MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 43 / 1977

Zahlentheorie

6.11. bis 12.11.1977

Die diesjährige Tagung über Zahlentheorie, insbesondere elementare und analytische Zahlentheorie, fand wieder unter der Leitung der Herren H.-E. Richert (Ulm), W. Schwarz (Frankfurt) und E. Wirsing (Ulm) statt. Da die Teilnehmerzahl (56 Zahlentheoretiker aus 15 Ländern) und die Zahl der Vorträge (44) gegenüber 41 bei der Tagung im November 1975 (41) weiter gestiegen waren, wurde diesmal, anscheinend erfolgreich, eine neue Organisationsform ausprobiert: In den ersten drei Tagen stellte jeder der Vortragenden in einer Viertelstunde seine Hauptergebnisse vor, an den beiden letzten Tagen wurden in Parallel-Veranstaltungen ausführlichere Vorträge gehalten, nachdem jeder Teilnehmer eine Wunschliste mit den Themen eingereicht hatte, über die er mehr erfahren wollte. Daneben blieb noch genügend Zeit für persönliche Diskussionen.

Teilnehmer :

P. Bundschuh, Köln
H. Delange, Orsay
J. M. Deshouillers, Bordeaux
H. G. Diamond, Urbana
R. Freud, Budapest
W. L. Fouché, Leiden
J. Galambos, Philadelphia
E. Grosswald, Philadelphia
K.-B. Gundlach, Marburg
K. Győry, Debrecen
F. Halter-Koch, Essen
D. G. Hazlewood, San Marcos
E. Heppner, Frankfurt
J. G. Hinz, Marburg

M. N. Huxley, Cardiff
K. -H. Indlekofer, Paderborn
H. Iwaniec, Warschau
M. Jutila, Turku
J. Kátai, Budapest
J. Knopfmacher, Johannesburg
J. Kubilius, Vilnius
L. Kuipers, Mollens
D. Leitmann, Clausthal-Zellerfeld
L. Lucht, Clausthal-Zellerfeld
K. Mahler, Canberra
M. Mendes France, Bordeaux
H. Möller, Münster
Y. Motohashi, Tokyo
W. Narkiewicz, Wroclaw
B. Novák, Prag
J. Pintz, Budapest
R. A. Rankin, Glasgow
H. -E. Richert, Ulm
G. J. Rieger, Hannover
B. Saffari, Boulogne-Billancourt
H. Sarges, Marburg
W. Schaal, Marburg
Th. Schneider, Freiburg
P. G. Schmidt, Marburg
A. Schinzel, Warschau
W. Schwarz, Frankfurt
F. Schweiger, Salzburg
J. -P. Serre, Paris
H. Siebert, Ulm
J. Sliwa, Wroclaw
S. A. Stepanov, Moskau
C. L. Stewart, Bures-sur-Yvette
P. Szüsz, Stony Brook
R. Tijdeman, Leiden
K. Väänänen, Freiburg
R. C. Vaughan, London
B. Volkmann, Stuttgart
R. Wallisser, Freiburg
R. Warlimont, Regensburg
E. Wirsing, Ulm
D. Wolke, Freiburg

Vortragsauszüge

H. Delange: On integral-valued multiplicative functions

Let f be an integral-valued multiplicative function. We prove that for every integer $k \geq 2$ and every integer l, the set of those positive integers n for which $f(n) \equiv l \bmod k$ possesses a density, $\delta_{k,l}$ say.
1) For a given k, we give necessary and sufficient conditions for that density to be the same for all l's coprime to k.
2) Narkiewicz had conjectured that, if $\delta_{k,l} = \frac{1}{k}$ for every k and l,

then necessarily $f(n)=n$. We show that this is not true, even if we suppose f to be completely multiplicative and positive. This follows from the following

Theorem: Let f be an integral-valued completely multiplicative function satisfying $f(n)>0$ for every n. Then $\delta_{k,l}=\frac{1}{k}$ for every k and l if and only if the following conditions hold:

1. For each prime p, $f(p)$ is a power of a prime or 1
2. For every prime q and every positive integer r:

$$\sum_{ij=r}\left(i\sum_{f(p)=q^i}\frac{1}{p^j}\right)=\frac{1}{q^r}.$$

(Comptes rendus Acad. Sci. Paris, 1977, 1325-1327)

J. M. Deshouillers: Additive bases

Let A be a sequence of integers, $0\in A$, $1\in A$; A is said to be a basis if there exists h such that $hA:=\{a_{i_1}+\ldots+a_{i_n},\ a_{i_j}\in A\}=\mathbb{N}$. Several criteria implying that A is not a basis are discussed, among them the following

Theorem: Let f be a sub-additive ($f(m+n)\leq f(m)+f(n)$) function from $\mathbb{N}$ to $\mathbb{R}^+$.

(i) If f is not bounded and $\lim \frac{f(a_i)}{\operatorname{Max}_{n<a_{i+1}} f(n)}=0$, then A is not a basis.

(ii) If f is bounded and $\lim_{\varepsilon\to 0+}\bar{d}\{n\in\mathbb{N},\ f(n)\leq\varepsilon\}=0$

($\bar{d}$=upper density), then $\lim_{i\to\infty} f(a_i)=0 \Rightarrow A$ is not a basis.

Examples for such functions f are $f(n)=n$, $f(n)=\alpha_n\cdot n$ where $\alpha_n\searrow$, $f(n)=\sum_{k=0}^{\infty}\|2^{-k}\alpha n\|$, $f(n)=\sum_{k=0}^{\infty}\frac{\varepsilon_k(n)}{k+1}$ where $n=\sum_{k=0}^{\infty}\varepsilon_k(n)2^k$, $\varepsilon_k(n)\in\{0,1\}$.

(to appear in "Proc. Bordeaux Conf. on Number Theory", Springer Lecture Notes)

H. J. Diamond: <u>A measure of the non-monotonicity of Euler's function</u>
(joint work with P. Erdös)

Set $F(n) = \#\{j<n, \varphi(j) \geq \varphi(n)\} + \#\{j>n, \varphi(j) \leq \varphi(n)\}$.
The value $F(n)$ counts the "reversals" from monotonicity at n.
The main result is the asymptotic formula $F(n)/n \sim h(\varphi(n)/n)$, as $n \to \infty$, for a function h which is strictly convex and satisfies $h(0)=1, h(1) = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} - 1 \approx .9436$. Here ζ is the Riemann zeta function.
We have been able to estimate the minimal value h_o of $h(u)$ and the point u_o where the minimum is achieved. This is done with the aid of an identity connecting $h(\frac{1}{2})$ with $h(\frac{1}{4})$, $h(\frac{1}{8})$, ... and numerical bounds of Charles R. Wall for the distribution function of $\varphi(n)/n$. We find that $.321 < h_o < .324$ and $.473 < u_o < .475$. (to appear in Pacific J. Math.)

W. Fouché: <u>On the largest prime factor of integers having r distinct prime divisors</u>

For $n \in \mathbb{N}$ let $w(n)$ be the number of distinct prime factors and $P(n)$ the largest such factor. Write for $r \in \mathbb{N}$ and $h \geq 1$

$$\pi_r(x,h) = \#\{n \leq x; w(n)=r,\ P(n) \leq x^{1/h}\}$$

and
$$\psi(x,h) = \#\{n \leq x;\ P(n) \leq x^{1/h}\}.$$

The following theorems are discussed:

<u>Theorem 1:</u> Given positive constants A, B, H with $H>1$. With every $x \geq x_o$ let an integer r and real number h be associated satisfying $A \log\log x \leq r \leq B \log\log x$ and $1 \leq h \leq H$. Write $\alpha(x)+1 = r/\log\log x$. Then $\pi_r(x,h) \sim \rho(\alpha(x);h)\,\pi_r(x,1)$ as $x \to \infty$, uniformly in h and r. Here ρ is the continuous real-valued function with domain $\{(\alpha,h); \alpha > -1, h > 0\}$ defined as follows: 1) $\rho(\alpha,h)=1$ for $\alpha > -1$, $0 < h \leq 1$,
2) $\rho(\alpha,h) = 1 - (\alpha+1)\int_1^h u^{-1}(1-u^{-1})^{\alpha}\,\rho(\alpha,u-1)\,du$ for $\alpha > -1$, $h > 1$.

Theorem 2: $$\frac{1}{\psi(x,h(x))} \sum_{\substack{n\le x \\ P(n)\le x^{1/h(x)}}} w(n) \sim \log\log x + h(x)\log h(x)$$

as $x\to\infty$, uniformly in $1\le h(x)\le(\log x)^{3/7}$.

