem 2: $f(n) < (3,4)^n \cdot n^{\frac{n}{2}}$.

This result improves a theorem of Erdös, Guy and Moon (1975).

Let $P(n)$ be the set of all non trivial partitions of n and ϕ be a function on $P(n)$ with $\phi(n=\Sigma n_i)\in\{n_i\in\mathbb{N}|i=1,\ldots,k\}\setminus\{1\}$. We say that a maximal chain $(p_1,\ldots,p_n)$ belongs to ϕ, if p_{i+1} comes from p_i by splitting one part of p_i of size $\phi(p_i)$ for $i=1,\ldots,n-1$.

Theorem 3: The number of chains belonging to ϕ is $c(n)$, where $c(n)$ is defined by $c(1)=1$ and $c(m) = \overline{\sum_{i=1,\ldots,[\frac{m}{2}]}} c(m-i)c(i)$.

Theorem 2 results from theorem 3.

M.R. Hoare: Markovian combinatorics and the special functions of discrete variable

It is possible to define combinatorial models having a Markovian character imposed by constraints between stages of a compound enumeration process. When based on the classical occupancy distributions (binomial and negative hypergeometric) this type of "dynamic occupancy-problem" has an intimate connection with the orthogonal polynomials of a discrete variable (Meixner and Hahn systems), specifically the latter prove to be eigenvectors of the transition matrices for the constrained trials, which are themselves modified convolutions of the original distributions. Moreover, the spectral representation of the same matrices takes the form of an Erdelyi-type bilinear expansion in the polynomials concerned.

Some of the simplest soluble processes of this type (the three and four-box problems) are described along with their transition operators, projection formulae and Erdelyi expansions. These models have practical application, e.g. in statistical mechanics and storage theory; mathematically their interest lies in the possibility of synthesizing kernels with "built-in" positivity as well as in providing an entirely new origin for the discrete special functions as eigenvectors of sum-operators.

E.H. Ismail: Weighted permutation problems and special functions

A number of illustrations are given of how to transform some combinatorial problems to equivalent problems involving special functions and vice versa. The main tool used is MacMahon's Master Theorem. For example, consider $n = n_1 + n_2 + n_3$ different objects given initially in three boxes, n_1 in the first, n_2 in the second and n_3 in the third. Rearrange these objects so that the number of objects in each box does not change. A result of Askey and Gasper is proved by showing that there are more rearrangements in which an even number of objects move to another box than there are rearrangements where an odd number of objects move to another box. Another interpretation of the same result can be formulated in terms of distances in Hamming schemes. Related results of Koornwinder will also be phrased in combinatorial language and will be proved combinatorially in some special cases.

R. Jamison: Dimensions of hyperplane spaces over finite fields

In the study of hyperplane coverings of a vector space V over a finite field F, the "reduced" covers are of natural interest - those coverings with hyperplanes which have no proper subcovers. Such covers can be rather large and the attempt to find an upper bound on their size led to the following construction. Let K be any field and in the K-linear space of all functions from V into K take the span $L_*(V,K)$ of the characteristic functions of all hyperplanes in V.

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If char K $\nmid$ char F, then this space consists of all functions from V to K. But if char K = char F, then $L_*(V,K)$ degenerates and its dimension gives a bound on the size of maximal reduced covers. This maximal size also can be interpreted as a combinatorial invariant of a certain notion of "convexity" over finite fields. As such it yields analogues of the classical theorems of Caratheodory and Helly for ordinary convex sets.

T.H. Koornwinder: Orthogonal polynomials as zonal spherical functions

First, harmonics and spherical functions are introduced on a general compact homogeneous space G/K , where (G,K) is a Gelfand pair. Then the special case of the Johnson scheme (i.e. the class of all N-subsets of a set of cardinality v) is considered. This is the homogeneous space $\mathfrak{S}_v / \mathfrak{S}_N \times \mathfrak{S}_{v-N}$. The spherical functions turn out to be the Hahn polynomials of a certain order (a result due to Delsarte). Finally, returning to the general case, we discuss two types of product formulas with positive kernels for spherical functions, and we emphasize the relationship with association schemes.

A. Lascoux: q-analogues et fonctions de Schur

Some identities with determinants of binomial coefficients have a q-analogue, which is the specialization of an identity between symmetric polynomials (when one takes for values of the indeterminates the successive power of q). We want to give some of these identities which are closely linked with algebraic-geometric problems.

K. Leeb: <u>Some problems and conjectures in the area of chain decompositions and packing</u>

Packing of Π() and Π() is known. Pack $\pi(K_m^n)$, i.e. cardinal products of complete bipartite graphs.

