MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 35|1977

Mathematische Aspekte der Methode der finiten Elemente

28. 8. bis 3. 9. 1977

Tagungsleiter: J. Nitsche (Freiburg)

Die Tagung diente dem Zweck, den gegenwärtigen Stand sowie neue Ergebnisse und Trends des Gebietes der 'Methode der finiten Elemente' aufzuzeigen.

Bedingt durch die begrenzte Teilnehmerzahl war die Auswahl der Einzuladenden ein besonderes Problem; mit Unterstützung in- und ausländischer Kollegen wurde ein gewisser Kompromiß erreicht.

Das Programm umfaßte 29 gleichberechtigte Vorträge von je 40 Minuten Dauer, es wurde erreicht durch den dankenswerten Verzicht einiger Teilnehmer auf einen Vortrag. Daneben blieb genügend Raum für Gedankenaustausch und Kontakte.

Herrn Professor Barner gilt unser Dank, daß gewisse Probleme - nicht zuletzt finanzieller Natur - gelöst wurden. In gleicher Weise gilt unser Dank den Mitarbeiterinnen und -arbeitern des Instituts für die vorzügliche Organisation und Betreuung.

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Vortragsauszüge

I. BABUŠKA: A Posteriori Error Estimates in the Finite Element Method

A posteriori error estimates for general finite element method based on general bilinear form will be discussed. The error estimates are in some sense optimal and based on completely local analysis of the computed solution. The results will be applied also in connection with (optimal) mesh generator.

G.A. BAKER: Finite-Element Methods for the Navier Stokes Equations

For Ω a bounded polyhedral domain in R^N, $N=2$, or $N=3$, we consider the problem of obtaining via non-standard Galerkin methods, finite element approximations for the initial boundary value problem, for the Navier Stokes equations. A vector valued function $u: [0,T] \to R^N$ and a scalar $p : [0,T] \to R^1$ are sought satisfying

$$u_t - \nu\Delta u + (u\cdot\mathrm{grad})u + \mathrm{grad}\, p = f \quad ,$$

and $\mathrm{div}\, u = 0$ in $\Omega x(0,T]$, with $u = 0$ on $\partial\Omega x(0,T]$ and $u(\cdot,0) = u^0$. f and u^0 are given vector valued functions and $\nu > 0$ denotes the coefficient of kinematic viscosity, a constant.

Using a Lagrange multiplier method the constraint $\mathrm{div}\, v = 0$ is aliviated in the discrete problems. Both semidiscrete and fully discrete approximations are obtained with optimal L^2 rates of convergence for the flow and the pressure. The fully discrete scheme requires the solution of linear systems of equations at each time level and is second order accurate in the time descretization.

J.H. BRAMBLE: Multistep-Galerkin Methods for Parabolic Equations

Some multistep-galerkin methods for parabolic initial-boundary value problems are considered. These methods are not in the usual category of linear multistep methods. Stability, accuracy and efficiency of these methods is discussed.

F. BREZZI: Finite Element Approximations of the von Kármán Equations

The von Karman equations, given by

$$(E)\quad \begin{cases} \Delta^2\varphi = -1/2\ [w,w] \text{ in } \Omega\ , \\ \Delta^2 w = [w,\varphi] + f \text{ in } \Omega\ , \\ \varphi, w \in H_0^2(\Omega)\ , \end{cases} \qquad \begin{aligned} &\Delta \overset{\text{df}}{=} \text{Laplace operator}, \\ &[u,v] \overset{\text{df}}{=} u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy} \end{aligned}$$

with f specified and $\Omega \subseteq \mathbf{R}^2$, determine one of the simplest problems of nonlinear elasticity theory and apply to the bending of a thin, elastic, isotropic plate. We prove that, if (φ,w) is an isolated solution of (E) , the discretised problem, under suitable assumptions, has a unique solution (φ_h,w_h) in a neighborhood of (φ,w) , and the error $(\varphi,w) - (\varphi_h,w_h)$ is asymptotically optimal. Moreover, the Newton iterates converge quadratically to (φ_h,w_h). We show that the "suitable assumptions" cover various kinds of finite element approximations, including the H_0^2-conforming f.e.m., the mixed approach of Herrmann-Hellan-Johnson, and part of the most used non-conforming methods.