R. Freud : On sets characterizing additive arithmetical functions

A set $\mathcal{A}=\{a_1<a_2<\dots\}\subseteq\mathbb{N}$ is said to be a U-set for additive functions if $f(a_k)=0$, $k=1,2,\dots$, f additive, implies $f(n)=0$ for all $n\in\mathbb{N}$. We determine the maximal possible "rate of growth" of the elements of a U-set: If $\mathcal{A}$ is a U-set, and $a_{k+1}/a_k^2=:e_k$, then $\underline{\lim}\, e_k\le 1$ and moreover $\underline{\lim}\,(e_1\dots e_k)=0$. Conversely, to any numbers α_k with $\alpha_k\ge 2^{-k}$, $\underline{\lim}\,(\alpha_1\dots\alpha_k)=0$ we can construct an $\mathcal{A}$ for which $e_k\ge\alpha_k$ holds and even the convergence of $\sum_{k=1}^{\infty} f(a_k)$ implies $f(n)=0$. We have similar results when characterizing the $f=0$ function by weaker conditions (e.g. the convergence of $f(a_k)$ or $f(a_k)-f(a_{k-1})$). We also show that we can achieve $a_{k+1}>g(a_k)$ with an arbitrary g, at a characterizing set $\mathcal{A}$, if we put suitable additional conditions on f. (to appear in Acta Math. Hung. and Acta Arithm.)

J. Galambos: Sequences of prime divisors

Let $n=q_1^{\alpha_1}q_2^{\alpha_2}\dots q_w^{\alpha_w}$ be the basic representation of n where $q_1<q_2<\dots<q_w$. Evidently $q_j=q_j(n)$ and $w=w(n)$. A number of results of the following nature are presented: with suitable chosen sequences $A(x,N)$ of real numbers,

$$\lim_{N\to\infty}\frac{1}{N}\left|\{n\le N; q_j(n)<A(x,N)\}\right| = F(x),$$

where $F(x)$ is a distribution function and the limit is in the sense of weak convergence. Here j may also depend on n and/or N. Special emphasis is on the following choices of j: (i) $j=w(n)-k$ with $k\ge 0$ fixed; (ii) $j\sim\log\log N$; (iii) $j<(1-\varepsilon)\log\log N$ but $j=j(N)\to\infty$ with N and (iv) j is the largest integer not exceeding a given function $g(N)$. The results

are essentially different for the above cases both in the actual form of $F(x)$ and in the magnitude of the normalizing constant $A(x,N)$. Extensions are given which permit to compare the "large" prime divisors of n and $n+1$, as well as an application to the distribution theory of additive functions.

The paper unifies and somewhat extends the results of my recent papers in Publ. Math. Debrecen 23 (1976), 263-266; J. London Math. Soc. 13 (1976), 360-362; Acta Arithm. 31 (1976), 213-218 and J. Number Theory 9 (1[illegible]), 338-341.

E. Grosswald : Nichtfortsetzbare, durch Dirichletsche Reihen darstellbare Funktionen mit Eulerprodukt .

Im Zusammenhang mit der Riemannschen Vermutung wurde unter anderem versucht, Funktionen zu bilden, deren Nullstellen sich so verhalten wie die der Riemannschen ζ-Funktion, entweder unter Annahme oder unter Leugnung der Riemannschen Vermutung. Es bestand die Hoffnung, dann feststellen zu können, unter welcher der beiden Alternativen diese Funktionen bekannte analytische Eigenschaften von $\zeta(s)$ aufweisen. Diese Versuche (Rademacher, Rubel und Straus) sind leider bis jetzt nicht von Erfolg gekrönt worden. Die gegenwärtige Arbeit scheint darauf hinzuweisen, daß nicht viel Aussicht besteht, mit dieser Methode Fortschritte machen zu können. Es werden nämlich Funktionen erklärt, we[illegible]e genau dieselben komplexen Nullstellen wie $\zeta(s)$ und einen Pol 1. Ordnung bei $s=1$ mit Residuum r, $\frac{1}{2} \leq r \leq 1$ besitzen, jedoch nicht über $\sigma=0$ fortsetzbar sind und keine Funktionalgleichung haben.
(gemeinsam mit F. J. Schnitzer, erscheint in Pacific Journal)

K.-B. Gundlach : Die Darstellungen einer Zahl als Summe von Quadraten

Schreibt man mit einer passenden Linearkombination $h_r(z)$ von Eisensteinreihen und einer Spitzenform $h_r^+(z)$

$$\vartheta^{2r}(z) = \left(\sum_{m=-\infty}^{\infty} e^{\pi i m^2 z}\right)^{2r} = h_r(z) + h_r^+(z) ,$$

so erhält man durch Koeffizientenvergleich aus den Fourierentwicklungen die Anzahl $A_{2r}(m)$ der Darstellungen einer natürlichen Zahl m als Summe von $2r$ Quadraten in der Form $A_{2r}(m)=P_{2r}(m)+R_{2r}(m)$, worin $P_{2r}(m)$ (der Koeffizient von $h_r(z)$) eine Teilerfunktion und $R_{2r}(m)$ ein Restglied ist. Eine Spurbildung, angewendet auf Eisensteinreihen zu geeigneten Hilbertschen Modulgruppen, erlaubt es jetzt, Spitzenformen mit einfachen, leicht berechenbaren Fourierkoeffizienten zu Untergruppen der Modulgruppe zu konstruieren und eröffnet damit die Möglichkeit, auch Restglieder $R_{2r}(m)$ durch Teilerfunktionen (in gewissen Zahlkörpern) auszudrücken (erscheint in Glasgow Math. J.)

K. Györy: Some effective results on polynomials with algebraic integer coefficients and S-unit discriminants

Let L be an algebraic number field of degree $n \geq 1$ with ring of integers $\mathbb{Z}_L$. Denote by D_L the absolute value of the discriminant of L . Let $\mathfrak{p}_1, \ldots \mathfrak{p}_s$ be $s \geq 0$ distinct prime ideals of L lying above rational primes $\leq P$ (for $s=0$ let $P=1$) and write $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Let U_S' denote the multiplicative semigroup of those integers of L which are not divisible by any prime ideal different from $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Let δ be a non-zero integer in L with $|\mathrm{Norm}_{L/Q}(\delta)| \leq d$.

Theorem: Let $f \in \mathbb{Z}_L[x]$ be a monic polynomial with degree $k \geq 3$ and discriminant $D(f) \in \delta U'_s$. Then there exist an $a \in \mathbb{Z}_L$, an $\eta \in U'_s$ and an $f^* \in \mathbb{Z}_L[x]$ such that $f(x) = \eta^k f^*(\eta^{-1}(x+a))$ and the maximum absolute value $\overline{|f^*|}$ of the conjugates of the coefficients of f^* satisfies

$$\overline{|f^*|} < \exp\left\{c_1\left[c_2(s+1)^{24}((D_L d^{1/k} P^{(s+1)n})^{3/2}(5^{skn}\log(D_L dP))^{n+1})^m\right]^{sm+4}\right\}$$

where $m = k(k-1)\cdot(k-2)$ and c_1, c_2 are effectively computable positive constants depending only on n and k.

This is a generalization of some of our results obtained previously in the case $s=0$. The theorem can be extended to not monic polynomials with vanishing discriminant. There are several applications to polynomials, to algebraic numbers and to Diophantine equations (see my papers "On polynomials with integer coefficients and given discriminant, V (to appear) and "On discriminant form and index form equations (to appear, joint paper with Z. Z. Papp)) .

E. Heppner : <u>Benachbarte multiplikative Funktionen</u>

Sei $M := \{ f: \mathbb{N} \to \mathbb{C},\ f \text{ multiplikativ},\ f \not\equiv 0 \}$. Bezüglich der Faltung "*" ist M eine abelsche Gruppe mit Einselement $e(e(1)=1, e(n)=0$ für $n>1)$. Die zu f inverse Funktion bezeichnen wir mit $\check{f}$. Die Faltung multiplikativer Funktionen spielt eine wichtige Rolle bei der Herleitung asymptotischer Formeln für $\sum_{n \leq x} f(n)$. Es wird folgendes Problem behandelt: gegeben $f, g \in M$. Wann existiert $h \in D_\beta$ mit $g = f*h$? $(D_\beta := \{h \in M,\ \sum_{n=1}^{\infty} \frac{|h(n)|}{n^\beta} < \infty\},\ 0 < \beta \leq 1)$. Wegen $h(p) = g(p) - f(p)$ ist die folgende Bedingung offenbar notwendig:

Def.: Für $0 < \beta \leq 1$ heißen f und g "β-benachbart", wenn $\sum_p \frac{|f(p)-g(p)|}{p^\beta} < \infty$.

