MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 26/1977

Representations of Semisimple Lie Groups

26.6. bis 2.7.1977

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The preceeding few years have seen the fruition of Harish-Chandra's monumental program, whose aim was a complete understanding of the Plancherel decomposition of $L^2(G)$, for semisimple Lie groups G. At the same time, several important questions that were left open by Harish-Chandra have been answered, and certain aspects of his program have been looked at from other points of view. The mathematicians responsible for this process of completion and clarification are geographicly scattered, and many of them felt that it would be useful to gather in one place, in order to get a comprehensive view of the present state of the subject. We are grateful to the Mathematische Forschungsinstitut Oberwolfach for making such a gathering possible, in virtually ideal surroundings.

By agreement among the participants, the lectures were fewer in number, but of much longer duration than is usual at mathematical conferences. This enabled the lecturers to describe their work in considerable detail. The lectures were followed, and often interrupted, by lively discussions.

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MATRIX COEFFICIENTS OF ADMISSIBLE REPRESENTATIONS

by

W. Casselman

1. Analytic systems of differential equations of finite codimension.

Let W be a finite-dimensional complex vector space, $E = \mathrm{End}_{\mathbb{C}}(W)$, and X an open subset of $\mathbb{C}^n$. A linear holomorphic differential operator on W-valued functions on X is a linear combination of operators of the form

$$\sum F_{\underline{m}}(x)\,(\partial/\partial x)^{\underline{m}} \quad (\underline{m} \in \underline{N}^n)$$

where each $F_{\underline{m}}$ is a holomorphic E-valued function on X. If $\{D_\alpha\}_{\alpha\in A}$ is a set of such operators, it corresponds to a system of partial differential equations for holomorphic W-valued functions Φ: $D_\alpha\Phi = 0$ for all $\alpha \in A$. Of course if Φ satisfies this sytem and $\{L_\alpha\}_{\alpha\in A}$ is a set of operators defined on some open subset $X' \subseteq X$, then on X' the function Φ also satisfies $\Sigma\, L_\alpha D_\alpha \Phi = 0$. The best way to discuss the situation is in terms of sheaves: for each $x \in X$ let $\mathfrak{D}_x$ be the <u>germs</u> of linear holomorphic differential operators on W-valued function defined in the neighborhood of x, and let $\mathcal{J}_x$ be the left $\mathfrak{D}_x$-ideal generated by $\{D_\alpha\}$. I say that the system of equations has <u>finite codimension</u> at x if $\mathfrak{D}_x/\mathcal{J}_x$ is a finitely generated module over $\mathcal{O}_x$, the germs of holomorphic functions at x. There seems to be no very concrete way to phrase this, but there are some fairly concrete conditions which are at least sufficient if not necessary (see §3.3) and which may be stated in terms of the symbols of the $\{D_\alpha\}$.

<u>The set of</u> $x \in X$ <u>where the system has finite codimension is the complement of an analytic subset</u> (3.1.11). If this set is non-empty, therefore, it is open and dense in X. On this set the $\mathcal{O}_x$-modules $\mathcal{D}_x/\mathcal{J}_x$ are free (3.1.10), and their rank locally constant. In other words, one has a vector bundle over this set such that $\mathcal{D}_x/\mathcal{J}_x$ consists of the germs of its local sections at x. The set where $\mathcal{D}_x/\mathcal{J}_x$ is not finitely generated is called the <u>singular set</u> of the system. A classical example: the system corresponding to

$$D_1 = x_2\partial/\partial x_1 - x_1\partial/\partial x_2$$

$$D_2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$$

on $\underset{\sim}{C}^2$. Here $\mathcal{D}_x/\mathcal{J}_x$ has rank 2 precisely where $x_1^2 + x_2^2 \neq 0$. (On $\underset{\sim}{R}^2$, D_1 is an infinitesimal rotation and D_2 the Laplacian.)

Suppose that X is connected and simply connected, and that the system has finite codimension on all of X. Let R_x be the fibre of the corresponding vector bundle at x, which is in a natural way a module over E. <u>The correspondence which takes</u> Φ <u>to</u> $D\Phi(x)$ <u>induces an isomorphism of the space of</u>

solutions of the system $\{D_\alpha \Phi = 0\}$ with $\mathrm{Hom}_E(R_x, W)$ (3.4.3). In other language, this is a more or less well known fact.

The set of singularities of the system may itself be highly singular as an analytic space, and this is a source of much difficulty. The least complicated case is when it is a divisor with only normal crossings--i.e. when locally it looks like the union of several coordinate hyperplanes--and in general one must apply results of Hironaka to reduce to this situation. It may be worthwhile to keep in mind that in the applications to matrix coefficients that the singular set will be the union on a maximal split torus of the walls of the Weyl chambers and certain hyperplanes "at infinity", in which case resolution of singularities is rather simple.

If the singular set is locally a divisor with normal crossings, then the system is said to have a regular singularity at x if the solutions of the system (which may be multi-valued in the neighborhood of x) are of moderate growth. Coordinatizing and assuming $x = 0$ for convenience, this means that there exist a finite number of complex exponents $\underset{\sim}{s} \in \underset{\sim}{C}^n$ and integral exponents $\underset{\sim}{m} \in \underset{\sim}{N}^n$ such that the solutions are finite combinations of functions of the form

$x_1^{s_1} \ldots x_n^{s_n} (\log x_1)^{m_1} \ldots (\log x_n)^{m_n} \Phi$, where Φ is holomorphic at 0 (3.5.1). One defines regular singularities in the general case by resolution of the singular set: one says x is a regular singular point if the system at any point in the resolution lying over x is one for a certain lifted system. <u>The points where</u> $\mathcal{J}$ <u>has neither finite codimension nor a regular singularity form on analytic subset</u> (2.2.5).

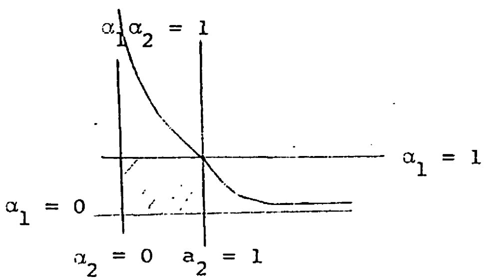
It is generally difficult to decide directly whether or not a given point is a regular singularity or not, and therefore it is extremely useful to know that if $\mathcal{J}$ <u>has regular singularities at the places where the singular set is a non-singular divisor, then it has regular singularities everywhere</u>. This is essentially a consequence of the well known result in the theory of complex variables which says that a function holomorphic in a set of codimension ≥ 2 is holomorphic everywhere. The condition of regularity at the points where the singular set is a non-singular divisor is often simple to verify in applications, because one often has a <u>quasi-simple</u> singularity (see §2.4).

Another generalization of a classical result is that <u>if</u> $\mathcal{J}$ <u>has a regular singularity at</u> x <u>then any formal power series solution of the system at</u> x <u>is actually convergent</u> (3.5.2). If the singular set is a divisor with normal crossings at x then the proof of this is essentially the same as that of the classical result for ordinary differential equations.

2. Let G be a semi-simple group with finite center and such that $Ad(G) \subseteq G^{ad}(\mathbb{C})$. Let K be a maximal compact, $G = KAN$ a Iwasawa decomposition. Let $\mathfrak{g}$, etc. be the corresponding complexified Lie algebras. If (τ,U) is a finite-dimensional representation of $K \times K$, let $C^\infty_\tau(G)$ be the corresponding space of τ-spherical functions on G. For any $X \in Z(\mathfrak{g})$ and $F \in C^\infty_\tau(G)$, XF lies again in $C^\infty_\tau(G)$. There exists a unique differential operator $\Pi_\tau(X)$ on A_{reg} such that this diagram commutes:

$$\begin{tikzcd} C^\infty_\tau \arrow[r, "X"] \arrow[d, "\text{resolution}"] & C^\infty_\tau \arrow[d, "\text{resolution}"] \\ C^\infty(A_{reg}, U^{K_m}) \arrow[r] & C^\infty(A_{reg}, U^{K_m}) . \end{tikzcd}$$

Now consider $A^{\cdot}$ to be embedded in $\mathbb{C}^\Delta$ by means of a set Δ of single roots for AN. Let $\mathfrak{N}_0$ be the union of the coordinate hyperplanes $\{\alpha = 0\}$ $(\alpha \in \Delta)$ and $\mathfrak{B}_s$ the union of the hypersurfaces $\{\gamma^2 = 1\}$ (γ a positive root). Let $\mathfrak{N} = \mathfrak{N}_0 \cup \mathfrak{B}_s$. For $SL_3(\mathbb{R})$ one has the following picture on the reals:



The shaded region is the Weyl chamber $A^- = \{a \in A \mid |\alpha(a)| < 1\}$ for all $\alpha \in \Delta\}$. Note that the roots are multiplicative characters from A to the positive roots, and extend to monomials on $\mathbb{C}^\Delta$.