P.G. CIARLET: Approximation of 3-d Models by 2-d Models

Usually, a plate theory is derived from a three-dimensional model by making a priori assumptions on the form of some of the unknowns (the displacement vector and the stress tensor). In ajoint work with Ph. Destuynder, the author has developped a method in which (i) no a priori assumption is needed, (ii) the standard two-dimensional linear plate model and the standard a priori assumptions are simultaneously obtained, (iii) error estimates are systematically obtained. One of the motivations behind this work is to provide a natural supplement to the traditional error estimates relative to finite element approximations of two-dimensional plate models.

J. DESCLOUX: Numerical Approximation of the Spectrum of Non-Compact Operators

Let U and V be complex Hilbert spaces with $U \subset V$ (continuous non-compact injection), $a : U \times U \to \mathbb{C}$ be a sesquilinear continuous and coercive form, $A : U \to U$ be the operator defined by $a(Au,v) = (u,v)_V \ \forall\, u,v \in U$. Let $\{U_h\}$ be a family of finite dimensional subspaces of U and $A_h : U_h \to U_h$ be defined by $a(A_h u,v) = (u,v)_V \ \forall\, u,v \in U_h$.

The spectrum of A is "well" approximated by the spectrum of A_h if:

P1: $\exists\, \varepsilon_h, \ \lim_{h \to 0} \varepsilon_h = 0$ such that $\forall\, u \in U_h, \exists\, w \in U_h$ with

$$\|Au-w\|_U \le \varepsilon_h \|u\|_U \quad ,$$

P2: $\forall\, u \in U, \ \exists\, u_h \in U_h$ with $\lim_{h \to 0} \|u-u_h\|_U = 0$.

One discusses P1, P2 on examples using finite element subspaces. The fundamental example is $U = V = H^m(\Omega)$, $\Omega \subset \mathbb{R}^h$, where A is the "multiplication" operator. The conditions can be verified for some partial differential operators arising from problems in magneto-hydrodynamics. However, they cannot likely be satisfied for general cases.

T. DUPONT: (with R.E. Bank):
An Optimal Order Procedure for Solving Elliptic Finite Element Equations

Two iterative methods for solving the algebraic equations that result from the application of finite element methods to elliptic problems will be presented. One of these methods can be shown to give the answers (to an appropriate accuracy) in $O(N)$ operations, where N is the number of unknowns. Such a method is said to be optimal order in terms of work.

Both of these methods can be used in rather general geometric situations. The optimal order procedure is related to multigrid methods considered by Federenko, Bakhvalov, Brandt, and others.

R.S. FALK: Error Estimates for a Class of Inverse Problems

In this talk we consider the approximation of a class of inverse problems in which the problem is to determine an unknown coefficient in a differential equation whose general form is known. One simple model of such problems is to determine a constant a and a function $u(x)$ satisfying:

$$-a\frac{d^2u}{dx^2} + cu = f\ ,\ 0 < x < 1\ ,\quad u(0) = g_1\ ,\ u(1) = g_2\ ,\ \text{and}$$

$$-au'(0) = g_3 \quad \text{where} \quad c, f, g_1, g_2\ ,\ \text{and}\quad g_3$$

are assumed known. We give conditions under which both this problem and a simple approximate problem are well posed and then derive an estimate for the error between a and its approximation.

