MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 26/1979

Funktionalgleichungen

17.6. bis 23.6.1979

Die 17. internationale Tagung über Funktionalgleichungen fand vom 17.6. bis 23.6.1979 im Mathematischen Forschungsinstitut Oberwolfach statt, unter der Leitung der Herren J. Aczél (Waterloo, Ontario; z.Zt. Graz), W. Benz (Hamburg) und A. Ostrowski (Basel). Die Tagung wurde von Herrn Ostrowski eröffnet, der die Teilnehmer im Namen der Leitung willkommen hieß.

Es waren 41 Teilnehmer aus 11 Ländern anwesend. Leider konnten keine Kollegen aus Jugoslawien, Rumänien und der Sowjetunion kommen.

Der Verlauf der Tagung gab Zeugnis von der lebhaften Weiterentwicklung der Theorie und Anwendungen von Funktionalgleichungen. Die Mehrzahl der Vorträge (34 in englischer, 2 in französischer Sprache) ließ sich in folgenden Themengruppen zusammenfassen: Translations- und Iterationsgleichungen, allgemeine Gleichungen vom Cauchy-Typ, konditionale Gleichungen, funktionelle Differential- und Integralgleichungen, allgemeine Differenzengleichungen, Beziehungen zur Algebra, Geometrie und Topologie sowie Anwendungen auf Informationstheorie und auf die Theorie metrischer Wahrscheinlichkeitsräume.

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Die Sitzungen "Offene Probleme und Bemerkungen", deren Tradition schon im Jahre 1962 mit der ersten Funktionalgleichungs-Tagung inauguriert wurde, erfreuten sich besonders lebhaften Zuspruchs. Dabei konnten sogar einige der hier gestellten Probleme in einer der folgenden Sitzungen vollständig oder teilweise gelöst werden.

Im Schlußwort dankte Herr Aczél der Institutsleitung sowie den Teilnehmern für ihren Beitrag zum Erfolg der Tagung.

Kurzfassungen der Vorträge sowie der Problemstellungen und Bemerkungen folgen (getrennt voneinander) in chronologischer Reihenfolge.

H. H. KAIRIES: The Jackson factorial functions and their functional equations

F. H. Jackson defined an extension of the Γ-function, $\Gamma_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\Gamma_\alpha(x) = (1-\alpha)^{1-x} \prod_{n=0}^{\infty} \frac{1-\alpha^{n+1}}{1-\alpha^{n+x}} \qquad \text{for } \alpha \in (0,1),$$

$$\Gamma_\alpha(x) = (\alpha-1)^{1-x} \alpha^{\frac{x(x-1)}{2}} \prod_{n=0}^{\infty} \frac{1-\alpha^{-(n+1)}}{1-\alpha^{-(n+x)}} \qquad \text{for } \alpha \in (1,\infty).$$

Γ_α satisfies the functional equation

$$\text{(F)} \quad f(x+1) = \frac{\alpha^x - 1}{\alpha - 1} f(x) .$$

It is shown that $\log \circ \Gamma_\alpha$ can be characterized as Nörlund's principial solution of

$$\text{(D)} \qquad \varphi(x+1) - \varphi(x) = \log \frac{\alpha^x - 1}{\alpha - 1} ,$$

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which is normalized by $\varphi(1) = 0$. The actual computation is done by means of Krull's uniqueness theorem for convex solutions of (D). This yields analogs of Bohr-Mollerup's theorem for Γ_α.

Two other characterization theorems are given. One involves the complete monotonicity property of $(\log \circ \Gamma_\alpha)''$, the other an interpolation equation of the form

$$\Gamma_\alpha(x)\ \Gamma_\alpha(x+\tfrac{1}{2}) = \Gamma_\alpha(\tfrac{1}{2})\, Q_\alpha(x)\, \Gamma_\alpha(2x) .$$

B. CHOCZEWSKI: Discontinuous solutions of a linear functional equation

The results presented in the talk have been obtained by M. Kuczma and the speaker. We consider the linear homogeneous iterative functional equation

$$(1) \qquad \varphi(f(x)) = g(x)\, \varphi(x)$$

in an interval $I = [0, b)$ of reals, with the origin as the only fixed point of the function f in I. For a case where equation (1) possesses in I the only continuous solution $\varphi(x) = 0$, $x \in I$, we prove the existence of a unique one-parameter family of solutions to (1), which, being discontinuous at the origin, are determined by a particular asymptotic behaviour at this point.