<u>Main Theorem:</u> <u>Let</u> I <u>be an ideal of</u> $Z(\mathfrak{g})$ <u>of finite codimension. There exists a neighborhood</u> U <u>of</u> $\bar{A}$ <u>in</u> $\mathbb{C}^\Delta$ <u>such that the system of operators generated by</u> $\{\Pi_T(X) \mid x \in I\}$ <u>is</u> w <u>defined and of finite codimension on</u> $\underset{\sim}{U} - \underset{\sim}{U} \cap \mathfrak{m}$, <u>and determines an ideal on all of</u> $\underset{\sim}{U}$ <u>which has regular singularities along</u> $\mathfrak{m}$.

For $\underset{\sim}{s} \in \mathbb{C}^\Delta$, $\underset{\sim}{m} \in \mathbb{N}^\Delta$, define

$$\alpha^{\underset{\sim}{s}}(a) = \prod \alpha(a)^{s_\alpha} \qquad (\alpha \in \Delta)$$

$$\log^{\underset{\sim}{m}}\alpha(a) = \prod \log^{m_\alpha}\alpha(a) \quad (\alpha \in \Delta).$$

It is an immediate corollary of the above, and the definition of regular singularity that there exists a finite set $s \subseteq \mathbb{C}^\Delta$ which are integrally inequivalent and a unique finite set of functions $\Phi_{\underset{\sim}{s},\underset{\sim}{m}}$ such that if $\Phi \in G^\infty$ is annihilated by $\mathcal{J} \subseteq Z(\mathfrak{g})$ of finite codimension then on A it has an expansion

$$\Phi = \sum \Phi_{\underset{\sim}{s},\underset{\sim}{m}}\alpha^{s}\log^{m}\alpha$$

where the $\Phi_{\underset{\sim}{s},\underset{\sim}{m}}$ are holomorphic in a neighborhood of the origin in $\mathbb{C}^\Delta$.

<u>Second Main Theorem:</u> <u>The</u> $\Phi_{\underset{\sim}{s},\underset{\sim}{m}}$ <u>are actually holomorphic in a</u>

<u>neighborhood of</u> $\bar{A}$ <u>in</u> $\mathbb{C}^{\Delta}$.

The step from the expansion near 0 to that on all of A is almost trivial; the crucial lemma is the observation that if X is an open subset of $\mathbb{C}^{\Delta}$ and Y a non-singular connected hypersurface, and ϕ a function holomorphic on the union of X - Y and any open subset of Y, then it is holomorphic on all of X.

This result seems to be a larger part of Harish Chandra's "along the walls" theory.

3. A second application is to prove that any admissible representation of (g,K) extends to a representation of G. This reduces to showing that such a representation has matrix coefficients; constructing these amounts to noting the fact that the representation of g gives one the Taylor's series expansion of matrix coefficients at 2, and then applying the result that found solutions of regular singularities systems converge. (In between one uses G = KAK to reduce to a problem on A.)

CLASSIFICATION OF PRIMITIVE IDEALS IN A SEMI-SIMPLE COMPLEX ENVELOPING ALGEBRA

by

Michel Duflo

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. A (two-sided) ideal $I \subset U(\mathfrak{g})$ is called primitive if it is the kernel of some irreducible representation of $\mathfrak{g}$. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. If I is primitive, $I \cap Z(\mathfrak{g})$ is a maximal ideal of $Z(\mathfrak{g})$ and defines a character of $Z(\mathfrak{g})$. Let χ be a character of $Z(\mathfrak{g})$ and denote by $\operatorname{Prim} U(\mathfrak{g})_\chi$ the set of primitive ideals which correspond to χ. It is a finite set ordered by inclusion, and Dixmier proved that it is finite. The purpose of the lecture was to describe some results on this set which can be obtained using the theory of admissible representations of the complex semi-simple simply connected group G with Lie algebra $\mathfrak{g}$, mainly Zelobenko's classification of admissible G-modules and Hirai's description of the irreducible factors of the principal series representations.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be a Borel subalgebra. For each $\lambda \in \mathfrak{h}^*$ define the Verma module $M(\lambda)$, its irreducible quotient $L(\lambda)$ and $I(\lambda)$ the kernel of

$L(\lambda)$ in $U(\mathfrak{g})$. Let χ_λ be the central character of $M(\lambda)$. Let W_λ be the set of Weyl group elements w such that $w\lambda - \lambda$ is a sum of roots.

<u>Theorem</u>: The map $W_\lambda \to \operatorname{Prim} U(\mathfrak{g})_{\chi_\lambda}$ defined by $w \mapsto I(w\lambda)$ is surjective.

<u>Corollary</u>: Let $X \mapsto {}^tX$ be the anti-isomorphism of $\mathfrak{g}$ such that ${}^tH = H$ for $H \in \mathfrak{h}$, and extends it to $U(\mathfrak{g})$. Then ${}^tI = I$ for any prime ideal of $U(\mathfrak{g})$.

The proof of the theorem also provides the following result: $\# \operatorname{Prim} U(\mathfrak{g})_{\chi_\lambda} \leq \#$ involutions in W_λ.

It is part of a very interesting conjecture by Jantzen that if λ is regular and W_λ a permutation group, there is equality. For other types of Weyl groups W_λ the conjecture has to be modified.

The connection between admissible G-modules and ideals comes from the fact that the complixified Lie algebra of G, considered as real, is equal to $\mathfrak{g} \times \mathfrak{g}$, and that $U(\mathfrak{g})/U(\mathfrak{g})\ker \chi_\lambda$, considered as a $\mathfrak{g} \times \mathfrak{g}$ module, is a spherical principal series.

CHARACTERS, ASYMPTOTICS AND OSBORNE CONJECTURE

by

Henryk Hecht
(joint results with W. Schmid)

Let G be a connected semisimple real Lie group with finite center K a maximal compact subgroup of G. Let (π,V) be a quasisimple representation of finite length. It turns out that one can read off leading exponents of K-finite matrix coefficients of π from the formula for Θ_π--the global character of π--on a maximally split Cartan subgroup of G. To make this statement precise we need more notation. Let $P = MAN$ be a minimal parabolic subgroup of G with $M \subset K$. We extend A to a Cartan subgroup B of G. Clearly $B = TA$ (direct product), where $T = B \cap M$. We denote, as usual, by $\mathfrak{g}_0, \mathfrak{k}_0, \mathfrak{m}_0, \mathfrak{a}_0, \mathfrak{n}_0, \mathfrak{b}_0, \mathfrak{t}_0$ the Lie algebras of G, K, M, A, N, B, T, respectively. Omitting subscript "0" means "complexification." In the root system Σ of $(\mathfrak{g}_0, \mathfrak{a}_0)$ we choose a positive root system Σ^+ if and only if $(-\alpha)$ roots space is contained in $\mathfrak{n}_0$. Let A^+ be the corresponding chamber of A. We choose a system of positive roots Φ^+ for the root system Φ of $(\mathfrak{g}, \mathfrak{b})$, which is compatible with Σ^+. We note that we may identify the root system $\Phi_{\mathfrak{m}}$ of $(\mathfrak{m}, \mathfrak{t})$ with a subset of Φ, and we put $\Phi^+_{\mathfrak{m}} = \Phi^+ \cap \Phi_{\mathfrak{m}}$.

A typical matrix coefficient $(\pi(g)u,\tilde{u})$ ($u \in V$, $\tilde{u} \in \tilde{V}$ are K finite vectors in V and its dual respectively), restricted to A^+, has an expansion

$$e^{-\rho} \sum_{\gamma} p_{\gamma}^{u,\tilde{u}}(\log a)e^{\gamma}(a)$$

where p_γ are polynomials on $\mathfrak{a}$, and $\gamma \in \mathfrak{a}^*$. ($\rho = \frac{1}{2} \Sigma_{\alpha\in\Sigma^+}\alpha$.) We say that γ is <u>an exponent of</u> π if $(\pi(g)u,\tilde{u}) \neq 0$ for some $u,\tilde{u}$ K-finite. We write $\gamma_1 < \gamma_2$ if $\gamma_2 - \gamma_1$ is a sum of roots in Σ^+. Maximal exponents with respect to this partial ordering, as $u,\tilde{u}$ vary, exist and are finite. We call them <u>leading exponents of</u> π <u>on</u> A^+.

For each leading exponent the MA module $(V/\mathfrak{n}V)_\gamma$ is nonempty. Let τ be its irreducible quotient, and $\varkappa(\tau) \in \sqrt{-1}\,\mathfrak{t}_0^*$ be $\Phi^+_{\mathfrak{m}}$ dominant and correspond to the infinitesimal character τ. We call $\lambda = \varkappa(\tau) = \gamma$ a <u>complete leading exponent</u> of π with respect <u>to</u> Φ^+. We order complete leading exponents lexicographically by integral cones spanned by Σ^+ and $\Phi^+_{\mathfrak{m}}$. Let $\mathcal{E}(\pi)$ be the set of all complete leading exponents of π.