Sei $G_\beta := \left\{ f \in M, \sum_p \frac{|f(p)|^2}{p^{2\beta}} < \infty, \sum_{p,\alpha \geq 2} \frac{|f(p^\alpha)|}{p^{\alpha\beta}} < \infty \right\}$,

$$G_\beta^* := \left\{ f \in G_\beta, \sum_{\alpha=0}^{\infty} \frac{f(p^\alpha)}{p^{\alpha s}} \neq 0 \text{ für } \operatorname{Re} s \geq \beta \right\},$$

$$M_\lambda := \left\{ f \in M, \sum_{n \leq x} |f(n)|^\lambda = 0(x) \right\}.$$

<u>Satz 1 :</u> $f \in G_\beta^*$, $g \in G_\beta$, f und g β-benachbart $\Rightarrow g = f * h$ mit $h \in D_\beta$.

<u>Satz 2 :</u> $\lambda > 1$, $\beta > \frac{1}{2} + \frac{1}{2\lambda} \Rightarrow M_\lambda \subset G_\beta$

Damit folgt aus Satz 1 ein ebenfalls hier vorgetragenes Ergebnis von L. Lucht. Der Vorteil von G_β gegenüber M_λ liegt vor allem in dem angenehmen Verhalten gegenüber der Faltung:

<u>Satz 3</u> (i) $f, g \in G_\beta \Rightarrow f * g \in G_\beta$

(ii) $f \in G_\beta^* \Rightarrow \check{f} \in G_\beta$

(iii) $D_\beta = \{ f \in G_\beta, f \text{ und } e \ \beta\text{-benachbart} \}$.

Satz 1 folgt damit direkt aus $h = g * \check{f}$.

(gemeinsame Arbeit mit W. Schwarz).

J.G. Hinz : <u>Eine Erweiterung des nullstellenfreien Bereichs der Heckeschen Zetafunktion</u>

Es sei K ein algebraischer Zahlkörper vom Grade n über $\mathbb{Q}$. Für ein ganzes Ideal $\mathfrak{a}$ aus K bedeute $N\mathfrak{a}$ seine Norm. Ferner bezeichne χ die Charaktere der engeren Idealklassengruppe modulo eines festen Ideals $\mathfrak{q}$ in K.

Satz : Bei passendem $c=c(K)>0$ liegen im Bereich

$$\sigma \geq 1-\frac{c}{M(\mathfrak{q},t)}, \quad M(\mathfrak{q},t):=\max\left\{\log N\mathfrak{q}, (\log(|t|+3))^{2/3}\cdot(\log\log(|t|+3))^{1/3}\right\}$$

außer der eventuell vorkommenden einfachen reellen Ausnahmenullstelle keine Nullstelle irgendeines $\zeta_K(\sigma+it,\chi)$, $\chi \bmod \mathfrak{q}$.
Zum Beweis wird im wesentlichen die von Mitsui [J. Math. Soc. Japan 20 (1968)] für die Dedekindsche Zetafunktion entwickelte Methode verallgemeinert. Der Satz erlaubt Anwendungen auf die Verteilung von Primidealen in einer Idealklasse $\mathfrak{k}$ der engeren Idealklassengruppe mod $\mathfrak{q}$.

M. N. Huxley: lim inf$(p_{n+1}-p_n)\log p_n$ is a little smaller than hitherto

Let p_n denote as usual the n-th prime number, $E_r := \liminf_{n\to\infty}(p_{n+r}-p_n)/\log p_n$.

Let N be a large positive integer, $n=1,2,\ldots,$

$$Z_r(2n)=\sum_{\substack{p_i-p_j=2n\\ p_i\leq N,\, i\equiv j \bmod r}}\log p_i \log p_j .$$ The small sieve upper bound

$$Z_r(2n)\leq Z_1(2n)=c \text{ . conjectured value of } Z_1(2n) \tag{1}$$

gives $E_r \leq r-\frac{1}{2c}$. Bombieri and Davenport using the circle method or large sieve showed, for any real weights $u(-k),\ldots,u(k)$ and in the case $r=1$

$$r\sum_{n=1}^{2k} Z_r(2n)\sum_{a-b=n}u(a)u(b)\geq N\left|\sum u(n)\right|^2-\frac{1}{2}\left(r-\frac{1}{2}\right)N\log N\sum u^2(n)$$

$$+ \text{ small error term .} \tag{2}$$

By itself, with equal weights this gives $E_r \leq r-\frac{1}{2}$. Bounds (1) and (2) can be used in combination. The optimal weights are given by a difference-

differential equation, piecewise sine and constant. This gives $E_r \leq \frac{2r-1}{4} \left\{ 2r-1+\frac{\pi}{4} \right\}$. A third inequality can be obtained by taking a second set of weights $w(-k),\ldots,w(k)$ with $w(-n)=-w(n)$. Then (2) becomes

$$r \sum_{n=1}^{2k} Z_r(2n) \sum_{a+b=n} w(a)w(b) \leq \frac{1}{2}(r-\frac{1}{2}) N \log N \sum w^2(n) + \text{error} .$$

Luckily the optimal choice of weights and parameters is simple. In particular the former estimate $E_1 \leq 0.446$ is improved to $E_1 = 0,4425\ldots$. (sent to "Mathematika")

K.-K. Indlekofer: <u>Grenzverteilungen und Eindeutigkeitsmengen additiver Funktionen</u>

Es sei $\mathcal{A} = \{a_m\}$, $a_m \in \mathbb{N}$, $f:\mathbb{N} \to \mathbb{R}$ stark additiv und für Primzahlen p:

$$\delta_p(a_m) = \begin{cases} 1 & p \mid a_m \\ 0 & \text{sonst} \end{cases} ; \quad \text{außerdem sei} \quad f_r := \sum_{p \leq r} f(p)\, \delta_p .$$

Im W-Raum $(\Omega_n, \mathcal{P}(\Omega_n), P_n)$ mit $\Omega_n = \{a_1,\ldots,a_n\}$, $P_n(\{a_m\}) = 1/n$ seien die folgenden Bedingungen erfüllt:

(I) $P_n(\delta_{p_1} = \ldots = \delta_{p_l} = 1) = \dfrac{\varrho(p_1)\ldots\varrho(p_l)}{p_1\cdots p_l} + o(1)$, $n \to \infty$.

(II) $\exists$ Folge $r_1(n) \to \infty$ mit $E_n(f_{r_1} - A_{r_1,\varrho})^2 \leq c\, B_{r_1}^2$ und

$$A_{r,\varrho} := \sum_{p \leq r} \frac{f(p)\,\varrho(p)}{p}, \quad B_r^2 := \sum_{p \leq r} \frac{f^2(p)}{p}, \quad A_r := A_{r,1} .$$

(III) $\exists$ Folge $r_2(n) \to \infty$ mit $E_n(f_{r_2} - A_{r_2})^2 \leq c\, B_{r_2}^2$.

Satz 1 : Es gelte I , II , III , B_r^2 konv., $r=\max(r_1, r_2)$.
$\Rightarrow$ ($f_r(a_m)$ besitzt Grenzverteilung $\Leftrightarrow$ $A_{r_2,\rho}$, A_{r_2} konvergieren).
Unter den Voraussetzungen I, $a_m=0(m)$, $\sum_{\substack{m \\ a_m=k}} 1=0(1)$ für $k \in \mathbb{N}$,

lassen sich notwendige und hinreichende Bedingungen für die Existenz der Grenzverteilung von $f(a_m)$ angeben (Satz 2) . Wie üblich heiße $\mathcal{A}$ eine E-Menge (für (vollständig) additive Funktionen) wenn gilt
$f(\mathcal{A})= 0 \Rightarrow f=0$ und C-Menge wenn $\lim_{m\to\infty} f(a_m)$ ex . $\Rightarrow f = 0$.

Satz 3 Sei $\mathcal{A}$ wie in Satz 2, $\rho(p)\neq 0$ für alle p . $\Rightarrow$ $\mathcal{A}$ ist C-Menge für vollst. additive Funktionen.
Zum Schluß werden E-Mengen für vollständig additive Funktionen durch die Darstellbarkeit natürlicher Zahlen $n=a_1^{\alpha_1} \dots a_\ell^{\alpha_\ell}$ ($\alpha_i \in \mathbb{Q}$) charakterisiert.