K. BARON: On systems having continuous solutions

Suppose that X is a compact metric space and let Y be a finite dimensional Banach space. Fix a $\xi \in X$ and denote by C the Banach space of all continuous functions mapping X into Y and vanishing at ξ with the supremum norm. Given a positive $(m \times m)$-scalar matrix A and a con-

tinuous self-mapping f of X, let S be the set of all functions $F \in C^m$ such that the system of functional equations

$$\varphi(x) = A\varphi[f(x)] + F(x)$$

has a solution $\varphi \in C^m$. The following problem is considered. Under what conditions is the set S dense ? It turns out that this is the case when the characteristic roots of the matrix A are less than or equal to one in absolute value, f fulfills the Lipschitz condition with the constant one and $\rho(f(x), \xi) < \rho(x, \xi)$ holds for all $x \in X \setminus \{\xi\}$. Note that under these conditions $S \neq C^m$, in general.

M. SABLIK: Differential solutions of functional equations in Banach spaces

Let φ_o be a formal solution of the functional equation

(1) $$\varphi(F(x)) = g(x, \varphi(x)) ,$$

where $F \in C^m(U,X)$, $g \in C^m(U \times V, Y)$, $0 \in \operatorname{Int} U = U \subset X$, $\varphi_o(0) \in \operatorname{Int} V = V \subset Y$; X, Y are Banach spaces.

Let $r_s(L)$ denote the spectral radius of the linear operator L. Assume

(2) $F(0) = 0$, $F'(0)$ is a linear bijection,

(3) $0 \in \operatorname{Int} W = W \subset U \Longrightarrow \exists V\ (0 \in V = \operatorname{Int} V \subset W \wedge F^{-1}(V) \subset V)$,

(4) $\exists c \geq 0\ (\| (F^{-1})^{(m)}(x) - (F^{-1})^{(m)}(0) \| = \mathcal{O}(\|x\|^c) \wedge$

$$\| g^{(m)}(x,y) - g^{(m)}(0, \varphi_o(0)) \| = \mathcal{O}(\|x\|^c + \|y - \varphi_o(0)\|^c), \quad x \to 0,\ y \to \varphi_o(0))$$

(5) $|\sigma(g'_Y(0, \varphi_o(0)))| \cap \langle (r_s(F'(0)^{-1}))^{-m}, r_s(F'(0))^m \rangle = \emptyset$,

(6) $r_s(g'_Y(0, \varphi_o(0))) < r_s(F'(0)^{-1})^{-m-c}$,

(7) $\exists M \, (\| g^{(m)}(x,y) - g^{(m)}(x,\bar{y})\| \leq M \| y - \bar{y}\|, \; x \in U, \; y, \bar{y} \in V).$

Under some assumptions on the norm in X (fulfilled e.g. in real Hilbert spaces) we have

Theorem 1. Assume (2), (5), (7). Then (1) has a local solution in C^m.

Theorem 2. Assume (2), (3), (4), (6), (7). Then (1) has a local solution φ in C^m, which fulfills $\| \varphi^{(m)}(x) - \varphi^{(m)}(0)\| = O(\|x\|^c)$. This solution is unique.

W. BENZ: A functional equation and Liouville's theorem on angle preserving mappings

Given a domain $D \subset \mathbb{R}^n$, $n \geq 3$, and a function $f : D \rightarrow \mathbb{R}^n$ ($f \in C^3(D)$, $J(f) \neq 0$ in D). Then f induces a mapping

$$\bar{f} : D \times \mathbb{P}^{n-1} \longrightarrow \mathbb{R}^n \times \mathbb{P}^{n-1}$$

on line elements, $\mathbb{P}^{n-1}$ the projective (n - 1)-space. Angle preservance means local preservance of elliptic distances in $\mathbb{P}^{n-1}$. We treat the functional equation of elliptic motions conditionally obtaining two results. One consequence in geometrical terms is that a given angle $0 < \alpha < \pi$ needs to be preserved "in general position" $\mu(n)$ times in order to maintain Liouville's theorem, where $\mu(n) \in \mathbb{N}$ is associated to the dimension of $\mathbb{R}^n$. It is $\mu = 5$ (but not 4) in case $n = 3$ and $\alpha = \frac{\pi}{2}$.