Assume now that π is quasisimple, with regular and real infinitesimal character. This restriction is in no way restrictive, but simplifies arguments. Let $\Delta = \Pi_{\alpha\in\Phi^+}(e^{\alpha/2} - e^{-\alpha/2})$. Then, on T^0A^+ we have a formula

$$\Delta \cdot \Theta_\pi = \sum_{w \in W_\Phi} a(w) e^{w\lambda} .$$

Here W_Φ is the Weyl group of Φ, and $a(w)$ are constants. We assume that λ is Φ^+ dominant.

<u>Theorem</u>: $\mathcal{E}(\pi)$ coincides with the set of maximal elements in $\{w\lambda : a(w) \neq 0\}$.

In order to prove this theorem we have to introduce a special class of representations. Let π be irreducible. We say that it is of <u>large growth</u> if it has a complete leading exponent which is strictly dominant with respect to Φ^+.

<u>Proposition 1</u>: π is of large growth if and only if it appears in character formula (1) with multiplicity 1.

The proof consists of remarking that Θ_π can be written as a sum of characters induced from the tempered characters, with precisely one term contributing to e^λ. What is involved is Langland's classification and formula for inducing characters.

Let us write $\Theta_\pi = \Theta_\pi(\underset{\sim}{\lambda})$, where $\underset{\sim}{\lambda}$ indicates all the data involved in defining Θ_π. If μ is a weight of a finite dimensional representation we can define an invariant eigen-distribution $\Theta_\pi(\underset{\sim}{\lambda} + \mu)$, which on $T^0 A^+$ is defined by formally replacing λ by $\lambda + \mu$. We know that eigendistributions so obtained are virtual characters. We call a virtual character

Θ of large growth, if some representation of large growth contributes to it. Denote by $m(\Theta)$ the sum of multiplicities of large growth summands of Θ.

Proposition: Let μ be a very large highest weight of a finite dimensional representation of G. Then $a(w) = m(\Theta(\lambda + w^{-1}\mu))$ for each $w \in W_{\Phi}$.

The proof is implied by Proposition 1 and the following statements.

Proposition: $\mathcal{E}(\pi)$ coincides with the set of maximal elements in $\{w\lambda : m(\Theta(\lambda + w^{-1}\mu)) \neq 0\}$.

The proof involves standard techniques of knowing representations with finite dimensional representations, coherence of property of large growth inside one Weyl chamber for Φ (follows from Proposition 1) and from a theorem of Miličić, asserting that leading exponents of a representation of finite length depend only on irreducible subquotients involved, but not on the composition series. Now Theorem A follows immediately from Propositions 2) and 3).

Theorem B (Osborne conjecture): For a regular $b \in T^0A^+$

$$\Theta_\pi(b) = \frac{\Sigma_q\, \chi_{MA}(H_q(n, V))(b)}{\det(I - Ad_n(b))}$$

$\chi_{MA}(\ldots)$ denotes the character of MA module $(\ldots)$.

Proof: $H_q(\mathfrak{n},V)$ are finite dimensional for each q, so the right hand side formula, call it Θ'_π, makes sense. Clearly, $\Delta\cdot\Theta'_\pi$ can be written in the form

$$\sum_{w\in W_\Phi} a'(w)e^{w\lambda}.$$

By formal reasons Θ'_π behaves well under tensoring, so in the above formula we can shift λ by a weight of a finite dimensional representation. This essentially reduces the proof to showing that $a(1) = a'(1)$. Put $\tilde{\lambda} = \lambda|\mathfrak{a}_0$. We know that $H_0(\mathfrak{n},V)$ is a semisimple MA module, so, by Proposition 2, $a(1) = \text{length } H_0(\mathfrak{n},V)_{\tilde{\lambda}}$. Also, $a'(1) = \Sigma_q(-1)^q \text{ length } H_q(\mathfrak{n},V)_{\tilde{\lambda}}$. By a theorem of Casselman and Schmid, $H_q(\mathfrak{n},V)_{\tilde{\lambda}} = 0$ for $q \geq 1$. This concludes the proof of Theorem B.

PRODUCTS OF SPHERICAL FUNCTIONS

by

Fritz Mayer-Lindenberg

Let G be a semi-simple connected Lie group with finite center and $G = KAN$ be a Twasawa decomposition. For each real linear form γ on $\mathfrak{a}$ = Lie A the corresponding spherical function φ_γ on G is given by

$$\varphi_\gamma(g) = \int_K a(gk)^{i\gamma-\rho}\, dk.$$

a denoting the Twasawa-projection onto A. Note that

$$|\varphi_\gamma(g)| \leq \varphi_0(g) = \int_K a(gk)^{-\rho} dk.$$

<u>Theorem</u>: For $\alpha,\beta,\gamma \in \mathfrak{a}'$ the function $\varphi_\alpha\varphi_\alpha\varphi_\gamma$ is integrable on G and the number $s(a,\alpha,\gamma) = \int_G \varphi_\alpha\varphi_\beta\overline{\varphi_\gamma}\,dg$ depends analytically on γ.

Now just $\mathfrak{A} = L^{\wedge}(G,\varphi_o(g)dg)_K$. For $f \in \mathfrak{A}$, $\gamma \in \mathfrak{a}'$ define

$$\hat{f}(\gamma) = \int_G f(g)\overline{\varphi_\gamma(g)}\,dg.$$

Let A denote the Fourier algebra $L^2(G) \times L^2(G)$. Then the usual inversion formula generalises as follows

<u>Theorem</u>: For $f \in \mathfrak{A} \cap A$ the function $\hat{f}$ is integrable with respect to the Plancherel measure $|c(\gamma)|^{-2}d\gamma$ on $\mathfrak{a}'$, and

$$f = \int_{\mathfrak{a}'} \varphi_\gamma \hat{f}(\gamma)\,|d(\gamma)|^{-2}d\gamma.$$

We apply this to the function $\varphi_\alpha\varphi_\beta$ $(\alpha,\beta \in \mathfrak{a}')$ and get

$$\varphi_\alpha\varphi_\beta = \int_{\mathfrak{a}'} \varphi_\gamma \quad s(\alpha,\beta,\gamma)\,|c(\gamma)|^{-2}d\gamma$$

<u>Corollary</u>: For $\alpha,\beta \in \mathfrak{a}'$ let $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$ be the corresponding unitary G-spaces of the spherical principal series. Then $\mathcal{H}_\alpha \otimes \mathcal{H}_\beta$ contains a G-subspace equivalent to $L^2(G/K)$.

Decomposing products of spherical functions defines on the dual space of any Gelfand pair (G,K) a sort of Hypergroup structure, that allows there the translation of functions on the dual. The Plancherel measure can be characterized to be the only translation invariant measure. A similar statement holds for Plancherel measures in general.

JACQUET MODULES FOR REAL REDUCTIVE GROUPS AND LANGLAND'S CLASSIFICATION OF REPRESENTATIONS

by

D. Miličić

Let G be a real reductive Lie group of Harish-Chandra's class. Denote by $\mathfrak{g}_0$ its Lie algebra and by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. Fix a maximal compact subgroup K of G such that its intersection with the identity component G_0 of G is a maximal compact subgroup of G_0 and K meets every component of G.

A <u>smooth representation</u> (π,V) of $(\mathfrak{g},K)$ is a $(\mathfrak{g},K)$-module such that

(i) as a K-module V is a direct sum of irreducible finite-dimensional smooth K-modules,

(ii) the actions of $\mathfrak{g}$ and K on V are compatible i.e. the actions of the Lie algebras of K corresponding to K-action and $\mathfrak{g}$-action coincide and

$$\pi(k)\pi(X)\pi(k^{-1}) = \pi((\mathrm{Ad}\,k)(X)), \quad k \in K,\ x \in \mathfrak{g}.$$

A smooth representation (π,V) of $(\mathfrak{g},K)$ is <u>admissible</u> if for every irreducible finite-dimensional smooth K-module U the space $\mathrm{Hom}_K(U,V)$ is finite-dimensional.

Every admissible representation of $(\mathfrak{g},K)$ is the direct sum of admissible representations of finite length.

A smooth representation (π,V) of $(\mathfrak{g},K)$ is <u>$z(\mathfrak{g})$-finite</u> if the kernel of the restriction of π on the center $z(\mathfrak{g})$ of the enveloping algebras of $\mathfrak{g}$ is an ideal of finite codimension in $z(\mathfrak{g})$.

The following result is a simple modification of a result of Harish-Chandra.

<u>Theorem 1:</u> A finitely generated $z(\mathfrak{g})$-finite smooth representation of $(\mathfrak{g},K)$ is an admissible representation of finite length.