Pack parametersets, especially partitions.

Spernerize parametersets, partitions. Spernerize partitions according to fine type, not just number of classes.

Lower ideals in ${}^d\omega$ cease to be Sperner at $d \geq 3$, yet Π of lower ideals in ${}^2\omega$ (Purdy) or even Π of Sperner lower ideals in ${}^d\omega$ could be.

<u>{Lower ideals in ${}^d\omega$} by $\subseteq$: Sperner:</u>

1) Gauss-coefficients = (lower ideal in ${}^2\omega \subseteq$ principal ideal).

Two approaches: a) Periodic solution on Tori $\underbrace{n \cdot 2^{\alpha}}_{n-1}$ prime factors

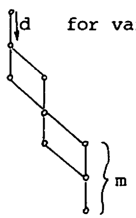
b) Packing by several layers.

2) Lower ideals in ${}^2\omega$ with distinct rows: Erdős-Lindström conjecture for code order.

Possible generalizations: a) Sequence of English number system stores:

$10 \mapsto 01$ replaced by $m\nu \mapsto 0\nu{+}1$

where $\nu+1 \leq n$

b) Iterations of  for various m,d:

"skew cube"

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{Lower ideals in ${}^{d}\omega^{\supseteq A}_{\subseteq B}$} by $\subseteq$: Spernerize!

Spernerize Π bipartite graphs with LYM.

Pack {lower ideals in ${}^{d}\omega$} by $\subseteq$.

Pack Ehresmann quintets of the simplest kinds.

Spernerize Ehresmann quintets according to fine type, not just rank.

H. Lenz: Numerical estimates for block designs and latin squares

Most of the recursive methods of Bose, Shrikhande, Hanani, Wilson, and others for the existence proof of block designs or orthogonal latin squares can be extended to numerical estimates of different resp. non-isomorphic designs of given parameters. The following basic idea is used: In the (0,1)-incidence matrix of a given incidence structure with u points the ones in each of the b (number of blocks) columns are replaced by auxiliary matrices $A_1,\dots,A_k$. If there are at least x choices for the auxiliary matrices then there are at least x^b choices in total, hence at least $v!^{-1}x^b$ non-isomorphic extended incidence structures (with v points). This simple method may be applied to block designs, quadruple systems, and transversal designs (that is to orthogonal latin squares). See the results of Doyen-Valette (MZ 120) and R.M. Wilson (MZ 135) on Steiner triple systems; applications of the basic idea to other structures appear promising.

I.G. Macdonald: Plane partitions

A survey of some results (some true, some false, some conjectural) on generating functions for plane, solid, ... partitions. A proof of MacMahon's conjecture on the generating function for symmetrical plane partitions.

V. Strehl: A combinatorial interpretation of modified Ghandi-polynomials

Modified Ghandi-polynomials were introduced by Dumont in his work on combinatorial interpretations of the Genocchi-numbers. Here a new interpretation is presented: these polynomials are - up to a factor of the form 2^{2n} - the enumerator polynomials for peaks in alternating permutations according to their height. The 2^{2n}-factor reflects symmetry properties of binary trees. The proof uses the symmetry results obtained by Dumont/Foata, and explicit expressions - given by Carlitz - for some relevant generating functions.

G. Viennot: Fonctions équidistribuées sur le groupe des permutations

Le but de cet exposé est de démontrer une conjecture de D. Foata et M.P. Schützenberger sur l'équidistribution de certaines fonctions définies sur le groupe des permutations.

Certains problèmes d'énumération de permutations sont relatifs à la succession des montées et descentes appelée forme (ou "up-down sequence") et qui est la suite des relations > ou < satisfaites par deux valeurs consécutives. D. Foata et M.P. Schützenberger ont découvert et démontré une remarquable propriété d'équidistribution des avances des permutations ayant une forme donnée (une avance étant une montée de la permutation inverse). En considérant la notion duale coavance ils ont conjecturé qu'il y avait équidistribution simultanée des avances et des coavances des permutations ayant une forme donnée, c'est-à-dire que dans un certain sens ces deux variables sont distribuées de façon indépendante (voir exposé aux journées de Bordeaux, Combinatoire et Informatique, Juin 1975, publications de l'université de Bordeaux I).
Nous prouvons cette conjecture en donnant des formules exactes de dénombrement des permutations ayant une forme donnée et un système donné d'avances et coavances. Nous retrouvons en particulier, après les formules de Carlitz, Niven et Foulkes, le dénombrement des permutations ayant une forme donnée, ainsi qu'avec Françon celles ayant un système donné de pics, creux, doubles montées et doubles descentes.

V. Strehl (Erlangen)