H. Iwaniec : Applications of sieve methods to diophantive problems

Theorem 1 : Let Θ be any positive irrational number and let $Q(\eta)=x_1^2 + x_2^2 - \Theta(x_3^2 + x_4^2)$. For every $\varepsilon > 0$ there exists an integral η such that $0 < |Q(\eta)| < \varepsilon$.

Theorem 2: Let K/Q be a cubic normal extension. If m satisfies some necessary congruence conditions and if m is sufficiently large, then $m=N\mathfrak{a} + N\mathfrak{b}$ where $N\mathfrak{a}$ and $N\mathfrak{b}$ are norms of integral ideals of K .

Theorem 3 : Let f(x) be a quadratic polynomial and let g(y, z) be a quadratic form. If discriminants of f and g generate different quadratic fields and if f and g satisfy some necessary congruence conditions then

$$\frac{X}{\sqrt{\log X}} \ll \sum_{\substack{x \le X \\ f(x)=g(y,z)}} 1 \ll \frac{X}{\sqrt{\log X}} ,$$

provided X is sufficiently large. (to appear in "Proc. Bordeaux Conf. on Number Theory") .

M. Jutila : Statistical Deuring-Heilbronn phenomenon

Let χ_1 be a real primitive character mod k and let $\beta = 1-\delta$ be a real zero of $L(s, \chi_1)$. Let $k^{\log\log k} \le Q \le e^{\delta^{-a}}$, where a (and b, c below) is a positive constant. Define

$$K(Q) = \{\chi \mid \chi \text{ and } \chi\chi_1 \text{ primitive}, \chi \bmod q , Q \le q \le 2Q\}$$

$$\varphi(s, \chi) = L(s, \chi) L(s, \chi\chi_1) .$$

Suppose that $\delta \le (\log k \log\log k)^{-1/a}$ in order that the interval for Q be non-empty.

Theorem: All $\chi \in K(Q)$, except possibly $Q^2 \delta^b$, satisfy the following conditions:

(i) If $\rho = \beta + i\gamma$ is a zero of $\varphi(s, \chi)$ such that $|\gamma| \le \delta^{-c}$, then ρ is simple and $\beta = \frac{1}{2}$.

(ii) If $\frac{1}{2} + i\gamma_j$, j=1, 2, are two zeros of $\varphi(s, \chi)$ such that

$$|\gamma_j| \le \delta^{-c} , \text{ then } \gamma_1 - \gamma_2 = \frac{2\pi m}{\log(Q^2 k)} \left(1 + 0\left(\frac{\log k}{\log Q}\right)\right) ,$$

where m is a non-zero integer.

It does not seem plausible that the zeros of $\varphi(s, \chi)$ in reality distribute so regularly as the condition (ii) implies. However it is perhaps not quite easy to show that it leads to a contradiction. (accepted by Acta Arithm.)

I. Kátai : On the growth of arithmetical functions .

The following questions were stated and partially solved jointly with P. Erdös.

For a non-negative strongly additive $g(n)$ let $f_k(n) := \max_{j=1,\dots,k} g(n+j)$.

It is obvious that $f_k(0) \le f_k(n)$ for every k and n . Let $\delta(1,\varepsilon) :=$

$$:= \sup_{x \ge 1} \frac{1}{x} \left| \{ n \le x \; ; \text{ for at least one } k \ge 1 : f_k(n) > (1+\varepsilon) f_k(0) \} \right| .$$

Problem 1: On what conditions does the relation (1): $\delta(1,\varepsilon) \to 0 \; (1 \to \infty) \; \forall \varepsilon > 0$ hold? It is obvious that $\sum \frac{1}{p} \min(1, g(p)) < \infty$ is necessary. We were unable to decide if the condition $\sum \frac{1}{p} g(p) < \infty$ was necessary or not. More precise results were proved in the case $g(p) \searrow 0$.

Let $\omega(n)$ be the number of distinct primes of n ,

$O_k(n) = \max_{j=1,\dots,k} \omega(n+j)$, $o_k(n) = \min_{j=1,\dots,k} \omega(n+j)$. Let $\psi(z) = z \log \frac{z}{e} + 1$.

Let $\varrho(u)$ be the inverse of $\psi(z)$ for $z \ge 1$ and $\bar{\varrho}(u)$ the inverse of $\psi(z)$ in $0 \le z \le 1$, $\alpha_{k,n} = \log k / \log\log n$, $\delta(n) = \sup_{k \ge 1} \frac{O_k(n)}{\varrho(\alpha_{k,n})} - 1$.

We proved that $\delta(n) \to 0 \; (n \to \infty)$, apart from a set of n's having zero density.

Problem 2 : Is it true that $\delta(n) = 0((\log\log n)^{-\gamma})$ for almost all n , if γ is an arbitrary constant $< 1/2$?

Concerning $o_k(n)$ we proved a weaker result .

J. Knopfmacher: Generalized Partitions and Modules over Polynomial Rings

A problem on the asymptotic enumeration of finite modules over polynomial rings over a finite field leads to the investigation of the coefficients $a_q(n)$ in the series

$$\sum_{n=o}^{\infty} a_q(n)z^n = \prod_{r=1}^{\infty}(1-qz^r)^{-1} \quad , \quad q > 1 \quad \text{fixed} .$$

Theorem 1: For any fixed $N \geq 2$

$$a_q(n) = \sum_{r<N} A_r(n)\, q^{n/r} + 0(q^{n/N}) \quad \text{as} \quad n \to \infty \quad ,$$

where

$$A_r(n) = \frac{1}{r}\sum_{m=o}^{r-1} e^{-\frac{2\pi imn}{r}} \prod_{\substack{t=1 \\ t\neq r}} Z_q(e^{\frac{2\pi imt}{r}} q^{-\frac{t}{r}}) \quad ,$$

$$Z_q(y) = (1-qy)^{-1} .$$

Theorem 2 : For $q \geq 12,59517\ldots$, $a_q(n) = \sum_{r=1}^{\infty} A_r(n)q^{n/r}$, $n>o$ fixed , the series being divergent otherwise.

These theorems can be extended to cover more general algebraic enumeration questions (apart from the given bound for divergence), and together with such extensions are based on work by J. N. Ridley and D. B. Sears, as well as the speaker. (earlier version appeared in Bull. London Math. Soc. 1976) .

J. Kubilius : On the remainder term in the central limit theorem for additive arithmetic functions

Let $f(m)$ be a real-valued strongly additive arithmetic function. Denote

$$B_n^2 = \sum_{p \le n} \frac{f^2(p)}{p} \quad , \quad f_n(m) = \frac{f(m)}{B_n} \quad ; \quad A_n := \sum_{p \le n} \frac{f_n(p)}{p}, \quad \varrho_n = \sum_{p \le n} \frac{|f_n^3(p)|}{p} .$$

Using analytic methods the following theorem is proved.

Theorem: If $\varrho_n \to 0$ as $n \to \infty$ and there exist two positive constants c_1 and c_2 such that

$$\sum_{\substack{p \le n \\ |f_n(p)| > c_1 \varrho_n}} \frac{\ln p}{p} \le \frac{c_2 \varrho_n^2}{\ln^2 1/\varrho_n} \ln n$$

then the frequency if positive integers $m \le n$, for which $f_n(m) < A_n + x$, equals

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du + O(\varrho_n) ,$$

uniformly in x. Some results obtained by using the sieve method were given too. (to appear in Lietuvos matematikos rinkinys, 1978).

L. Lucht : Über benachbarte multiplikative Funktionen

Es sei M die Menge der multiplikativen Funktionen $f: \mathbb{N} \to \mathbb{C}$ mit $f(1)=1$, M_λ für $\lambda \ge 1$ die Menge der $f \in M$ mit $\sum_{n \le x} |f(n)|^\lambda \ll x$. Funktionen $f, g \in M$ mit $\sum_p \frac{|g(p) - f(p)|}{p}$ heißen benachbart. Für $f \in M$, $p \in \mathbb{P}$ (prim), $s \in \mathbb{C}$ bezeichnen $\varphi_f(p, s)$

die Dirichletsche Reihe

$$\sum_{\nu=0} \frac{f(p^\nu)}{p^{\nu s}} .$$

Auf der Faltungsgleichung $g=h*f$ beruht eine oft verwendete Schlußweise, die gewisse summatorische Eigenschaften von f (etwa Mittelwertaussagen) auf g zu übertragen gestattet, falls $\sum \frac{|h(n)|}{n}$ konvergiert. Hinreichende Bedingungen dafür liefert der folgende

<u>Satz :</u> Es seien $f, g \in M_\lambda$ für ein $\lambda > 1$, benachbart, $\varphi_f(p, s) \neq 0$ für $\mathrm{Re}\, s \geq 1$ und $g=h*f$. Dann konvergiert absolut

$$\sum \frac{h(n)}{n} = \prod_p \frac{\varphi_g(p, 1)}{\varphi_f(p, 1)} .$$

Sätze dieser Art und Anwendungen gaben Delange 1961 für $f, g \in M$ vom Betrage ≤ 1, allgemeiner Schwarz 1973 .
(erscheint in Archiv der Mathematik) .