Let P be a parabolic subgroup of G. Let θ be the Cartan involution of G corresponding to K. Put $L = P \cap \theta(P)$. Then L is the θ-stable Levi factor of P, and if N is the unipotent radical of P, $P = L\cdot N$. Put $K_L = K \cap L$. Then K_L is a maximal compact subgroup of L and the pair (L,K_L) has the same properties as the pair (G,K).

Denote by $\mathfrak{l}$ and $\mathfrak{n}$ the complexified Lie algebras of L and N respectively. For an admissible representation of finite length (π,V) of $(\mathfrak{g},K)$ put $V_{\mathfrak{n}} = V/\mathfrak{n}\cdot V$ and denote by $\pi_{\mathfrak{n}}$ the corresponding $(\mathfrak{l},K_L)$-action. Then $(\pi_{\mathfrak{n}},V_{\mathfrak{n}})$ is a smooth representation of $(\mathfrak{l},K_L)$.

A simple consequence of Theorem 1 is the following result.

<u>Theorem 2:</u> Let (π,V) be an admissible representation of finite

length of $(\mathfrak{g},K)$. Then $(\pi_{\mathfrak{n}},V_{\mathfrak{n}})$ is an admissible representation of finite length of $(\mathfrak{l},K_L)$.

$(\pi_{\mathfrak{n}},V_{\mathfrak{n}})$ is <u>the Jacquet module</u> of (π,V) corresponding to the parabolic subgroup P. The Jacquet functor $(\pi,V)\rightsquigarrow(\pi_{\mathfrak{n}},V_{\mathfrak{n}})$ maps admissible representations of finite length of $(\mathfrak{g},K)$ into admissible representations of finite length of $(\mathfrak{l},K_L)$.

Let (σ,U) be an admissible representation of finite length of $(\mathfrak{l},K_L)$. By Casselman's realizability theorem (σ,U) is the space of K-finite vectors of a certain representation of L on a complete locally convex space. Let $\mathrm{Ind}(\sigma|P,G)$ be the admissible representation of finite length of $(\mathfrak{g},K)$ on the space of the induced representation. The following result is a simple observation due to Casselman and Jacquet.

<u>Theorem 3</u> (Frobenius reciprocity): Let (π,V) be an admissible representation of finite length of $(\mathfrak{g},K)$ and (σ,U) an admissible representation of finite length of $(\mathfrak{l},K_L)$. Then

$$\mathrm{Hom}_{(\mathfrak{g},K)}(V,\mathrm{Ind}(\sigma|P,G)) = \mathrm{Hom}_{(\mathfrak{l},K_L)}(V_{\mathfrak{n}},\sigma).$$

Let P_0 be a minimal parabolic subgroup of G lying inside of P. Then $P_* = L \cap P_0$ is a minimal parabolic subgroup of L. Denote by $\mathfrak{n}_0$ and $\mathfrak{n}_*$ the complexified Lie algebras of the unipotent radicals of P_0 and P_* respectively. Then

$$(\pi_{\mathfrak{n}_0}, V_{\mathfrak{n}_0}) = ((\pi_{\mathfrak{n}})_{\mathfrak{n}_*}, (V_{\mathfrak{n}})_{\mathfrak{n}_*}).$$

By the relation between the leading exponents of admissible representations and embeddings into the nonunitary principal series representations for the minimal parabolics, this gives us control over the asymptotic behavior of matrix coefficients of Jacquet modules.

Let (π, V) be an irreducible admissible representation of $(\mathfrak{g}, K)$. By choosing a right parabolic subgroup P of G and a right irreducible essentially tempered quotient (σ, U) of $(\pi_{\mathfrak{n}}, V_{\mathfrak{n}})$ it is possible to give a canonical construction of Langland embeddings and give a proof of Langland's classification of irreducible representations of $(\mathfrak{g}, K)$ which is a natural extension of Casselman's proof of the subrepresentation theorem.

A GEOMETRIC CONSTRUCTION OF THE DISCRETE SERIES

by

W. Schmid

Let G be a connected semisimple Lie group which contains a compact Cartan subgroup H, and K a maximal compact subgroup, with $H \subset K$. The Lie algebras of G, K, H will be denoted by $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$, and their complexifications by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$. Further notation:

Φ = root system of $(\mathfrak{g}, \mathfrak{h})$

Φ_c, Φ_n = sets of compact, resp. noncompact roots

Λ = weight lattice of H

W = Weyl group of $(\mathfrak{k}, \mathfrak{h})$.

For simplicity, assume that G is linear and that one half of the sum of the positive roots lies in Λ, i.e. G is "acceptable" in the sense of Harish-Chandra. To each nonsingular $\lambda \in \Lambda$ corresponds a discrete series representation π_λ, with character Θ_λ, and every discrete series representation is isomorphic to one of these [1].

Using Harish-Chandra's fundamental results, one can give a geometric realization of the π_λ. Let me briefly recall the relevant statement [2]. I keep fixed a particular nonsingular $\lambda \in \Lambda$, and define

Φ^+ = system of positive roots determined by

ρ_c, ρ_n = half sum of the positive compact, resp. noncompact roots.

Then $\mu = \lambda - \rho_c$ is dominant with respect to $\Phi_c \cap \Phi^+$, and hence is the highest weight of an irreducible K-module V_μ (one may have to go to a two-fold covering of G to insure $\mu \in \Lambda$). The adjoint homomorphism $K \to SO(\mathfrak{g}/\mathfrak{k})$ gives rise to two half spin modules S_+, S_-, which shall be labelled so that ρ_n is a weight of S_+. Let $\mathcal{V}_\mu$, $\mathcal{S}_+$, $\mathcal{S}_-$ be the homogeneous vector bundles over G/K whose fibres at the identity coset equal V_μ, S_+, S_- (as K-modules). There exists an essentially unique G-invariant, elliptic, first order differential operator

$$C^\infty(\mathcal{V}_\mu \otimes \mathcal{S}_+) \underset{D}{\overset{D}{\rightleftarrows}} C^\infty(\mathcal{V}_\mu \otimes \mathcal{S}_-)$$

with $D^* = D$, the <u>Dirac operator</u>. The L^2 kernels of D on $\mathcal{V}_\mu \otimes \mathcal{S}_+$, $\mathcal{V}_\mu \otimes \mathcal{S}_-$ are Hilbert spaces, on which G acts unitarily, by translation. Denote these kernels by $\mathcal{H}_\lambda^+$, $\mathcal{H}_\lambda^-$; then $\mathcal{H}_\lambda^-$ vanishes, whereas $\mathcal{H}_\lambda^+$ is nonzero, irreducible and realizes the discrete series representation π_λ.[2]

As mentioned above, previous proofs of these facts were based on Harish-Chandra's construction of the discrete series. In recent joint work, Atiyah and I have reversed the process: we prove directly that $\mathcal{H}_\lambda^+ \neq 0$, and then deduce the main results about the discrete series.

According to Borel and Harish-Chandra, there exists a discrete, torsion-free subgroup $\Gamma \subset G$, with $\Gamma\backslash G$ compact. The bundles $\mathcal{V}_\mu \otimes \mathcal{S}_+$, $\mathcal{V}_\mu \otimes \mathcal{S}_-$ and the differential operator D descend to the compact manifold $\Gamma\backslash G/K$. Atiyah's L^2-index theorem[3], coupled with the usual index theorem for elliptic operators on compact manifolds, gives an explicit formula for a suitably defined, normalized Γ-index of the operator D on G/K. More concretely,

(1) $$\operatorname{vol}(\Gamma\backslash G) \cdot \operatorname{index}_\Gamma D = d(\lambda)$$

(= dimension of the irreducible, finite dimensional G-module of highest weight $\lambda - \rho$). Re-interpreting the Γ-index in terms of the Plancherel decomposition of

$L^2(G)$, one finds

$$\text{(2)}\qquad \begin{aligned} &\mathrm{vol}(\Gamma\backslash G)\cdot \mathrm{index}_\Gamma D = \\ &\int_{i\in\hat{G}} \{\dim(H_i^*\otimes V_\mu\otimes S_+)^K - \dim(H_i^*\otimes V_\mu\otimes S_-)^K\}\, dm(i)\ , \end{aligned}$$

where

$\hat{G}$ = unitary dual of G

dm = Plancherel measure on $\hat{G}$

H_i = representation space of $i \in \hat{G}$

$(\ldots)^K$ = space of K-invariants in ...

Let Θ_i be the character of H_i. One knows that the restriction of Θ_i to K is closely related to the K-character of H_i. In particular, the integrand in (2) is, up to sign, the coefficient of e^λ in $\Theta_{i\,|G}$, multiplied by the Weyl donominator. I denote the coefficient by $n_\lambda(i)$; then

$$\text{(3)}\qquad \int_{i\in\hat{G}} n_\lambda(i)\, dm(i) = (-1)^q\, d(\lambda)\ ,$$

with $q = \frac{1}{2}\dim G/K$. At this point Γ has completely disappeared from the scene. If $n_\lambda(i) \neq 0$, H_i must have infinitesimal character χ_λ, in Harish-Chandra's notation, and thus must be one of finitely many representations $i \in \hat{G}$. Consequently the integration (3) extends over a finite subset of $\hat{G}$ -- in other words, it reduces to a sum over the discrete series. Since $d(\lambda) \neq 0$, this sum cannot be zero. In particular, the discrete series is non-empty.