G. GEYMONAT: Spectral Approximations and Error Bounds with Mixed and Hybrid Methods

I will report on some results obtained at Torino on the existence and approximation of eigenvalues and eigenvectors with mixed f.e.m. for the abstract problem: Find $\lambda \in \mathbb{R}$ and $(u,\Psi) \in V \times W$ such that

$$\begin{cases} a(u,v) + b(v,\Psi) = 0 & \forall\ v \in V \\ -b(u,\varphi) = \lambda(\Psi,\varphi)_H & \forall\ \varphi \in W \end{cases}$$

where $W \subseteq H$ with compact imbedding. Under the general assumptions of Brezzi-Raviart the optimal error estimates can be proved and the numerical tests for the plate problem confirm the results.
I will also report on the approximation of the eigenvalue problem corresponding to the hybrid f.e.m.: Find $\lambda \in \mathbb{R}$ and $(u,\Psi) \in V \times W$

$$\begin{cases} a(u,v) + b(v,\Psi) = \lambda(u,v)_K & \forall\ v \in V \\ b(u,\varphi) = 0 \quad \forall\ \varphi \in W & \end{cases}$$

with $V \subset H$.

W.W. HAGER: A Mixed Method for Variational Inequalities

Previously Brezzi, Hager, and Raviart analyzed the error in the primal finite element approximation to two variational inequalities - the obstacle problem and the unilateral problem. We now consider mixed finite element approximations. The mixed method, previously utilized by Raviart and Thomas for the solution of second order, elliptic problems, is extended to treat a class of elliptic variational inequalities. For both model problems, we prove $O(h)$ convergence in L^2 for function values and gradients using piecewise "constant" elements. Furthermore using "linear" elements, we prove $O(h^{3/2-\epsilon})$ convergence for the obstacle problem. Finally we show that there is no reduction in the convergence rate when the skin between the triangulated domain and the true domain is ignored.

R. HAVERKAMP: The Method of Rayleigh-Ritz for Boundary Value Problems of Singular Ordinary Differential Equations

A numerical method is described for the solution of the boundary value problem

$$u'' + au' - bu + f = 0 \quad \text{in } (0,1) , \quad u(0) = u(1) = 0$$

and the associated eigen-value problem where a and b have singularities of first and second order, respectively. Approximating functions in the Ritz procedure are appropriately weighted finite elements. It is shown that up to a factor $\log \frac{1}{h}$ on compact sub-intervals of $(0,1]$ the same order of convergence is achieved as in the regular case.

W. HÖHN (H.D. MITTELMANN): The Discrete Maximum Principle for Finite Elements

A triangulation of a plane domain is said to satisfy the strong discrete maximum principle (d.m.p.) with respect to a subspace of discrete harmonic Lagrange finite element functions if it satisfies a d.m.p. locally. Necessary and sufficient conditions are given on the local geometry of the triangulation for piecewise quadratic functions. Globally they reduce to the equilateral and the standard rectangular mesh in the interior of the domain. Counter examples are given for cubic functions and for the continuous d.m.p. for piecewise quadratics.

P. JAMET: Galerkin-Type Approximations Which are Discontinuous in Time for Parabolic Equations in a Variable Domain

Let $\Omega(t)$ be a given time-dependent bounded domain in $\mathbb{R}^m$ for $0 \leq t \leq T$ and $\Gamma(t)$ be its boundary. We consider problems of the type

$$
1) \quad
\begin{aligned}
&\text{a)}\ \frac{\partial u}{\partial t} - A\,u = f && \text{in } \mathcal{O}_T = \{(x,\ t)\ ;\ x \in \Omega(t)\ ,\ 0 < t < T\}\\
&\text{b)}\ Bu = 0 && \text{on } \Sigma_T = \{(x,\ t)\ ;\ x \in \Gamma(t)\ ,\ 0 < t < T\}\\
&\text{c)}\ u = u^0 && \text{in } \Omega(0)\ ,
\end{aligned}
$$

where A is an elliptic operator of order 2μ , B is a boundary operator, $f \in L^2(\mathcal{O}_T)$ and $u^0 \in L^2(\Omega(0))$ are given functions.