K. Mahler: <u>Eine Klasse transzendenter Dezimalbrüche.</u>

Sei $\{\alpha(1), \alpha(2), \ldots\}$ eine beliebige Folge natürlicher Zahlen. Man schreibe hinter dem Dezimalkomma nacheinander

jede der 1-Ziffern-Zahlen $1, 2, \ldots, 9$ $\alpha(1)$ mal ,
jede der 2-Ziffern-Zahlen $10, 11, \ldots, 99$ $\alpha(2)$ mal ,
usw. Dann ist der so erhaltene Dezimalbruch transzendent.
(Communications on Pure and Appl. Math. <u>29</u> (1976), 717-725) .

K. Mahler: On a class of non-linear functional equations

Let p be a prime and (c_{hk}) a symmetric $(p+1)x(p+1)$ matrix with complex elements. One studies the set of all formal Laurent series

$$f(z) = \sum_{k=m}^{\infty} a_k z^k \qquad (m \gtreqless 0)$$

which satisfy a functional equation

$$(1) \qquad f(z^p)^{p+1} + f(z)^{p+1} + \sum_{h=o}^{p} \sum_{k=o}^{p} c_{hk} f(z^p)^h f(z)^k = 0 \ .$$

(Equations of this type occur in the theory of modular functions).

It is possible to decide on how many constants these $f(z)$ are dependend . Of particular importance are the basic solutions

$$h(z) = \frac{1}{z} + \sum_{h=1}^{\infty} b_h z^h \ ,$$

since all solutions $f(z)$ can be expressed in terms of $h(z)$.
For $p=2$ and $p=3$ the coefficients c_{hk} in (1) can be evaluated explicitly in terms of $\frac{p(p+3)}{2} - 2$ coefficients of $h(z)$, and this implies algebraic recursive formulae for the b_h in which the c_{hk} do not occur explicitly. This implies a simple set of recursive formulae for the Fourier coefficients of the modular function $j(w)$ which seems to be new. A main result states that the formal series f and h in fact converge in a neighbourhood of $z=0$. Hence (1) allows these functions to be continued into $|z| < 1$. It is, however, possible, that the functions have infinitely many algebraic branch points in $|z| < 1$, where these points approximate all points of $|z| = 1$. An example for this situation is given . (J. Austr. Math. Soc. 22 (ser. A) (1976), 65-120) .

M. Mendes France: On van der Corput's difference theorem (joint work with T. Kamae)

Let $H \subset \mathbb{N} = \{1, 2, 3, \ldots\}$. Let $P(H)$ be the set of trigonometric polynomials f with spectrum in H and such that $f(0)=1$, $|f(x)| \leq 1$. Let $P^*(H)$ be the smallest set containing $P(H)$ which is stable by pointwise convergence.

Theorem: Suppose $\forall \delta > 0$, $\exists f \in P^*(H)$ such that $\mathrm{Re}(f) \geq -\delta$. Then the two following properties hold :

1) $(u_{n+h} - u_n)$ equidistributed mod 1 for all $h \in H$ $\Rightarrow$ u_n is equidistributed mod 1.

2) If $A \subset \mathbb{N}$ is such that $(A - A) \cap H = \emptyset$, then A has density 0.

Examples of such H's

a) $H = \mathbb{N}$. This is von der Corput's theorem.

b) $H = a\mathbb{N}$ where $a \in \mathbb{N}$. This result leads to a theorem of Delange.

c) $H = (J - J)^+ = \{0 < i-j, i \in J, j \in J\}$ where $J \subset \mathbb{N}$ is infinite.

d) H is the set of quadratfrei numbers.

e) H is the set of powers of ν (where $\nu \geq 1$ is a given integer).

This last result contains a theorem of Sarközy.

Y. Motohashi : Remarks on the large sieve.

A sort of the hybrid of the Selberg sieve and the large sieve is proved which is too complicated to be stated here in detail. From the result,

combined with some ideas due to Selberg and myself, the prime number theorem of Linnik-Fogels-Gallagher comes out quite straight forwardly. Further this new method dispenses with the zero-density argument, the Deuring-Heilbronn phenomenon and Turán's power-sum method. (to appear in Inventiones math.)

H. Möller: Über Hasses Verallgemeinerung des Syracuse-Algorithmus (Kakutanis Problem).

Let m, d be integers with $m \geq 1$, $d \geq 2$ and $(m,d)=1$. Let further N_d be the set of integers n with $d \nmid n$ and R_d be a complete residue system (mod d) without multiple of d. Then the sequence of iterated functions $(H^k)_{k \in \mathbb{N}_0}$

with $H=H(m,d,R_d): \; N_d \longrightarrow N_d \;,\; x \longmapsto \dfrac{mx-r}{d^{\alpha}}$ where $r \in R_d$

and $\alpha \in \mathbb{N}$ are uniquely determined, is called "Hasse's algorithm".

Theorem: If $m < d^{d/d-1}$, then the set $\mathcal{Z} = \mathcal{Z}(m,d,R_d) :=$

$$:= \{n \in N_d \cap \mathbb{N} \; ; \; \text{there exists } k \in \mathbb{N} \text{ such that } H^k(n) < n\} \cup \{n \in \mathbb{N}, d \mid n\}$$

has natural density 1, i.e. $\lim\limits_{x \to \infty} \frac{1}{x} \operatorname{card} \{n \in \mathcal{Z}, n \leq x\} = 1$.

The author conjectures that Hasses algorithm is periodic for every $n \in N_d$ if and only if $m < d^{d/d-1}$.

(to appear in Acta Arithmetica)

W. Narkiewicz : Normal orders for functions associated with factorisation.

Let K/Q be finite and let $f(n)$ be the number of factorisations of n into irreducibles in K. Let also $g(n)$ be the number of such factorisations with different length. P. Turán asked whether f, g can have nondecreasing normal orders (NN0).

Theorem 1 (J. Rosinski, J. Sliwa) : If the class number h of K is ≥ 2, then f cannot have a NN0. (to appear in Bull. Acad. Polon. Sci).

Theorem 2 (W. Narkiewicz, J. Sliwa) : If $h \geq 3$ then g has $C(K) \cdot \log\log n$ for NN0. (to appear in Colloq. Math.)

Theorem 3 : The function $\log f(n)$ has $C(f,K) \log_2 n \log_3 n$ for NN0. (to appear in Acta Arithmetica)

Theorem 2 has been found independently by S. Allen and P. Pleasants (to appear in Acta Arithmetica)

B. Novák : Recent results in lattice point theory

Let $Q(u) = \sum_{j,k=1}^{r} a_{jk} u_j u_k$ be a positive def. quadratic form with integral coefficients and det D. Let further α_j, b_j, $j=1,2,\ldots,r$ be real numbers. We put

$$P(x) = \sum_{\substack{Q(u_j) \leq x \\ u_j \equiv b_j (1)}} e^{2\pi i \sum_{j=1}^{r} \alpha_j u_j} - M \frac{x^{\frac{r}{2}} \delta e^{2\pi i \cdot \sum_{j=1}^{r} \alpha_j b_j}}{\Gamma\left(\frac{r}{2}+1\right)},$$

$M = \frac{\pi^{r/2}}{\sqrt{D}}$, where $\delta = 1$ if all numbers α_j are integers,

$\delta = 0$ otherwise .

<u>Theorem:</u> Let, for $\varrho > 0$, $P_\varrho(x) = \frac{1}{\Gamma(\varrho)} \int_0^x P(t)\,(x-t)^{\varrho-1} dt$.