If λ is very nonsingular, one can use infinitesimal arguments to show that at most one $i \in \hat{G}$ can contribute to the sum [4]. In this way, one recovers the known statements about $\mathcal{H}_\lambda^+$, $\mathcal{H}_\lambda^-$: $\mathcal{H}_\lambda^-$ vanishes, whereas $\mathcal{H}_\lambda^+$ is non-zero, irreducible, square integrable, with a character whose restriction to H is given by Harish-Chandra's formula -- at least for "most" $\lambda \in \Lambda$.

Implicit in (3) is also a formula for the sum of the discrete series character which correspond to the infinitesimal character χ_λ, each multiplied by its formal degree. Specifically, if $\tilde{\Theta}_\lambda$ denotes this sum, and if $\lambda_1, \cdots, \lambda_N$ is an enumeration of the $\Phi_c \cap \Phi^+$-dominant conjugates of λ,

$$(4) \qquad \tilde{\Theta}_\lambda|_H = (-1)^q \, d(\lambda) \sum_{j=1}^{N} \frac{\sum_{w \in W} \operatorname{sgn} w \, e^{w\lambda_j}}{\prod_{\alpha \in \Phi, (\alpha,\lambda) > 0} (e^{\alpha/2} - e^{-\alpha/2})} .$$

Properly interpreted, this formula and its derivation remain valid even if λ is singular: in this case, the restriction of $\tilde{\Theta}_\lambda$ to H vanishes.

Characters of discrete series representations are tempered, and hence their numerators grow at most polynomially on any Cartan subgroup [1]. A slightly strengthened version of Harish-Chandra's arguments shows that the numerators actually tend to zero at infinity. Because of the "matching conditions" for invariant eigendistributions, this can happen only if the support of the character in question meets a compact Cartan subgroup. This line of reasoning also proves that G has a non-empty discrete series only if it contains a compact Cartan subgroup.

These properties of discrete series characters make it possible to extend the construction of the discrete series representations from the "very non-singular" case to the general case, and to prove exhaustion at the same time; the main ingredients of the argument are the identity (4) and Zuckerman's tensoring technique.

[1] Harish-Chandra, Acta Math. 116 (1966), 1-111.

[2] Parthasarathy, Ann of Math. 96 (1972), 1-30.

[3] Atiyah, Astérisque 32-33 (1976), 43-72.

A NEW PROOF OF THE BOREL-WALLACH-ZUCKERMAN VANISHING THEOREM

by

Wilfried Schmid

Let G be a connected, simple Lie group, with maximal compact subgroup K, and π a non-trivial, irreducible, unitary representation of G on a Hilbert space H. According to Borel-Wallach and Zuckerman (independently),

(1) $\quad H^k(\mathfrak{g},\mathfrak{k};H_o) = 0, \quad$ for $k < \mathrm{rk}_{\mathbb{R}} G$;

here $\mathfrak{g}$ and $\mathfrak{k}$ denote the complexified Lie algebras of G and K, and H_o is the space of K-finite vectors in H, on which $\mathfrak{g}$ acts by the differential of π. As has been known since Matsushima, any vanishing theorem of this type implies a vanishing theorem for the cohomology of discrete, co-compact subgroups of G (cf. Zuckerman's lecture).

Let $G = KAN$ be an Iwasawa decomposition, and $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ the corresponding decomposition of $\mathfrak{g}$. For simplicity, I shall assume that $\mathfrak{a}$ is a Cartan subalgebra -- the general case is only slightly more complicated.

<u>(2) Lemma</u> If the cohomology groups $H^k(\mathfrak{n}, H_o)$ vanish up to some integer k_o, then so do the relative cohomology groups $H^k(\mathfrak{g},\mathfrak{k};H_o)$.

The proof is essentially formal and consists of a string of spectral sequences. Because of Wigner's lemma, one may assume that π has the same infinitesimal character as the trivial representation. Thus, according to Casselman-Osborne, the $\mathfrak{n}$-cohomology groups of H_o are sums of generalized $\mathfrak{a}$-eigenspaces corresponding to weights of the form $w\rho - \rho$, with $w \in W$ = Weyl group of $(\mathfrak{g}, \mathfrak{a})$, and $\rho = \frac{1}{2}$ trace $\mathrm{ad}|_{\mathfrak{n}}$. Let $l(\dots)$ be the length function on W, and n the dimension of $\mathfrak{n}$.

<u>(3) Lemma</u> Suppose $H^n(\mathfrak{n}, H_o)_{w\rho-\rho} = 0$, for all $w \in W$ of length $l(w) \leq k_o$. Then $H^{n-k}(\mathfrak{n}, H_o)_{w\rho-\rho} = 0$ whenever $l(w) \leq k_o + k$. In particular, $H^k(\mathfrak{n}, H_o) = 0$ for $k < k_o$.

The two ingredients of the proof are i) the corresponding statement about the $\mathfrak{n}$-homology of modules in Bernstein-Gelfand-Gelfand's category $\mathcal{O}$, which follows from a tensoring argument, and ii) the identification of the $\mathfrak{n}$-cohomology of H_o with the dual of the $\mathfrak{n}$-homology of the $\mathfrak{g}$-module $H_o^{[\mathfrak{n}]}$ = $\mathfrak{n}$-finite part of H_o^*, which belongs (almost) to the category $\mathcal{O}$; this is a result of Casselman, which he described in his informal lecture.

The matrix coefficients of a non-trivial, irreducible, unitary representation of a simple group vanish at infinity (this is due to Howe; cf. the talk of Vogan-Speh). On the other hand, the asymptotic behavior of the K-finite matrix coefficients can be read off from the $\mathfrak{a}$-action on $H^n(\mathfrak{n}, H_o)$ (Miličić). In particular, if the matrix coefficients of π vanish at infinity,

(4) $H^n(\mathfrak{n}, H_o)_{w\rho-\rho} = 0$, if $l(w) < \mathrm{rk}_{\mathbb{R}}\, G$.

Evidently (2-4) imply the vanishing theorem (1).

COMPOSITION SERIES FOR GL(n,ℝ) AND CLASSIFICATION OF UNITARY REPRESENTATIONS FOR GL(4,ℝ)

by

B. Speh

Let P be a cruspidal parabolic subgroup of $GL(n,\mathbb{R})$ with Langland's decomposition $P = MAN$, π a discrete series representation of M, e^{ν} a character of AN and $U(P,\pi,\nu)\text{-}\mathrm{ind}_P^G(\pi \otimes e^{\nu})$. We give necessary and sufficient irreducibility criteria for the representations $U(P,\pi,\nu)$. Reducibility occurs iff the differential of the character is contained in a union of hyperplanes, which satisfy certain invariance conditions under the Weyl group of the parabolic. The proof is based on a reduction to computations for certain subgroups, which play a role similar to that of the rank 1 cone (this was observed independently by N. Wallach and G. Zuckerman). We show that the unitary dual of $GL(4,\mathbb{R})$ consists of a

a, unitarily induced principal series representations,

b, complementary series representations

c, limits of complementary series representations

d, unitarily induced degenerate series representations

e, complementary series representations for degenerate series representations

f, limits of unitarily induced degenerate series,

e, the one dimensional unitary representation.

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THE ISOMORPHISM BETWEEN THE SPACE OF AUTOMORPHIC FORMS AND THE SPACE OF INTERTWINING OPERATORS

by

J. Szmidt

We first formulate the theorem in some general context. Let G be a unimodular Lie group, K its compact subgroup and (τ, E) a finite dimensional unitary representation of K. $E_\tau \to G/K$ will be the induced vector bundle. We assume that it is given an elliptic differential operator

$$A: C^\infty(E_\tau) \longrightarrow C^\infty(E_\tau)$$

commuting with the action of G. Let $L^2(E_\tau)$ be the space of square integrable sections and $\mathcal{H}^2(E_\tau)$ the L^2-kernel of this operator. We denote by $\mathcal{H}(E_\tau)$ the whole kernel of the operator A.

Now let Γ be a closed subgroup of G with compact quotient G/Γ and denote by $\mathcal{H}(\Gamma;E_\tau)$ the subspace of $\mathcal{H}(E_\tau)$ consisting of Γ-invariant sections. We have the unitary representation $(\pi^\tau, \mathcal{H}^2(E_\tau))$ of G being the action of the group on sections. On this representation we put the following restrictions:

1. It is irreducible and K-finite one,

2. there exists a dense, G-invariant subspace $\Phi \subset \mathcal{H}^2_\infty(E_\tau)$ of the space of smooth vectors such that the

functions

$$G \ni x \longrightarrow \|\varphi(x)\|_E \quad \text{for} \quad \forall \varphi \in \Phi \quad .$$

are integrable ones on the group manifold.