We divide the interval $[0,T]$ into N sub-intervals $(t^n\ ,\ t^{n+1})$ and approximate u by a function u_h which is continuous in each strip $t^n < t \leq t^{n+1}$ and which admits discontinuities at the times $t = t^n$, $0 \leq n < N$; we use an integral relation obtained by multiplying (1.a) by a test function φ and integrating by parts in each strip $t^n \leq t \leq t^{n+1}$. Unconditional stability is proved and a general error estimate in $L^2(0,\ T\ ;\ H^\mu(\Omega(t))$ and $L^2(\Omega(t^n))$ for each n is established. These results are applied to space-time finite element methods : each strip $(t^n,\ t^{n+1})$ is partitioned into $(m+1)$-simplices or $(m+1)$-dimensional prisms.

C. JOHNSON: A Mixed Equilibrium Finite Element Method for Problems in Continuum Mechanics

I described some joint work with Bertrand Mercier on equilibrium finite element methods for problems in continuum mechanics. I presented a mixed element based on piecewise linear approximation of stresses and displacements and discussed its application to problems in elasticity, plasticity, and fluid mechanics.

A. LOUIS: Acceleration of Convergence for Finite Element Solutions of Linear Differential Equations on Irregular Meshes

Let A be a linear elliptic differential operator of order $2m$ with variable coefficients and u_h be the finite element approximation to $Au = f$ in Ω with zero boundary conditions. In a joint work with F. Natterer the author has developped a method for calculating from u_h for each $z \in \Omega_o \subset\subset \Omega$ an approximation $\bar{u}_h(z)$ to $u(z)$ with $|\bar{u}(z)-u_h(z)| = O(h^{k+r-2m})$ where $r = \min(k,4m)$ and k is the order of the finite elements. In contrast to earlier results of Bramble and Schatz we need not work on a regular mesh but we have to compute global averages of u_h.

F. NATTERER: Ill-Posed Problems and Finite Elements

Let A be an operator in $L_2(G)$, G a domain in R^d, such that the norms $\|Ax\|$, $\|x\|_{H^{-a}}$ are equivalent. The equation $Ax = y$ is solved by the least squares procedure using suitable finite elements S_h as trial functions. If $x \in H^t$, then the estimate $\|x-x_h\| \leq h^t\|x\|_{H^t} + h^{-a}\|y-\tilde{y}\|$ holds if the computations are carried out with an approximation $\tilde{y}$ to y. As an application it is shown that x can be calculated from $y(t_1),\dots y(t_n)$ and $\|x\|_{H^t}$ up to an accuracy $O(n^{-t/d})$. It is also shown that this estimate is sharp. The surprise is that this accuracy does not depend on the degree of the ill-posedness as measured by the number a.

J.C. NEDELEC: Finite Element Approximations for Some Singular Integral Equations

Let V be a Hilbert space and $A \in \mathfrak{L}(V,V^*)$ of the form $A = J + K$, J symmetric and $(Jq,q) \geq \alpha\|q\|_V^2$, $J^{-1}K \in \mathfrak{L}(V,W)$, $W \subset V$ and compact into V. Then if we choose $V_h \subset V$ such that $\inf_{v_h \in V_h} \|u-v_h\|_V \leq c(h)\,\|u\|_W$ and A_h is an approximate operator of A such that

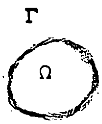
$$|(A_h u_h, v_h)_h - (Au_h, v_h)| \leq c(h)\, \|u_h\|_V\, \|v_h\|_V \quad \text{and} \quad c(h) \to 0$$

then

$$\|u-u_h\|_V \leq c \left\{ \inf_{v_h \in V_h} \|u-v_h\|_V + \sup_{w_h \in V_h} \frac{|(A_h u_h, w_h) - (Au_h, w_h)|}{\|w_h\|_V} \right.$$

if A^{-1} exists and we solve $Au = f$ and $(A_h u_h, v_h)_h = (f, v_h)$.