Let n be a maximal pos. integer, $\varrho + n < \frac{r}{2} - 1$. Then

$$P_\varrho(x) = \frac{M}{\Gamma(\varrho+1)} \sum_{j=0}^{n-1} (-1)^j \frac{\Gamma(\varrho+j+1)}{j!} \sum_{\substack{h \text{ int.} \\ h \neq 0,\ 0<k\leq\sqrt{x} \\ (h,k)=1}} x^{\frac{r}{4}-\frac{j}{2}-\frac{1}{2}} \cdot$$

$$\cdot \frac{S_{h,k}\, e^{iax}}{k^r (ia)^{\varrho+1+j}} \frac{J_{r/2-j-1}(2\sqrt{bx})}{b^{r/4-j/2-1/2}} + O(x^{\frac{r}{4}+\frac{\varrho}{2}})$$

where $a = a_{h,k} = \frac{2\pi h}{k}$, $b = \tilde{b}_k = \frac{\pi^2 R_{h,k}}{k^2}$ and

$R_{h,k} = \min_{(m_j)} Q(m_j + k\alpha_j - 2h \sum_{l=1}^{r} a_{jl} b_l)$.

(if $b=0$ we put $\frac{J_{r/2-j-1}(2\sqrt{bx})}{b^{r/4-j/2-1/2}} = \frac{x^{\frac{r}{4}-\frac{j}{2}-\frac{1}{2}}}{\Gamma(\frac{r}{2}-j)}$) ,

$S_{h,k}$ are generalized Gauss's sums .

This theorem generalizes well known Petersson's formulae (see for example Walfisz, Gitterpunkte in mehrdimensionalen Kugeln, 1957, and author's paper in Acta Arithm. XIII(1968)) . (sent to Acta Arithm.)

J. Pintz: On the oscillation of the remainder term of the prime number formula

Let $\Delta(x) := \sum_{n \le x} \Lambda(n) - x$. Then we assert as sharpenings of results of Turán and Ingham resp.:

Theorem 1: If $\rho_0 = \beta_0 + i\gamma_0$ is a non-trivial zero of $\zeta(s)$, $\varepsilon > 0$, and $T > c(\rho_0, \varepsilon)$ (effective lower bound depending on ρ_0 and ε), then there exist $x_1, x_2 \in [T, T^{s_0 \log \gamma_0}]$ such that

$$\Delta(x_1) > (1-\varepsilon)\frac{x_1^{\beta_0}}{|\rho_0|} \quad \text{and} \quad \Delta(x_2) < -(1-\varepsilon)\frac{x_2^{\beta_0}}{|\rho_0|} .$$

Theorem 2: If $\zeta(s) \neq 0$ in $\sigma > 1 - \eta(|t|)$, where $\eta(t)$ is a continuous decreasing function for $t \ge 0$, $0 < \varepsilon < 1$ and $w(x) := \min_{t \ge 1} (\eta(t)\log x + \log t)$,

then $$\frac{\Delta(x)}{x} = 0\left(\frac{1}{e^{(1-\varepsilon)w(x)}}\right).$$

This can be converted as

Theorem 3: If $\zeta(s)$ has infinitely many zeros in $\sigma > 1 - g(\log t)$, where $g(u)$ is a continuous decreasing function and $g'(u) \nearrow 0$, $0 < \varepsilon < 1$ and $w(x) := \min_{u \ge 0} (g(u)\log x + u)$, then

$$\frac{\Delta(x)}{x} = \Omega_{\pm}\left(\frac{1}{e^{(1+\varepsilon)w(x)}}\right).$$

Let $V(T)$ denote the number of sign changes of $\Delta(x)$ in the interval $[2, T]$. Lower bounds for $V(T)$ were proved earlier by S. Knapowski and P. Turán.

We assert

<u>Theorem 4:</u> $V(T) > c_o \dfrac{\log T}{(\log\log T)^3}$ for $T > T_o$,

<u>Theorem 5:</u> $V(T) > c_1 \dfrac{\sqrt{\log T}}{\log\log T}$ for $T > T_1$,

where c_o, c_1, T_1 are effective, but T_o is ineffective .

(to appear in different articles in Acta Arithm.)

R.A. Rankin: <u>The vanishing of Poincaré series</u>

For positive integral m the Poincaré series

$$G_k(z,m) = \sum_T e^{2\pi i m\, T(z)} (cz+d)^{-k} ,$$

where $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, span the space of cusp forms of weight k for the full modular group. The dimension μ_k of this space is approximately $k/12$ and is zero for $k=4, 6, 8, 10$ and 14 so that, for these values of k , $G_k(z,m)$ vanishes identically. For $k=12$ and $k \geq 16$ (k even) $\mu_k > 0$, and it is known that the $G_k(z,m)$, for $1 \leq m \leq \mu_k$, span the space and so do not vanish identically .
It is shown that, for every positive ε , there exist positive constants A and B such that, if $k \geq A$ (k even), then $G_k(z,m) \not\equiv 0$ for $1 \leq m \leq Bk^{2-\varepsilon}$.
This result can be sharpened slightly.

G.J. Rieger : Packungen mit Kugeln unterschiedlicher Größe

We start with the euclidean space $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. For $z \in \mathbb{C}$, $d \in \mathbb{R}_+$ denote by $B(z,d)$ the ball with center $\langle z, \frac{d}{2} \rangle$ and diameter d . Let $\varrho := \frac{1}{2}(1+\sqrt{-3})$, $H := \mathbb{Q}(\varrho)$, $L := \{ m+n\varrho \, ; \, m,n \in \mathbb{Z} \}$. H is an euclidean field; every $z \in H$ can be written as $z = \alpha/\beta$ with $\alpha \in L$, $0 \neq \beta \in L$, $(\alpha, \beta) = 1$; $F(z) := B(z, |\beta|^{-2})$ is called Fordsphere over z . Let $F := \bigcup_{z \in H} F(z)$; different spheres of this union have no interior point in common . Let $V_1 := \bigcup_{z \in L} F(z)$.

To V_1 we add all balls $B(z,d)$ with maximal d such that the $B(z,d)$ have no interior point in common with V_1 ; this gives $V_2 (\supseteq V_1)$. We proceed in this way to get V_2 , $V_3, \ldots$. Let $V := \bigcup_{m \geq 1} V_m$.

Theorem 1 : $F = V$. The proof uses very special properties of H . For $\eta \in \mathbb{R}_+$ denote by $D(\eta)$ the triangle with the corners $\langle 0, \eta \rangle$, $\langle 1, \eta \rangle$, $\langle \varrho, \eta \rangle$; denote by $r(\eta)$ the area of $D(\eta) \cap V$; the area of $D(\eta)$ is $\frac{1}{2}\sqrt{3}$; let therefore $r^*(\eta) = 2r(\eta)/\sqrt{3}$.

Theorem 2: For $0 < \eta < 1$ we have $r^*(\eta) = \frac{2}{3L(2)} + O(\sqrt{\eta})$, where $L(s)$ is the L-function to the non-principal character mod 3 .
Observe $\frac{2}{3L(2)} \approx 0,85$. An analogue of Theorem 1 does not hold in $\mathbb{Q}(\sqrt{-1})$. (to appear in Journ. f. d. R. u. A. M.) .

B. Saffari: Bounds for the coefficients of real trigonometric polynomials

Let $f(t) = \sum_{k=0}^{n} a_k \cos(kt + \alpha_k)$ where the a_k and α_k are real and $a_n \neq 0$.

Let $M := \max_{t \in \mathbb{R}} |f(t)|$. Let p run over the set of odd primes and

p_k denote the k th odd prime ($p_1=3, p_2=5, \ldots$). Then

$$|a_k| \leq M \text{ if } k>\frac{n}{3} \; ; \; |a_k| \leq \frac{2}{\sqrt{3}} M \text{ if } k>\frac{n}{5} \; ;$$

$$|a_k| \leq \frac{M}{\cos\frac{\pi}{6}\cos\frac{\pi}{10}} \quad \text{if } k>\frac{n}{7} \; , \ldots$$

and, in general, $|a_k| \leq M / \prod_{p \leq p_k} \cos(\frac{\pi}{2p})$ if $k > \frac{n}{p_{k+1}}$.

These inequalities are interesting only when $p_{k+1} \leq 19$, because for $p_{k+1} \geq 23$ we have $\prod_{p \leq p_k} \cos(\frac{\pi}{2p}) < \frac{\pi}{4}$, so that the above inequality becomes less good than the (best possible) inequality $|a_k| \leq \frac{4}{\pi} M$ $(k=1,2,\ldots,n)$. The method of proof suggests the possibility that some of the above inequalities (for $p_{k+1} \geq 5$) could be improved upon, but only very slightly, since otherwise the optimality of $|a_k| \leq \frac{4M}{\pi}$ would be contradicted. (joint work with S.K. Pichorides, to appear within "three notes on trigonometrical polynomials", Publ. Math., d'Orsay, Université de Paris-Sud)

A. Schinzel: <u>The number of irreducible factors of a polynomial</u>

Let for a given polynomial $f \in \mathbb{Z}[x]$, $|f|$ be its degree and $\|f\|$ the sum of squares of the coefficients, $\Omega(f)$ the number of irreducible factors of f counted with multiplicities, $\Omega_1(f)$ the number of irreducible non-cyclotomic factors, $\Omega_2(f)$ the number of irreducible non-reciprocal factors . The following results hold:

<u>Theorem 1:</u> $\Omega_2(f) \leq \frac{\log \|f\|}{2 \log \vartheta_o}$, (ϑ_o the real root of x^3-x-1) .