H. SCHWERDTFEGER: Über die Symmetrie der (r+1)-Punkte-Invariante einer r-fach transitiven Gruppe

Beweis des folgenden Satzes: Sei $f = f(x_0, x_1, \ldots, x_r)$ die Invariante von $r + 1$ Punkten der genau r-fach transitiven Transformationsgruppe G_r, die

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in dem Raum X operiert. Wenn f in Bezug auf die Untergruppe H der symmetrischen Gruppe S_{r+1} symmetrisch ist, so muß H ein Normalteiler von S_{r+1} sein. So folgt, daß Symmetrie von f in Bezug auf H nur vorkommen kann (aber nicht muß), wenn r = 3 . Sei G_3 die projektive Gruppe der geraden Linie über einem Körper mit mehr als drei Elementen; die Invariante, das Doppelverhältnis, ist symmetrisch in Bezug auf die Vierergruppe.

K. SIGMON: Problems on distributive topological groupoids

Considered here is the question of the structure of a cancellative distributive ($x \cdot yz = xy \cdot xz$ and $xy \cdot z = xz \cdot yz$) topological groupoid on an n-manifold. Since the structure of cancellative, idempotent, and medial ($xy \cdot uv = xu \cdot yv$) topological groupoids on any n-manifold is known, the usual line of attack is to show that the distributive groupoid must actually be medial.

Problem. Suppose M is a cancellative, distributive topological groupoid on either an n-manifold or a compact n-manifold with boundary. Must M be medial ?

The problem is known to have an affirmative answer in case M is a compact n-manifold, R, or an n-cell as well as when M has a differentiable multiplication. There remains, in particular, however, the non-compact case for $n \neq 2$. It is conjectured that M needs not to be medial for n = 3.

M.A. TAYLOR: Some groupoid identities

In his paper "Balanced identities in quasigroups" (1966) V.D. Belousov characterizes quasigroups satisfying balanced identities. Through the use of closure condition techniques, it is possible not only to simplify considerably the proofs of Belousov's results but also to generalize his work.

The generalization is effected on both the quasigroup structure and the type of identity to be satisfied by the groupoid.

J. DHOMBRES: Conditional Cauchy equations

Let G, F be groups, denoted additively, $\mathcal{C}$ some class of functions from G into F and Z be a non empty subset of the product $G \times G$. By t(Z), we denote the set of all z in G such that $z = x + y$, where $(x, y) \in Z$.

A Z-additive function $f : G \to F$ is such that $f(x+y) = f(x) + f(y)$ for all pairs $(x, y) \in Z$.

Condition Z is said to be $\mathcal{C}$ -redundant if any Z-additive function (in $\mathcal{C}$) is in fact G-additive. Condition Z is $\mathcal{C}$ -quasi-redundant if any Z-additive function (in $\mathcal{C}$) coincides on t(Z) with some additive function belonging to $\mathcal{C}$.

Some examples of $\mathcal{C}$ -redundancy and $\mathcal{C}$ - quasi-redundancy are given. For example, a generalization of Erdös' theorem in number theory.

If $G = F = \mathbb{R}$ and M is the class of monotonic functions, $H \times H$ is an M-quasi redundant condition for any subsemigroup H of $\mathbb{R}$.

Another example is a characterization of an inner product space by a condition of redundancy.

These examples are from a book to appear soon: Some aspects of functional equations, Lecture Notes, Chulalongkorn University, 1979.