We then give two applications to the equation of

(1) 

$$\Delta u = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_\Gamma = g$$

$$\begin{cases} u(y) = \dfrac{1}{4\pi} \displaystyle\int_\Gamma q(x) \frac{1}{|x-y|}\, d\gamma(x) , & y \in R^3 \\ g(y) = -\dfrac{q(y)}{2} + \dfrac{1}{4\pi} \displaystyle\int q(x) \frac{\partial}{\partial n_x} \left(\frac{1}{|x-y|}\right) d\gamma(x) & \end{cases}$$

(2) 

$$\Delta u + k^2 u = 0, \qquad u|_\Gamma = u_o$$

$$u_o(y) = \frac{1}{4\pi} \int_\Gamma q(x) \frac{e^{ik|x-y|}}{|x-y|}\, d\gamma(x) , \quad y \in R^3$$

where we use finite element approximations of the surface and of the unknown functions q .

J.E. OSBORN: <u>Approximation of Eigenvalues of Differential Equations with Rough Coefficients</u>

When the eigenvalues of a differential equation are approximated by the usual Ritz-Galerkin method, the accuracy of the approximations depends on the smoothness of the eigenfunctions. However, in many problems (e.g., problems arising in the study of composite materials) the coefficients in the differential equation, and hence the eigenfunctions, are rough. A method proposed by Nemat-Nasser which yields accurate approximations to the eigenvalues of such problems is discussed. Rate of convergence estimates and numerical computations are presented.

R. RANNACHER: A Natural Finite Difference Scheme Interpreted as a Perturbed Finite Element Method

The elliptic boundary value problem on a bounded domain $\Omega \subset R^2$

(D) $-\partial_i(a_{ik}\partial_k u) + au = f \quad \text{in} \quad \Omega \quad , \quad u = o \quad \text{on} \quad \partial\Omega \quad ,$

and its natural finite difference analogues

$$(D_h) \quad -\frac{1}{2} \sum_{+,-} D_i^{\mp h}(a_{ik}D_k^{\pm h}U_h) + aU_h = F_h \quad \text{in} \quad \Omega_h \ , \quad U_h = o \quad \text{on} \quad \partial\Omega_h \ ,$$

are considered. It is observed that the schemes (D_h) may be interpreted as a perturbed finite element method. This allows the application of new techniques guaranteing the order $O(h^{1-\varepsilon})$ of global pointwise convergence even for $u \in H^{2,2}(\Omega)$. This generalizes the known results derived by discrete maximum principles or discrete Sobolev and Morrey space methods. Further a refined boundary value approximation is constructed which leads especially for the Laplacian to the order $O(h^{2-\varepsilon})$ under such weak conditions on u allowing also discontinuous f .

P.A. RAVIART: Finite Element Approximation of First Order Systems

Let Ω be a bounded domain of $\mathbb{R}^n$ with boundary Γ . We consider the first order systems

$$(1) \qquad \sum_{i=1}^{n} A_i \frac{\partial u}{\partial x_i} + A_o u = f \quad \text{in} \quad \Omega$$

where the $p \times p$ matrices A_i , $0 \le i \le n$, satisfy Friedrich's conditions

$$A_i = A_i^* \ , \ A_o + A_o^* - \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i} \ge c_o I \ , \ c_o > 0 \ .$$

The boundary conditions are of the form

$$(2) \qquad (B - M)u = 0 \quad \text{on} \quad \Gamma \ ,$$

where $M + M^* \ge 0$, $B = \sum_{i=1}^{n} A_i \nu_i$ and $\nu = (\nu_1 , \ldots, \nu_n)$ denotes the outer normal to Γ . In this lecture, we present some general results obtained by Lesaint and the author concerning the finite element approximation of problem (1) , (2) when using different spaces for the trial functions and for the space functions. Applications to the numerical solution of the neutron transport equation and of the heat equation in non-cylindrical domains are given.

A. SCHATZ: L_∞ Estimates on Piecewise Smooth Domains

Consider the F.E.M. for the model problem $-\Delta u = f$ in Ω $u = g$ on $\partial\Omega$ where for simplicity Ω is a polygonal domain (not necc. convex). Then if S_r^h is a space of finite elements defined on a quasi-uniform partition of Ω we have

I) $$\|u-u_h\|_{L_\infty(\Omega)} \leq c\left(\ln\frac{1}{h}\right)^2 \inf_{\substack{\chi\in S_r^h \\ \chi=g_h}} \|u-\chi\|_{L_\infty(\Omega)}$$

where u_h is the F.E.M. solution of $u_h = g_h$ on $\partial\Omega$.