<u>Theorem 2:</u> If $f(o) \neq 0$: $\Omega(f(x^n)) \leq |f| \tau(n)$, ($\tau(n) := \sum_{d|n} 1$) .

<u>Theorem 3:</u> If $f(x) = a_o + \sum_{i=1}^{k} a_i x^{n_i}$, $(0 < n_1 < n_2 < \ldots < n_k)$, then

$$\Omega_1(f) \leq \begin{cases} \exp\left(\frac{\log 2 + o(1)}{\log_3 \|f\|} \log_2 \|f\|\right) & \text{if } k=1 \\ \frac{\log \|f\|}{\log \vartheta_o + o(1)} & \text{if } k=2 \\ \left(\frac{1}{2\log \vartheta_o} + \frac{1}{2 \log 2}\right) \log \|f\| & \text{if } k=3 \text{ and } |a_o|+|a_3| \geq |a_1|+|a_2| . \end{cases}$$

(to appear in Acta Arithm. <u>34</u> (1978)) .

F. Schweiger : <u>A Dirichlet series associated with number-theoretical endomorphisms</u>

Let (B, S) be a number theoretical endomorphism and τ be a probability measure on a σ-algebra of subsets of B . As usual denote $B(k_1, \ldots, k_q)$ a cylinder with digits $k_1, \ldots, k_q$. Then given a set E we put $f_q(\alpha, E) = \sum_{B(k_1, \ldots, k_q) \cap E \neq \emptyset} \lambda(B(k_1, \ldots, k_q))^{\alpha}$ and call the infinite series $D(E, \alpha) = \sum_{q=1}^{\infty} f_q(\alpha, E)$ the Dirichlet series of the set E. If dim E denote the Billingsley dimension of E than dim $E \leq \beta$, where $\beta := \inf\{\alpha \mid D(E, \alpha) < \infty\}$, is obvious. From a more complicated theorem by Stradner and Schweiger one can prove dim $E = \beta$ for certain special cases. The use of $D(E, \alpha)$ was inspired by papers of

Boyd and Smorodinsky .
(Anzeiger d. Österr. Akad. Wiss., Math. Naturw. Kl. Jg. 1973, 1-4; Schweiger + Stradner: Sitzungsber. Öst. Akad. Wiss., Math. Naturw. Kl. II (1971), 95-109 und 81(1973), 151-155).

J.-P. Serre: Applications of analytic number theory to elliptic curves and modular forms

These applications use mainly the Cebotarev density theorem, in the quantitative version given by Lagarios-Odlyzko, applied to non-abelian extensions of $\mathbb{Q}$. Sample results are:

(i) The density of the set of n with $\tau(n) \neq 0$ is positive.
(τ being the Ramanujan's function).

(ii) (Under GRH): If E is an elliptic curve over $\mathbb{Q}$ with at least one point of order 2 irrational, then the set of primes p such that the group of points of E modulo p is cyclic has a positive density. (for more details and related results see Durham Conf. Proc. 1975, ed. A. Fröhlich, Acad. Press 1977).

H. Siebert: Sieve methods and Siegel's zeros

From the linear Selberg sieve of W. B. Jurkat and H.-E. Richert follows $(\alpha \in (0,1))$:

$$A_\alpha(x,k,l) := \sum_{\substack{n \le x \\ n \equiv l(k) \\ p \mid n \Rightarrow p \ge x^\alpha}} 1 \begin{cases} \le \dfrac{x}{\varphi(k)\log x} \dfrac{e^{-\gamma}}{\alpha} F\left(\dfrac{1}{\alpha}\right)(1+o(1)) \\ \ge \dfrac{x}{\varphi(k)\log x} \dfrac{e^{-\gamma}}{\alpha} f\left(\dfrac{1}{\alpha}\right)(1+o(1)) \end{cases}, \quad k \le \exp(c\sqrt{\log x}) ,$$

where the functions $f(u)$, $F(u)$ are defined as the solutions of certain difference-differential equations. I prove that any improvement of the main term constants $(e^{-\gamma}/\alpha)f(\frac{1}{\alpha})$, $(e^{-\gamma}/\alpha)F(\frac{1}{\alpha})$ would lead to the proof of Siegel's theorem with effective constants. My proof gives also a direct arithmetic interpretation of $f(u)$, $F(u)$ and integral representations for these functions.

J. Sliwa : Remark on uniform distribution of sequences of integers

Let $\{x_n\}$, $\{y_n\}$ be sequences of integers and N_1, N_2 integers ≥ 2. The pair $\{x_n\}$, $\{y_n\}$ of sequences is called uniform distributed (u.d.) mod $\{N_1, N_2\}$ iff for each pair of integers (m_1, m_2) the set of those n for which $x_n \equiv m_1 \bmod N_1$, $y_n \equiv m_2 \bmod N_2$ has density $(N_1 N_2)^{-1}$.

Theorem: The pair $\{x_n\}$, $\{y_n\}$ is u.d. mod $\{N_1, N_2\}$ iff following conditions hold:

(i) If for some integers l_1, l_2 and all $n: N_1 N_2 | N_2 l_1 x_n + N_1 l_2 y_n$, then $N_1 | l_1$, $N_2 | l_2$.

(ii) The sequence $\left\{ \frac{N_2 l_1}{t(l_1, l_2)} x_n + \frac{N_1 l_2}{t(l_1, l_2)} y_n \right\}$ is u.d. mod $\frac{N_1, N_2}{t(l_1, l_2)}$

for all $0 \leq l_i \leq N_i$, $i = 1, 2$, where $t(l_1, l_2) = (N_2 l_1, N_1 l_2, [N_1, N_2])$.

This theorem is analogous to a theorem of H. Delange, which gives sufficient and necessary conditions for uniform distribution of a pair of sequences in case $x_n = f(n)$, $y_n = g(n)$, f, g being additive functions. A similar result can be obtained for weak uniform distribution. (to be sent to Colloq. Math.)

S. Stepanov: On Jacobsthal's sums

A survey of some results about estimates of Jacobsthal's sums is given. For a prime p. integer a with $(a,p)=1$ and $n \geq 1$ write $s=(p-1,n-1)$, $t=(p-1,n)$.

Theorem 1: $\left| \sum_{x=1}^{p} \left(\frac{x^n+ax}{p}\right) \right| \leq \sqrt{\frac{s}{t}}\, p$.

Theorem 2: For $l \geq 1$, $2 \mid l$, $p \equiv 1 \bmod l$ and $a \not\equiv y^l \bmod p$ we have

$$\left| \sum_{x=1}^{p} \left(\frac{x^n+ax}{p}\right) \right| \leq \frac{1}{\sqrt{2}} \sqrt{p}.$$

Let g be a primitive root mod p.

Theorem 3: Let $l \equiv 0 \bmod 4$, $p \equiv 1 \bmod 2l$. Then

$$S(g^{2i+1}) \equiv \sum_{j=0}^{\frac{l}{4}-2} \alpha_{ij} S(g^{2j}) \bmod p \quad \text{where}$$

$$S(g^k) = \sum_{x=1}^{p} \left(\frac{x^{l+1}+g^k x}{p}\right) \quad \text{and, what's more,}$$

$\det \|\alpha_{ij}\| \not\equiv 0 \bmod p$. (joint work with Postnikov; to appear in Trudy mat. inst. im. B. A. Steklova AH CCCP(1976); translated as Proc. of the Steklov institute).

C. L. Stewart: Infinite difference sets of sequences of positive integers

Let A be a sequence of positive upper density and let D be the set of non-negative integers which are representable in infinitely many ways

as the difference of two elements of A. Various results concerning sets D were discussed. In particular

Theorem 1: If $\bar{d}(A_i) \geq \varepsilon_i > 0$, for $i=1,\ldots,h$, then there exists a sequence A with $\underline{d}(A) \geq C_h$ for which $D=D_1 \cap \ldots \cap D_h$, where $C_1 = \varepsilon_1$ and $C_h = \prod_{i=1}^{h} (\varepsilon_i / 5 \log(h+1))$ for $h \geq 2$. Further let $\mathbb{D}$ denote the set of all difference sets associated with sequences of positive upper density. We have

Theorem 2: $\mathbb{D}$ is a filter on the set of subsets of the non-negative integers (submitted to Cand. J. Math.)