R. GER: Additive functions bounded above on subsets of graphs

We say that a set $T \subset \mathbb{R}^n$ belongs to the class A $(\mathbb{R}^n)$ provided each additive (or, equivalently, Jensen-convex) real functional bounded above on T is continuous. We deal with the (unpublished) problem posed by J. Smital: find conditions on a set $T \in A(\mathbb{R}^n)$ and a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ guaranteeing that

$$\text{the set} \qquad \varphi_T := \{(t, \varphi(t)) \in \mathbb{R}^{2n} : t \in T\} \text{ belongs to } A(\mathbb{R}^{2n}).$$

In particular, we give an affirmative answer to the following (unpublished) question of A. Lasota: does any set of positive linear Lebesgue measure on the (unit) circle belong to $A(\mathbb{R}^2)$?

M. KUCZMA: Topologically saturated non-measurable sets

A set $T \subset \mathbb{R}^n$ is said to be saturated non-measurable whenever $m_i(T) = m_i(\mathbb{R}^N \setminus T) = 0$, where m_i stands for N-dimensional inner Lebesgue measure. As a topological analogue, a set T in a topological space X is said to be topologically saturated non-measurable whenever neither T, nor $X \setminus T$, contains a second category Baire set.

In the theory of Cauchy's functional equation, saturated non-measurable sets occur fairly often. All such theorems have topological counterparts. Example: a theorem of Ostrowski says that if $f : \mathbb{R}^N \to \mathbb{R}$ is a discontinuous additive function, then for every non-degenerated interval $J \subset \mathbb{R}$, the set $f^{-1}(J)$ is saturated non-measurable. It turns out that, under the same conditions, the set $f^{-1}(J)$ ist topologically saturated non-measurable.

Let $X \subset \mathbb{R}^N$ be an uncountable Borel set. A Hamel basis H of $\mathbb{R}^N$ is called a Burstin basis relative to X iff H intersects every uncountable Borel subset of X. Theorem: Every Borel set $X \subset \mathbb{R}^N$, containing a Hamel basis, contains a Burstin basis. The proof of this theorem relies on the Zermelo axiom. If $X \subset \mathbb{R}^N$ is of full measure (i.e., $\mathbb{R}^N \setminus X$ is of measure 0), then every Burstin basis relative to X is saturated non-measurable. If $X \subset \mathbb{R}^N$ is residual (i.e., $\mathbb{R}^N \setminus X$ is of the first category), then every Burstin basis relative to X is topologically saturated non-measurable.

Let $\mathbb{R}^N = A \cup B$, where A is a G_δ of measure zero, and B is an F_σ of the first category (it is wellknown that such a decomposition exists). Let H_1 be a Burstin basis relative to A, and let H_2 be a Burstin basis relative to B. Then H_1 is topologically saturated non-measurable, but of measure zero, whereas H_2 is saturated non-measurable, but of the first category.

T.M.K. DAVISON: When does partial homogeneity imply bi-additivity ?

Suppose $F : V \times V \to k$ satisfies

$$F(x,y) + F(x+y, z) = F(x, y+z) + F(y,z)$$

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for all $x, y, z \in V$. Suppose there are elements (fixed $a, b \in k$) such that $F(ax, by) = ab\,F(x, y)$ for all $x, y \in V$. When does it follow that F is additive in each variable ? The classical case (Jordan-von Neumann) is when $a = 1$, $b = -1$, and $F(x, y) = f(x + y) - f(x) - f(y)$. Applications to the theory of the equations for quadratic forms have been noted.

J. RÄTZ: <u>On the decomposition of functions of several variables by means of an algebraic operation</u>

Let n be a positive integer, $n \geq 2$, $X_1, \ldots, X_n$ nonempty sets, $(Y, \cdot)$ a monoid, and $f : X_1 \times \ldots \times X_n \longrightarrow Y$. The main question is whether there exist $g_k : X_k \longrightarrow Y$ $(k = 1, \ldots, n)$ such that

$$f(x_1, \ldots, x_n) = g_1(x_1) \cdot \ldots \cdot g_n(x_n) \quad (\forall (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n).$$

It turns out that a functional equation, more precisely an exchange identity, and an invertibility condition are of central importance for the answer to the question. Results of M. Kuczma (1972) and K. Szymiczek (1973) now appear in a more general light. Furthermore, the decomposition of multiadditive $f : X_1 \times \ldots \times X_n \longrightarrow Y$ into additive $g_k : X_k \longrightarrow Y$ is discussed.