II) We next discussed an a priori estimate for discrete harmonic functions. Namely if u_h is a discrete harmonic function then

$$|u_h|_{L_\infty(\Omega)} \leq c \left|\ln\frac{1}{h}\right| |u_h|_{L_\infty(\partial\Omega)}$$

where the constant depends on the Lipschitz character of the boundary. This estimate can serve as a substitute for a maximum principle in some problems where the nature of the boundary is known.

III) L_∞ estimates for the finite element methods were discussed when singular functions are added to the subspace in order to increase the rate of convergence near the corners. Some peculiar behavior in L_∞ was discussed for this method.

The results of I and II are generalizable to piecewise smooth domains when isoparametric elements are used.

R. SCOTT: Applications of Banach Space Interpolation to Finite Element Approximation

We use the real method of interpolation developed by Lions and Peetre to derive two approximation results having applications in finite element theory. The first is a "simultaneous approximation" theorem proved jointly with J.H. Bramble that says that any subspace giving optimal approximation at one point in a Banach scale does so simultaneously at that point and any lower point as well. Applications for the scale of Sobolev spaces of functions having square integrable derivatives are given. The second result uses a scale of such Sobolev spaces incorporating boundary conditions that was studied by P. Grisvard. With this scale, we study the problem of approximating nonsmooth functions satisfying boundary conditions from subspaces that themselves satisfy boundary conditions. The scale itself incorporates a smoothing operator preserving boundary conditions without requiring any explicit construction.

E. STEPHAN: A Finite Element Method for the Biharmonic Equation in a Polygonal Domain

Using the results by Kondratiev the solution of the Dirichlet problem

$$\Delta^2 u = f \quad \text{in} \quad \Omega , \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on the boundary} \quad \partial\Omega$$

can be written in the form

$$u = \sum_{k=1}^{\nu} u_k + w ,$$

with $w \in H^4(\Omega)$ for given $f \in L_2(\Omega)$, where the exceptional functions u_k describe the singular behavior of the solution in a neighborhood of the vertex P_k . On the other hand for $f \in L_2(\Omega)$ there exists exactly one weak solution

$$u_o \in \overset{o}{H}{}^2_\beta(\Omega)$$

of the Dirichlet problem, if

$|\beta_k| < \frac{2}{3} \left(\sqrt{1+3\frac{\pi^2}{\theta_k^2}} - 1\right)$, $\beta_k > -4 \; (\beta \equiv \beta_1, \ldots, \beta_\nu)$, where θ_k is the angle at P_k . Therefore by studying the rate of convergence of a conforming finite element method in suitably weighted Sobolev spaces $\overset{o}{H}{}^2_\beta(\Omega)$, it can be shown that the use of different spaces of trial and test functions restricts the low rate of convergence to a neighborhood of each vertex of the polygonal domain Ω .

G. STRANG: Finite Elements and Optimization

We study, with H. Matthies and E. Christiansen, an infinite dimensional programming problem which arises in solid mechanics. It concerns the moment of collapse for a plastic structure which is subjected to increasing loads. The fundamental result is the duality between the static and kinematic theorems of limit analysis, and our contribution is to prove that a saddle point does exist; we can choose admissible sets for the stresses and displacements within which the collapse states can be found, and the sup-inf theorem becomes a genuine "minimax". Then we discuss the aprox- imation of the infinite problem by a family of discrete (finite) problems, and describe numerical experiments in which this dis- cretization is based on finite elements.