R. Tijdeman: Difference sets of sequences of integers

Let A denote an infinite, strictly increasing sequence of non-negative integers with upper density $\bar{d}(A)>0$. Denote the set of non-negative integers which occur as the difference of two terms of A by $D(A)$.

(i) There exists a sequence A such that $D(A)$ does not contain any arithmetical progression.

(ii) There exists a sequence A with $\bar{d}(A) \geq \frac{1}{8}$ such that $D(A)$ does not contain any factorial.

(iii) If $K=\{k_1, k_2, \ldots\}$ has the property $\liminf_{j \to \infty} \frac{k_{j+1}}{k_j} > 1$, then there exists a sequence A with $D(A) \cap K = \emptyset$.

(These and related results are joint work with C. L. Stewart: On finite difference sets, Math. Centr. Report ZW 100, Amsterdam, 1977).

R. C. Vaughan: An elementary method in prime number theory

An elementary method was discussed that enables one to establish to following

Theorem 1: Suppose that $(a,q)=1$, $|\alpha - \frac{a}{q}| \leq q^{-2}$, $\mathcal{L} = \log 2N$. Then

$$\sum_{n=1}^{N} \Lambda(n) e(\alpha n) \ll (Nq^{-1/2} + N^{4/5} + N^{1/2} q^{1/2}) \mathcal{L}^{7/2} .$$

Theorem 2: Suppose that α is irrational and $\|\gamma\| = \min_{n \in \mathbb{Z}} |\gamma - n|$.

Then there are infinitely many primes p such that

$$\|\alpha p - \beta\| \ll p^{-\frac{1}{4}} \cdot \log^8 p .$$

Theorem 3: Suppose that $Q \geq 1$, $Y \geq 1$, $\mathcal{L} = \log 2QY$. Then

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi}^{*} \max_{x \leq Y} |\psi(x,\chi)| \ll \mathcal{L}^4 (Y + Y^{5/6} Q + Y^{1/2} Q^2) .$$

(See: R. C. V.: Sommes analytiques des nombres (Comptes rendus); and R. C. V. + H. L. Montgomery: Exponential sums with multiplicative coefficients (sent to "Inventiones")).

R. Wallisser: Untere Schranken für Polynome in Werten der p-adischen Exponentialfunktion

Bezeichnet man mit $|\ |_p$ die durch $|p|_p = p^{-1}$ normierte p-adische Bewertung von $\mathbb{Q}$, mit $\mathbb{C}_p$ die p-adische Vervollständigung des alge-

braischen Abschlusses von $\mathbb{Q}_p$ und mit $\mathbb{A}$ die über $\mathbb{Q}$ algebraischen Elemente von $\mathbb{C}_p$, so gilt:

Seien $\alpha_1, \ldots, \alpha_s \in \mathbb{C}_p \cap \mathbb{A}$ und über $\mathbb{Q}$ linear unabhängig, $|\alpha_\delta|_p < p^{-1/p-1}$ $(\delta = 1, \ldots, s)$; sei $\mathbb{K} := \mathbb{Q}(\alpha_1, \ldots, \alpha_s)$, $[\mathbb{K}:\mathbb{Q}] = r$.

Sei $P \in \mathbb{Z}[X_1, \ldots, X_s]$, $\neq 0$, vom Gesamtgrad k und der Höhe H. Dann gibt es effektiv berechenbare Konstanten $c_1, c_2 > 0$, die nur von $k, p, s, \alpha_1, \ldots, \alpha_s$ abhängen, so daß für alle $a > \max(c_1, 6r \log H)$, die reine Potenzen von p sind, gilt:

$$|P(e^{a\alpha_1}, \ldots, e^{a\alpha_s})|_p > e^{-c_2 a}.$$

(sent to Math. Ann.)

R. Warlimont: <u>Die kleinste natürliche Zahl maximaler Ordnung mod m</u>

Sei $m \in \mathbb{N}$, $a \in \mathbb{Z}$, $(a, m) = 1$. Man setze

$$\vartheta_m(a) := \min\{h \mid h \in \mathbb{N}, a^h \equiv 1 \bmod m\},$$

$$\psi(m) := \max\{\vartheta_m(a) \mid a \in \mathbb{Z}, (a, m) = 1\},$$

$$g(m) := \min\{a \mid a \in \mathbb{N}, \vartheta_m(a) = \psi(m)\}.$$

Bewiesen wird

$$\sum_{m \leq x} g(m) \ll x^{1+\varepsilon}.$$

(erscheint in Monatshefte f. Math.)

E. Wirsing: Additive zahlentheoretische Funktionen mit beschränktem Wachstum

Satz : Ist $f: \mathbb{N} \to \mathbb{R}$ vollständig additiv, $f(n+1)-f(n)=o(\log n)$, so ist $f=\text{const.}\log$.

Aus der Voraussetzung $f(n+1)-f(n) \to 0$ hatte Erdös 1946 dieselbe Folgerung gezogen. Für beliebig additive Funktionen gelten schwächere Aussagen, die aber, wie der formulierte Satz, auch "bestmöglich" sind. Der Beweis geht, wie schon eine ältere Arbeit des Vortragenden, von der Bemerkung aus, daß

$$a_N := \frac{1}{N!} \prod_{n=1}^{N} (an+1) \quad \text{"fast ganz" ist, und}$$

daß $f(a_N) = \sum (f(a)+f(n)+\text{Rest}) - f(N) = Nf(a)+\text{Rest}$ ist. Da man f auch explizit über die Primfaktorzerlegung berechnen kann, erhält man nach einiger Umarbeitung

$$\frac{1}{\log p_o} f(p_o) = \sum_{x_1 < p_1 \leq x_1^2} w(p_o|p_1) \frac{1}{\log p_1} f(p_1) + \text{Rest} ,$$

wobei x_1 innerhalb gewisser Grenzen frei gewählt werden kann. Wird diese Formel nun iteriert, so hat man, abgesehen von Restgliedern, einen stochastischen Prozeß vor sich. Ein Gesetz der großen Zahl besagt für diesen Prozeß, daß die "kombinierte Übergangswahrscheinlichkeit" $W(p_o|p_n)$ in

$$\frac{f(p_o)}{\log p_o} = \sum_{x_n < p_n \leq x_n^2} W(p_o|p_n) \frac{f(p_n)}{\log p_n} + \text{Rest}$$

für großes n von p_o nur noch unerheblich abhängt, daß $f(p)/\log p$ also konstant ist. Dabei sichert der Primzahlsatz von Bombieri-Vinogradov die "Unabhängigkeit" der $w(p_\mu|\,.)$ und $w(p'_\nu|\,.)$ im notwendigen Umfang.

D. Wolke: Exponential sums and zeta-zeros

About 30 years ago P. Turán proved several theorems of the following type: Necessary and sufficient for the truth of the quasi Riemann hypothesis is the existence of $C_1, C_2 > 0$ such that

$$\left| \sum_{N_1 < p \leq N_2} e(t \log p) \right| \leq C_1 \exp(23(\log\log n)^2)\, N t^{-1/2} \qquad (e(\alpha) = e^{2\pi i \alpha})$$

for $C_2 \leq t^{10} \leq \frac{N}{2} \leq N_1 < N_2 \leq N$.

We have a similar theorem for "short" exponential sums:

Let T be sufficiently large, $U = T^{9/10}$, $\gamma > 0$, $X_1 = X_1(\gamma) = (\frac{T}{\gamma})^{1/\gamma}$,

$X_2 = (\frac{T+U}{\gamma})^{1/\gamma}$, $(\frac{X_2 - X_1}{X_1} \approx \frac{U}{T})$; $S(\gamma) = \sum_{X_1 < n \leq X_2} \Lambda(n) e(n^{\gamma})$.

Theorem: If $S(\gamma) \ll T^{-\varepsilon} X^{1-\gamma/2} \exp(24(\log\log X_1)^2)$ for some $\varepsilon > 0$

and $\frac{c}{\log^{7/4} T} < \gamma \leq \frac{1}{100}$, $(c > 0)$, then $\zeta(\sigma + i\tau) \neq 0$ for

$\sigma \geq 1 - \frac{c^*}{\log^{4/7} |\tau|}$ and $|\tau|$ sufficiently large.

Apart from the factor $T^{-\varepsilon}$ the upper bound can be proved by means of the Turán-Vinogradov-method, or by zero density results.

H. Siebert, Ulm