W. SANDER: <u>An operator-valued Cauchy-equation</u>

Let B denote a Banach space and let $\mathcal{L}(B, B)$ be the Banach algebra of bounded linear operators on B. Let X be a locally compact Hausdorff space and let $(X, \mathcal{G}, \mu)$ be a σ-finite regular measure space. Let $F : X \times X \rightarrow X$ be a continuous and $(\mathcal{G} \times \mathcal{G}, \mathcal{G})$-measurable globally solvable function [1], such that for all $x, y \in X$ the functions F_x, F^y are μ-preserving.

<u>Theorem A</u> (a) $T : X \rightarrow \mathcal{L}(B, B)$ uniformly (strongly) measurable in X

(b) $\forall\, x, y \in X : T[F(x, y)] = T(x) \cdot T(y)$

$\Longrightarrow$ T uniformly (strongly) continuous in X

The proof is based upon the following generalization of the well known fact, that measurable, finite-valued subadditive or convex functions are bounded above on compact sets.

Theorem B (a) $f, g, h : X \longrightarrow R$

(b) h is bounded above on a measurable set of positive measure

(c) $\forall\, x, y \in X$: $f[F(x,y)] \leq g(x) + h(y)$

$\Longrightarrow$ f is bounded above on each compact set of X

Reference:

[1] Sander, W., Glasnik Mat. 13 (33)(1978), 237 - 247.

G. L. FORTI: Existence and stability theorems for a class of functional equations.

Consider the functional equation (1) $g[F(x,y)] = H[g(x), g(y)]$, where $F : E \times E \to E$, $H : X \times X \to X$ are given functions, $g : E \to X$ is the unknown function, E is a set and (X, d) is a complete metric space. We put $G(x) = F(x,x)$, $K(u) = H(u,u)$, we assume that X is a modulus set for K and K is invertible; for every $x \in E$ we define $x^o = x$, $x^n = G^n(x)$. For every $f : E \to X$ and every $x, y \in E$ we set

$$\delta(x,y) = d\,(f[F(x,y)],\ H[f(x), f(y)]).$$

The following theorems hold:

Theorem. Assume that:

(i) there exists a function $k : R^+ \to R^+$, strictly increasing, super-additive, such that for every $u, v \in X$, $d(K(u), K(v)) = k(d(u,v))$, moreover for some constant $c > 1$, $k(t) \geq ct$ for every $t \in R^+$;

(ii) for every $u, v \in X$, $H[H(u,v), H(u,v)] = H[H(u,u), H(v,v)]$;

(iii) H is continuous;

(iv) for every $x, y \in E$, $F[F(x,y), F(x,y)] = F[F(x,x), F(y,y)]$.

If there is a function $f : E \to X$ such that for every $x, y \in E$, $\delta(x^n, y^n) = o(c^n)$ and the series $\sum_{n=1}^{\infty} c^{-n}\delta(x^{n-1}, x^{n-1})$ converges, and for some $\bar{x}, \bar{y} \in E$ it is $\lim_{n\to+\infty} \inf k^{-n}[d(f(\bar{x}^n), f(\bar{y}^n))] > 0$, then the equation (1) has a non constant solution.

Corollary. Under the hypotheses of the previous theorem, there exists a unique solution g of the functional equation (1) such that for every $x \in E$, $d(g(x), f(x)) \leq \sum_{n=1}^{\infty} k^{-n}[\delta(x^n, x^n)]$. Furthermore if E is a topological space, G is continuous and $\sup\{\delta(x, y) : x, y \in E\} < +\infty$, then the continuity of f implies that of g.

L. PAGANONI: On the solutions of a conditional equation

Consider the following conditional equation

$$(*)_{a,b} \qquad f(x+y) \neq f(x)+f(y)+a \implies f(x+y) = f(x)+f(y)+b$$

where $f : G \to H$, G and H semigroups and $a, b \in H$.

Here I present the general solution of this equation when G is equal to N or Z and H = Z (N is the class of the positive integers and Z the group of the integers).