F. STUMMEL: Convergence Conditions in Methods of Nonconforming Finite Elements

The lecture describes a series of recent results. First it is shown by a simple counterexample that success in Irons' patch test is not sufficient for convergence of nonconforming approximations. Neither is this necessary as may be seen from approximations of boundary value problems with variable coefficients or nonconforming elements satisfying the required continuity conditions only approximately or excepting a sufficiently small subset of nodal points. Next, a generalized and improved patch test (see Stummel, ZAMM 58, no 5) is explained. Using this test, conditions are stated ensuring the convergence of approximations of generalized elliptic boundary value problems with variable coefficients without particular regularity assumptions. The improved patch test is passed, for example, by the well-known nonconforming elements of Wilson, Crouzeit-Raviart, Adini, Morley, Freijs-de Veubecke and by Zienkiewicz's triangles in regular meshes. Finally, conditions are formulated which guarantee the fundamental discrete and collective compactness of natural embeddings of nonconforming, piecewise polynomial function spaces. On this basis, one can establish general stability theorems under coerciveness conditions, basic norm equivalences and the convergence of nonconforming approximations of eigenvalue problems.

V. THOMÉE: Some Interior Estimates for Semidiscrete Galerkin Approximations for Parabolic Equations

Consider a solution u of the parabolic equation

$$u_t + Au = f \quad \text{in} \quad \Omega \times [0,T] \quad ,$$

where A is a second order elliptic differential operator. Let $\{S_h;\ h \text{ small}\}$ denote a family of finite element subspaces of $H^1(\Omega)$ which permits approximation of a smooth function to order $O(h^r)$. Let $\Omega_o \subset \Omega$ and assume that $u_h : [0,T] \to S_h$ is an approximate solution which satisfies the semidiscrete interior equation

$$(u_{h,t},\chi) + A(u_h,\chi) = (f,\chi) \quad \forall \quad \chi \in \overset{o}{S}_h(\Omega_o) = \{\chi \in S_h,\ \text{supp}\ \chi \subset \Omega_o\},$$

where $A(.,.)$ denotes the bilinear form on $H^1(\Omega)$ associated with A. It is shown that if the finite element spaces are based on uniform partitions in a specific sense in Ω_0, then the difference quotients of u_h may be used to approximate derivatives of u in the interior of Ω_0 to order $O(h^r)$ provided certain weak global error estimates for $u_h - u$ to this order are available. This generalizes results proved for elliptic problems by Nitsche and Schatz [Math. Comp. 28 (1974), 937-958] and Bramble, Nitsche and Schatz [Math. Comp. 29 (1975), 677-688].

L.B. WAHLBIN: On the Finite Element Method in Plane Polygonal Domains

I shall report on some results obtained jointly with A. Schatz concerning error estimates in the maximum norm for the finite element solution of the problem

$$- Au = f \quad \text{in} \quad \Omega \, , \quad u = 0 \quad \text{on} \quad \partial\Omega$$

where Ω is a plane polygonal domain.

General finite element partitions are considered, in particular both quasi-uniform subdivisions and such that are refined systematically at vertices. For the latter kind, the amount of refinement necessary to obtain a desired rate of convergence is investigated in some detail.

B. WERNER: About Some Nonconforming Finite Elements Based on the Complementary Energy

Some nonconforming (displacement) methods for some linear elliptic boundary value problems are characterized whose solutions are conforming for the complementary energy principle yielding lower bounds for the energy.

The main condition which can be tested elementwise is a compatibility condition between the bilinear map $a(.,.)$ defining the boundary value problem and the used finite elements. It is a consequence of this condition that the nonconforming method using those finite elements satisfies a generalized patch test.

Examples for those elements are

1. The piecewise linear triangular element with the mean values along the edges as degrees of freedom (for $\Delta u = 0$ in Ω , $u = u^0$ on $\partial\Omega$) .

2. The Morley element

   and

3. some rectangular versions of these both elements.

M.F. WHEELER: A Local-Residual-Finite-Element-Method

P. Percell and M.F. Wheeler have defined and derived optimal error estimates for a local-residual-element method for elliptic boundary value problems. The procedure is local in the sense that the equations involving the differential operator are of the form

$$\int_{\tau} (LU - f)\Delta v \, dx = 0 ,$$

where τ is an element of the triangulation of the domain. The remaining equations which penalize jumps in value and normal derivative are independent of L and f and involve only one dimensional integrals.

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