Under these assumptions we have the following.

<u>Theorem:</u> There exists an algebraic isomorphism

$$L_G(\mathcal{H}^2(\mathbb{E}_\tau), L^2(G)\Gamma)) \cong \mathcal{H}(\Gamma;\mathbb{E}_\tau)$$

where the left hand side states for the space of intertwining operators.

This theorem has as its sources the work of K. Maurin and L. Maurin [2]. The proof goes as follows. Every intertwining operator $T: \mathcal{H}^2(\mathbb{E}_\tau) \to L^2(G/\Gamma)$ describes by the formula

$$\mathcal{H}^2_\infty(\mathbb{E}_\tau) \ni a \longrightarrow (Ta)(e) \qquad e \in G$$

the identity element a continuous functional on the space of smooth vectors and by the elliptic regularity theorem for A it can be represented by a smooth section of the bundle $\mathbb{E}_\tau$ which lies in $\mathcal{H}(\mathbb{E}_\tau)$ and is Γ-invariant. In the opposite direction if $w \in \mathcal{H}(\Gamma;\mathbb{E}_\tau)$ then the formula

$$(T\varphi)(g) = (\pi^\tau(g^{-1})\varphi|w) \quad \varphi \in \Phi, \ g \in G$$

gives an operator which is closable and by the Naimark theorem it is continuous.

Now if G is a semi-simple Lie group with non-empty discrete series, Γ a discrete subgroup with compact quotient G/Γ and as the operator A we take the Dirac operator or the complex Laplace-Beltrami operator and as $(\pi^{\tau}, \mathcal{H}^2(E_{\tau}))$ an integrable representation of G realized according to Wilfried Schmid [3,4] then we have the isomorphism between the spaces of Γ-invariant Dirac spinors (or the Γ-invariant harmonic forms) and the spaces of intertwining operators. In this way we have another proof of the part of Langlands conjecture [1] about equality at the multiplicities of integrable classes in $L^2(G/\Gamma)$ and the dimensions of the spaces of intertwining operators.

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[2] K. Maurin and L. Maurin, A generalization of the duality theorem of Gelfand-Piateckii-Sapiro and Tamagava automorphic forms, J. Fac. Sci. Univ. Tokyo 17 (1970), 331-339.

[3] W. Schmid, On the characters of the discrete series. The hermitian symmetric case, Inv. Math. 30 (1975), 47-144.

[4] __________, L^2-cohomology and the discrete series, Ann of Math. 103 (1976), 375-394.

SOME EXPLICIT FORMULAS OF DISCRETE SERIES CHARACTERS

by

Jorge Vargas

Let G be a connected, semisimple matrix Lie group, having both a compact Cartan subgroup B. Let K be a maximal compact subgroup of G containing B. From now on: For any subgroup of G, the corresponding Lie algebra will be denoted by the corresponding german letter. Moreover, if H is any subgroup of G, $H^{\mathbb{C}}$ will denote its complexification and if $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$, $\mathfrak{h}^{\mathbb{C}}$ will denote its complexification. Let $\Phi(\mathfrak{b})$ be the root system corresponding to the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$. Let Λ be the lattice of differentials of characters of B. Let Ψ denote a system of positive roots for Ψ and let $W(G,B)$ denote the group generated by the reflections about the compact roots in $\Phi(\mathfrak{b})$. Then, it follows from the Harish-Chandra construction of the discrete series representations of G, that there exists a unique invariant eigendistribution $\Theta(\Psi,\lambda)$ such that

$$1) \quad \Theta(\Psi,\lambda)\big|_{B\cap G_{reg}} = (-1)^q \frac{\sum_{w\in W(G,B)} \epsilon(w) e^{w\lambda}}{\prod_{\alpha\in\Psi}(e^{\alpha/2}-e^{-\alpha/2})} .$$

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2) $\Theta(\Psi,\lambda)$ is tempered if λ is dominant with respect to Ψ.

As usual G_{reg} denotes the set of semisimple regular elements in G, and $q = \frac{1}{2} \dim G/k$. We recall, that Atiyah and Schmid have given a complete independent proof of the last statement. Here we are concerned with the problem of finding global, explicit formulas for the functions $\Theta(\Psi,\lambda)$. For this end, we assume for simplicity that G is simple. The system of positive roots Ψ is said to have the Borel-de Siebenthal property, if precisely one simple root β for Ψ is semi-compact and β occurs at most twice in the highest root. We recall that Borel-de Siebenthal have proven the existence of such systems of positive roots. If β occurs once in the highest root Martens and Hecht, in their respective thesis have given completely explicit global formulas for these eigen-distributions. In the general situation Schmid has proven "the so-called stepping down formula" which reduces the computation to the following particular case.

a) G has both a compact Cartan subgroup B and a split Cartan subgroup A

b) we need to know $\Theta(\Psi,\lambda)$ restricted to the identity component of the split Cartan subgroup A.

We will now give a concrete formula in this situation and under the further assumption that Ψ has the Borel-de Siebenthal property. Before stating the formula we need more pieces of notation. Let d be an inner automorphism of $G^{\mathbb{C}}$ such that d maps $B^{\mathbb{C}}$ onto $A^{\mathbb{C}}$ isomorphically. Let C denote

$$C = \{x \in A^0 : e^{\gamma}(x) < 1 \ \forall \gamma \in d^*(\Psi)\}.$$

Finally, let W_U denote the group generated by the reflections about the compact simple roots for Ψ. Then

$$\Theta(\Psi,\lambda)|_C = (-1)^q \{\frac{|W(G,B)|}{|W_U|} \frac{\sum_{w \in W_U} \epsilon(Tw) e^{Tw\lambda}}{\Pi_{\alpha\in\Psi} e^{\alpha/2} - e^{-\alpha/2}}\},$$

where T belongs to the Weyl group of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$, and it will be described below. For this end, let $\Phi^0 = \Phi$, Φ^1 equal to set of roots in $\Phi(\mathfrak{b})$ orthogonal to β. Assume we have defined Φ^i and let Φ^{i+1} be the orthogonal in Φ^i, to the noncompact simple roots for $\Psi \cap \Phi^i$. We set

$$S = \{\gamma \in \Psi : \gamma \text{ is simple for } \Psi \cap \Phi^i \text{ for some } i\}.$$

It turns out that S is a strongly orthogonal spanning set for $\Phi(\ )$. The element T of the Weyl group of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ that shows up in the above formula, the unique element satisfying

1) T is a product of reflection about some roots in S, and if β occurs twice in the highest, then $T \neq 1$.

2) T takes any long simple root into a noncompact root.

3) $(Tw\lambda,\mu) \geq 0$ for every $w \in W_U$ and λ,μ dominant forms with respect to Ψ. We mention that the proof of the above formula uses the so-called Harish-Chandra matching conditions, and the "stepping-down" formula for discrete series characters, due to Schmid.

ALGEBRAIC IDEAS IN THE CLASSIFICATION OF TEMPERED REPRESENTATIONS

by

David Vogan

In order to classify the irreducible tempered representations of a semi-simple Lie group G, it suffices to exhibit the complete reduction of the following standard representations. Suppose $P = MAN$ is a cuspidal parabolic subgroup of G, $\delta \in \hat{M}$ is a discrete series representation, and $\nu \in \hat{A}$ is a unitary character. Then $\mathrm{Ind}_P^G(\delta \otimes \nu \otimes 1)$ decomposes into a direct sum of irreducible tempered representations; and every tempered representation arises in this way. The decomposition of this representation is described in terms of a certain finite group. We give a description of this group in algebraic terms, and relate it to the problem of computing the lowest K-types of the induced representation. These results can be applied in several ways (using intertwining operators, character theory, or more algebra) to prove the decomposition of the induced representation. (This decomposition in its final form is due to Knapp-Zuckerman; the relevant finite group was first described by Knapp.)

Suppose now that P is a Borel subgroup, and δ is trivial on the identity component of M. (The general case is treated by a reduction to this case.) Let W be the Weyl group of A in G, and W_δ the stabilizer of δ in W.

<u>Definition</u>: Δ_δ, the set of <u>good</u> <u>roots</u> of $\mathfrak{a}$ in $\mathfrak{g}$ with respect to δ, consists of the complex roots; together with those real roots α so that if m_α is a generator of M intersected with the three dimensional subgroup corresponding to $\pm\alpha$, then $\delta(m_\alpha) \neq -1$.