Let $\mathcal{H}$ be the class of solutions of $(*)_{0,1}$ for which $f(1) = 0$. Then $f : G \to Z$ (G = N or Z) is a solution of $(*)_{a,b}$ if and only if there exists an additive function $g : G \to Z$ and a function $f_1 \in \mathcal{H}$ such that $f = g - a + (b - a) f_1$.

So we have only to look for the solutions of $(*)_{0,1}$. For every $\alpha \in R$, define a function $\varphi_\alpha : G \to Z$ in the following way: $\varphi_\alpha(n) = [\alpha n]$ (here [x] denotes the integral part of x). If $E \subset G$, χ_E is the characteristic function of the set E. Then the two following theorems are true:

Theorem 1. $f : N \to Z$ is a solution of $(*)_{0,1}$ if and only if it can be represented in one of the following forms:

a) $f = \varphi_\alpha$ with $\alpha \in R$, b) $f = \varphi_\alpha - \chi_{qN}$ with $\alpha = \frac{p}{q} \in Q$, $(p, q) = 1$, $q > 0$.

Theorem 2. $f : Z \to Z$ is a solution of $(*)_{0,1}$ if and only if it can be represented in one of the following forms a) - g):

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a) $f = \varphi_\alpha$ with $\alpha \in R$, b) $f = \varphi_\alpha - \chi_{\{0\}}$ with $\alpha \in R \setminus Q$ or, if $\alpha \in Q$, $\alpha = \frac{p}{q}$, $(p, q) = 1$, $q > 0$, c) $f = \varphi_\alpha - \chi_{qN}$ d) $f = \varphi_\alpha - \chi_{-qN}$ e) $f = \varphi_\alpha - \chi_{qZ^+}$ f) $f = \varphi_\alpha - \chi_{-qZ^+}$ g) $f = \varphi_\alpha - \chi_{qZ}$.

B. SCHWEIZER: Some remarks on copulas

Let H be a two-dimensional joint distribution function with margins $H_1(x) = H(x, \infty)$, $H_2(y) = H(\infty, y)$ and copula C_H, so that $H(x, y) = C_H(H_1(x), H_2(y))$. Let f and g be non-decreasing functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(\infty) = g(\infty) = \infty$ and $f(-\infty) = g(-\infty) = -\infty$; and let $G(u, v) = H(f(u)), g(v))$. Then G is a joint distribution function and $C_G = C_H$. From this and the fact that $C_H(u, v) = H(H_1^{-1}(u), H_2^{-1}(v))$, it follows that any functional or "property" defined on the set of all two-dimensional joint distribution functions (of pairs of random variables, X and Y) which is invariant under increasing transformations (of X and Y) is a functional or "property" of the set of copulas (and independent of the individual distribution functions of X and Y). Hence, in this sense, the theory of order statistics may be viewed as the theory of copulas.

A. SKLAR: Further work on the representation of associative functions

The lecture consists of 4 parts:

1. Conditions more general than those previously known can be given to insure that, if T_1 is associative and T is defined by

$$T(x, y) = f(T_1(g(x), g(y))),$$

then T is associative.

2. In the process of filling a gap in the argument given by C.-H. Ling (Publ. Math. Debrecen 12 (1965), 189-212), G. Krause has produced a simpler proof of Ling's results under much weaker conditions.

3. M. J. Frank has extended the work of C. Kimberling (Publ. Math. Debrecen 20 (1973), 21-39) and the author (Aequationes Math. 14 (1976), 211-212) by showing that the representable associative functions of Ling and Krause

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(no. 2 above) are completely determined by their behaviour on appropriate pairs of short line segments.

4. Results on the representability of certain discontinuous associative functions (Aequationes Math. 14 (1976), 211-212) have been improved.

C. ALSINA: On ruled t-norms

In a joint work with Abe Sklar we have proved the following:

Theorem. Let f be a strictly increasing continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that f(0) = 0. The most general solution of the functional equation

$$f^{-1}\left[\frac{f(x)}{f(x+y)}\right] + f^{-1}\left[\frac{f(y)}{f(x+y)}\right] = 1, \quad x, y \in \mathbb{R}^+, \ x+y \neq 0,$$

is the power function $f(x) = x^c$, where $c > 0$ is an arbitrary positive constant.