Then Δ_δ is a root system, and its Weyl group $W(\Delta_\delta)$ is contained in W_δ. Define $R_\delta = W_\delta/W(\Delta_\delta)$; then $R_\delta = (\mathbb{Z}/2\mathbb{Z})^n$. Following Bernstein, Gelfand, and Gelfand ("Models of representations of compact Lie groups," Func. Anal. Appl.) we can define the set of <u>fine</u> K-<u>types</u> containing the M-types δ; we write this set as $A(\delta)$.

__Theorem:__ There is a natural simply transitive action of R_δ on $A(\delta)$. If $\mu \in A(\delta)$, then $\mu|_M$ is the sum of the M-types in the W orbit of $A(\delta)$ in $\hat{M}$, each occurring with multiplicity one.

__Definition:__ $R_\delta(\nu) = \{\sigma \in R_\delta | \sigma\cdot\nu = \sigma_0\cdot\nu$ for some $\sigma_0 \in W(\Delta_\delta)\}$. (Recall that $\nu \in \hat{A}$ is a unitary character.)

__Theorem:__ The number of irreducible components of $\mathrm{Ind}_P^G(\delta \otimes \nu \otimes 1)$ is $|R_\delta(\nu)|$.

This result follows fairly easily from the preceding one using some results in character theory, for example.

To treat the general case, one first constructs a smaller group, a Borel subgroup, and representations in $\hat{M}$ and $\hat{A}$ for this Borel; this smaller MA is just the Cartan subgroup associated to the original cuspidal parabolic. The set of K-types" given above for the small group gives rise to a set of K-types for the large group, which occur exactly once in the induced representation. This can be combined with character theory to obtain the reduction of the induced representation (cf. Zuckerman's second lecture).

A UNITARITY CRITERION

by

David Vogan and Birgit Speh

Langland's classification of the irreducible admissible representations of a semi-simple Lie group provides fairly precise information about the asymptotic behavior of matrix coefficients. In order to use this information in discussing questions of unitarity, the following theorem is useful. It was pointed out recently by Howe; Birgit Speh observed that it follows from an old result of T. Sherman.

<u>Theorem</u>: Let $G = KAN$ be a connected simple Lie group, π a non-trivial irreducible unitary representation of G on $\mathcal{H}$; $0 \neq H \in \mathfrak{a}$, the Lie algebra of A; and $v \in \mathcal{H}$. Then

$$\lim_{t\to\infty} \langle \pi(\exp)(tH))V,V\rangle = 0.$$

The proof has three steps. One produces a copy of the $ax + b$ group inside G; by the representation theory of this group, the theorem is true unless $\mathcal{H}$ contains vectors fixed by a certain one parameter subgroup of N. By studying π restricted to a copy of $SL(2,\mathbb{R})$, one sees that in this case contains vectors fixed by a one parameter subgroup of A. By considering $ax + b$ groups again, one sees that the subspace of vectors fixed by a one-parameter subgroup of A is G-stable. Since π is non-trivial, we are done.

REPRESENTATIONS THAT REMAIN IRREDUCIBLE ON PARABOLIC SUBGROUPS

by

Joseph A. Wolf

Physicists have known for some time that "ladder representations" of the conformal group $SO(2,4)$ remain irreducible on its Poincaré subgroup $R^{1,3} \cdot SO(1,3)$, cf. Mack-Todorov, J. Math. Phys. 10 (1969), 2078-2085. Sternberg and I ["Hermitian Lie Algebras and Metaplectic Representations," to appear in TAMS] noticed a similar phenomenon for ladder representations of $U(p,q)$--here note $SO(2,4) = SU(2,2)$. Since then I pushed the method and it seems to hold for quite a few "dual reductive pairs" in symplectic groups.

1. Irreducibility: Kobayashi [J. Math. Soc. Japan 20 (1968), 638-642] used a reproducing kernel argument to prove irreducibility theorems for homogeneous vector bundles. The same sort of argument[1] gives irreducibility on subgroups with some transitivity properties.

Theorem: Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on a complex manifold M, such that point evaluations are continuous. Let G be a locally compact group acting on M

[1]For simplicity I just state the case of trivialization line bundles. The extension to holomorphic vector bundles is straightforward.

such that the action lifts to a unitary representation π on $\mathcal{H}$ and such that the G-orbits are real analytic submanifolds of M. Let L be a subgroup of G such that M has a G-stable open set U that (i) meets every component of M, (ii) if $\mathcal{O} \subset U$ is a G-orbit then $\exists\, m \in \mathcal{O}$, $L(m)$ open in $\mathcal{O}$ and meets every component of $\mathcal{O}$. Then every closed $\pi(L)$-invariant subspace of $\mathcal{H}$ is $\pi(G)$-invariant. In particular if π is irreducible so is $\pi|_L$.

<u>Corollary</u>: If $\pi = \int \pi_\alpha d\nu(\alpha)$ direct integral of irreducibles then $\pi_\alpha|_L$ is irreducible for ν-almost-every α.

2. <u>Dual reductive pairs</u>. Inside a symplectic group $Sp(n;\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we consider subgroups $G = G_1G_2$ where (i) the G_i are closed reductive, (ii) each G_i is the centralizer of the other. Some examples: (I) $G = G_1 = Sp(n;\mathbb{F})$, $G_2 = \{1\}$; (II) $G = U(k,\ell)\cdot U(p,q)$ in $Sp((k+\ell)(p+q);\mathbb{R})$, from $\mathbb{C}^{k,\ell} \otimes_{\mathbb{C}} \mathbb{C}^{p,q} = \mathbb{C}^{kp+\ell q, kq+\ell p}$; (III) $G = GL(u;\mathbb{F})\cdot GL(v;\mathbb{F}) \subset Sp(uv;\mathbb{F})$, given as all $\begin{pmatrix} A\otimes B & 0 \\ 0 & {}^t(A\otimes B)^{-1}\end{pmatrix}$; (IV) $G = O(k,\ell)\cdot Sp(v;\mathbb{R}) \subset Sp((k+\ell)v;\mathbb{R})$, (V) $G = O(u;\mathbb{C})\cdot Sp(v;\mathbb{C}) \subset Sp(uv;\mathbb{C})$ by complexifying (IV); (VI) $G = SO^*(2u)\cdot Sp(k,\ell) \subset Sp(2u(k+\ell);\mathbb{R})$.

In a number of these cases G has a subgroup L, a bit smaller than a parabolic, which has the "more or less transitive on most of the G-orbits" property of the theorem in §1. Here

M is the complex vector space underlying the metaplectic representation in Fock-Bergman form, and π is the restriction to G of the metaplectic representation or one of its sub-representations. Some examples:

I. $G = Sp(n;\mathbb{E})$. The parabolic subgroups are the $P_{E_1,\ldots,E_t} = \{g \in G\colon gE_i = E_i \text{ for } 1 \leq i \leq t\}$ where $0 \neq E_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} E_t$ are totally isotropic $\mathbb{E}$-subspaces of $\mathbb{E}^{2n}$. Define

$$\chi_i\colon P_{E_1,\ldots,E_t} \longrightarrow \mathbb{E} \quad \text{by} \quad \chi_i(g) = \det(g|_{E_i/E_{i-1}})$$

and let L be any subgroup of G that contains $(\ker \chi_2) \cap \cdots \cap (\ker \chi_t)$ if $\dim_{\mathbb{E}} E_1 = 1$, contains $(\ker \chi_1) \cap \cdots \cap (\ker \chi_t)$ if $\dim_{\mathbb{E}} E_1 > 1$. Then L has an open orbit $\mathbb{E}^{2n} \setminus E_1^{\perp}$ on $\mathbb{E}^{2n}$.

II. $G = U(k,\ell)\cdot U(p,q) \subset Sp((k+\ell)(p+q);\mathbb{R})$ where $\min(k,\ell) \geq p + q$. Write $P_{E_1,\ldots,E_t}$ (resp. $P'_{F_1,\ldots,F_u}$) for the parabolic subgroups of the first (resp. second) factor defined by isotropic flags in $\mathbb{C}^{k,\ell}$ (resp. $\mathbb{C}^{p,q}$) as in I. L can be any subgroup of G containing a group of the form L_1L_2, $L_1 \subset U(k,\ell)$ and $L_2 \subset U(p,q)$, where, for some parabolic $P_{E_1,\ldots,E_t} \subset U(k,\ell)$.

(i) $(\ker \chi_2) \cap \cdots \cap (\ker \chi_t) \subset L_1$ if

$\dim E_1 = p + \epsilon$, $(\ker \chi_1) \cap \cdots \cap (\ker \chi_t)$ if $\dim E_1 > p + q$.

(ii) L_2 together with some $U(k',\ell')$, $k' + \ell' = p + q$, generates $\{g \in GL(p+q;\mathbb{C}) : |\det g| = 1\}$.

3. Applications. Irreducibility of ladder representations on parabolics is the case $U(k,\ell)\cdot U(1) \subset Sp(k+\ell;\mathbb{R})$, cf. Wolf and Sternberg cited above for maximal parabolics.

A VANISHING THEOREM

by

Gregg J. Zuckerman

Let G be a real simple Lie group, K a maximal compact subgroup $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G; (π,H) a nontrivial unitary representation of G; (σ,V) a finite dimensional irreducible representation of G; Ω the Casimir operator of $\mathfrak{g}$; and ℓ the real rank of G.

Theorem (also obtained independently by Borel-Wallach): If $\pi(\Omega) = \sigma^*(\Omega)$,

$$\mathrm{Hom}_K(\Lambda^i \mathfrak{p}, H \otimes V) = 0 \quad \text{if} \quad i < \ell .$$

Corollary: Suppose Γ is a cocompact discrete subgroup of G. If V is nontrivial, $H^i(\Gamma,V) = 0$ for $i < \ell$. If V is the trivial module,

$$H^i(\Gamma,V) \cong (\Lambda^i \mathfrak{p})^K \qquad \text{for} \quad i < \ell .$$

<u>Proof of Corollary:</u> Use Matsushima's Formula

$$H^*(\Gamma,V) \cong \bigoplus_{\substack{\pi\in\hat{G} \\ \pi(\Omega)=\sigma^*(\Omega)}} \mathrm{Hom}_K(\Lambda^*\mathcal{P},H_\pi\otimes V) \otimes \mathrm{Hom}_G(H_\pi,L^2(G/\Gamma))$$

($\hat{G}$ is the unitary dual of G).

Some earlier results:

1. (Matsushima): Given G, there exists a constant $c_G \geq \frac{1}{4}\ell$ such that $H^i(\Gamma,\mathbb{C}) \cong (\Lambda^i_{\mathcal{P}})^K$ for $i < c_G$. Sometimes c_G is much greater: if G is the restriction of scalars of complex E_8, $c_G = 14$ (Kaneyaki-Nagano, Osaka Math. J., 14, 1962).

2. (Ragunathan-Borel): If V is nontrivial, $H^i(\Gamma,V) = 0$ for $i < \ell$; if the highest weight of V is regular, and $\mathrm{rk}\, K = \mathrm{rk}\, G$, $H^i(\Gamma,V) = 0$ for $i < \frac{1}{2}\dim G/K$. Proofs were obtained by curvature methods.

3. (Garland): If G is a <u>p-adic</u> simple group and the residue characteristic is large, then for Γ cocompact discrete, $H^0(\Gamma,\mathbb{C}) = \mathbb{C}$ and $H^i(\Gamma,\mathbb{C}) = 0$ for $0 < i < \mathrm{rk}\, G$ (over the ground field). The proof was by an ingenious method of combinatorial curvature.

4. (Casselman): A new proof of Garland's theorem; the residue characteristic was removed, and the theory of continuous cohomology of topological groups was combined with the representation theory of p-adic semisimple groups.

5. The vanishing theorem for unitary representations of real simple groups was conjectured by Borel-Wallach by analogy with Casselman's work (fall, 1976); Ragunathan's work was checked for general groups (see 2); Wallach discovered a criterion for an admissible representation to be unitary (this was also discovered independently by B. Speh, in her work on $GL(n,\mathbb{R})$.)

We now sketch the proof of the vanishing theorem (the proof by Borel-Wallach is essentially the same).

Theorem (van Est, Hochschild-Mostow): Let E be a smooth locally convex complete G-module. Then

$$H^*_{cont}(G,E) \cong H^*_{diff}(G,E) \cong H^*(\mathfrak{g},K,E).$$

(Continuous cohomology is computed from continuous Eilenberg-MacLane cochains.)

Proposition (Matsushima, Kuga): Let $E = H^\infty_\pi \otimes V_\sigma$ (H^∞_π is the smooth vectors in H_π):

a) if $\pi(\Omega) \neq \sigma^*(\Omega)$,

$$H^*(\mathfrak{g},K,H^\infty_\pi \otimes V) = 0$$

b) if $\pi(\Omega) = \sigma^*(\Omega)$,

$$H^*(\mathfrak{g},K,H^\infty_\pi \otimes V) \cong \mathrm{Hom}_K(\Lambda^*\mathfrak{p},H^\infty_\pi \otimes V).$$

<u>Proof</u>: Finite dimensional Hodge theory.

<u>Lemma</u> (Wigner): E is a smooth, quasisimple G-module; V is finite dimensional irreducible. If the infinitesimal character χ_E of E is not equal to χ_{V^*}, then

$$H^*_{diff}(G, E \otimes V) = 0.$$

Hence, we must prove the vanishing of $H^i(G, H^\infty \otimes V)$ for $i < \ell$ and $\chi_H = \chi_{V^*}$. However, we have better understanding of admissible representations than we have of unitary representations: we have Langland's classification, discussed by Miličić in his talk at this conference. Each irreducible admissible G-module is isomorphic to the Langland submodule $J(E|P)$ of a full induced module $I(E|P)$, where $E = E^t \otimes e^\lambda$, E^t tempered and λ a real, negative form on the split component $\mathfrak{a}$ of $\mathfrak{p}$.

<u>Definition</u>: $\|J\| = \langle\lambda,\lambda\rangle$. (This is an intrinsic invariant of the module.)

<u>Observation 1</u>: If $J(E'|P')$ is another composition factor of $I(E|P)$, $\|J(E'|P')\| < \|J(E|P)\|$ (this was observed during Miličić's lecture).

<u>Observation 2</u>: If the matrix coefficients of J <u>decay</u>, so do the matrix coefficients of all other J' in I.

<u>Theorem</u> (also discovered in a special case by Delorme, who communicated the result to Borel): If $P = MN$, and E is simply admissible,

$$H^*(G, I(E|P) \otimes V) \cong \oplus H^*(M, E \otimes \delta_P \otimes H^*(N,V))$$

(δ_P) is the usual modular character of P).

<u>Remark on the proof</u>: We use Kostant's calculation of $H^*(N,V) \cong H^*(\mathfrak{n},V)$ to prove that the relavant spectral sequence collapses.

<u>Strategy</u>: Look at the collection $\mathcal{E}$ of J with

(i) matrix coefficient decay

(ii) $\chi_J = \chi_{V^*}$.

$\mathcal{E}$ is finite (by character theory), and partially ordered by $\|J\|$. We try an induction on $\|J\|$.

<u>Proposition</u> (partially known to Langlands in 1972): Let D be a smooth discrete series module. Let $q = \frac{1}{2}\dim G/K$. Then if $i \neq q$, $H^i(\mathfrak{g}, K, D \otimes V) = 0$.

<u>Proof</u>: Compute $\mathrm{Hom}_K(\Lambda^i \mathfrak{p}, D \otimes V)$ via Schmid's "weak Blattner" result (see Schmid's first talk in this conference).

<u>Proposition</u>: Let E be an essentially tempered smooth G-module:

$$H^i(G, E \otimes V) = 0 \quad \text{if} \quad |k-q| > \tfrac{1}{2}(\mathrm{rk}\, G - \mathrm{rk}\, K).$$

In particular,

$$H^i(G,E \otimes V) = 0 \quad \text{if} \quad i < rk_{\mathbb{R}}\, G_{der}.$$

<u>Proposition</u>: Assume $J(E|P) \in \mathcal{E}$. Then, $H^i(G,I(E|P) \otimes V) = 0$ if $i < rk_{\mathbb{R}}\, G_{der}$.

<u>Theorem:</u> Assume $J \in \mathcal{E}$:

$$H^i(G,J \otimes V) = 0 \quad \text{if} \quad i < rk_{\mathbb{R}}\, G_{der}.$$

<u>Proof</u>: We have an exact sequence

$$0 \hookrightarrow J \longrightarrow I \longrightarrow \text{Coker} \longrightarrow 0 .$$

The Coker is composed of lower J's; apply the long exact sequence in cohomology to obtain

$$H^i(G,\ C \otimes V) = 0 \quad \text{for} \quad i < rk_{\mathbb{R}}\, G.$$

Then apply the long exact sequence again to obtain the theorem from Proposition 3.

<u>Unitarity Criterion</u> (discussed during Miličić's lecture: see "earlier results", number 5): If G is real simple and J is nontrivial, unitary, then the matrix coefficients of J decay.

The vanishing theorem for unitary representations now follows.

CLASSIFICATION OF IRREDUCIBLE TEMPERED REPRESENTATIONS

by

Gregg J. Zuckerman

This was a report on the speaker's paper, (with A. W. Knapp), Classification of Irreducible Tempered Characters, Proceedings of the National Academy of Sciences, July 1976.

A new proof using a completeness theorem for a class of tempered invariant eigendistributions, and also using recent results on the minimal K-types of induced from discrete series representations (work of Vogan; see his talk in this conference) was briefly sketched. Character identities of Schmid and Hecht-Schmid were crucial in the speaker's joint work with Knapp. Vogan's work also sheds new light on Knapp's R-group (reducibility group).

Berichterstatter: W.Schmid