Using this result we show that if s(x, y) is a t-norm and if the graph of $\{(x,y) \mid s(x,y) > 0\}$ is a ruled surface then

$$s(x,y) = \operatorname{Max}\left(1 - \left((1-x)^k + (1-y)^k\right)^{\frac{1}{k}}, 0\right), \quad k > 0.$$

J. ACZÉL: Generalized information functions and measures of degrees 2 and 0

In solving the generalization

$$(*) \qquad f_1(x) + (1-x)^{\alpha} f_2\left(\frac{y}{1-x}\right) = f_3(y) + (1-y)^{\alpha} f_4\left(\frac{x}{1-y}\right) \quad (x, y \in [0,1[, \ x+y \leq 1)$$

of the fundamental equation of information, and in applications to inset entropies and to deviations (divergences, inaccuracies etc.), $\alpha = 2$ and 0 have been unsettled exceptional cases. In consequence of work done by the author and by Gy. Maksa, also these cases have now been cleared: If the powers of 0 are properly defined (which is not always obvious) then the results are similar to other $\alpha \neq 1$ cases in most instances - but not always: (*) has, for $\alpha = 2$, solutions which are nowhere continuous (The case $\alpha = 1$ has been solved first and its singular behaviour has known reasons.) The proofs are different from those for other α.

Z. MOSZNER: Solution générale de l'équation de translation sur le demi-groupe des éléments non-négatifs

On peut obtenir toutes les solutions de l'équation

$$F(F(\alpha, x), y) = F(\alpha, x+y),$$

où $F : \Gamma \times G^{+} \to \Gamma$, Γ est un ensemble arbitraire et G^{+} est un demi-groupe d'éléments non-négatifs d'un groupe G ordonné archimédien, remplissant la condition

$$\forall \alpha_1, \alpha_2 \in F(\Gamma, G^{+}): [F(\alpha_1, G^{+}) \cap F(\alpha_2, G^{+}) \neq \emptyset \Rightarrow F(\alpha_1, G^{+}) \subset F(\alpha_2, G^{+}) \text{ ou } F(\alpha_2, G^{+}) \subset F(\alpha_1, G^{+})]$$

selon la construction suivante:

1) Soit $f : \Gamma \to \Gamma$ une fonction telle que $\forall \alpha \in \Gamma : f(f(\alpha)) = f(\alpha)$.

2) Décomposons $f(\Gamma) = \bigcup_{k \in K} \Gamma_k$ aux ensembles Γ_k non-vides, disjoints et tels que pour chaque k de K il existe une décomposition invariante $\{W_{ik}\}_{i \in I_k}$ d'un intervalle Δ_k de G pour lequel $G^{+} \subset \Delta_k$ et $\bar{\bar{I}}_k = \bar{\bar{\Gamma}}_k$.

3) Soit $h_k : \{W_{ik}\}_{i \in I_k} \to \Gamma_k$ une bijection et définissons $\bar{h}_k : \Delta_k \to \Gamma_k$ de la manière suivante: $\bar{h}_k(x) = h_k(W_{ik})$ pour $x \in W_{ik}$.

4) Posons $F(\alpha, x) = \bar{h}_k(\bar{h}_k^{-1}(f(\alpha)) + x)$ pour $f(\alpha) \in \Gamma_k$.

La démonstration de ce théorème est sous presse dans Rocznik Nauk. Dydak. WSP w Krakowie. Toutes les décompositions invariantes de Δ_k sont caractérisés dans la note de Z. Moszner "Décompositions invariantes du demi-groupe des éléments non-négatifs du groupe archimédien", sous presse dans Tensor.

A. GRZĄŚLEWICZ: The translation equation on the loop

The loop is a groupoid with a unit 1 such that both equations $a \cdot x = b$, $y \cdot a = b$ possess unique solutions.

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Let X be an arbitrary set, $Q(\cdot)$ an arbitrary loop. We call a function $F : X \times Q \longrightarrow X$ the transitive solution of the translation equation if $F(F(x,\beta),\alpha) = F(x,\alpha\cdot\beta)$ for $x \in X$, $\alpha.\beta \in Q$ and if for every $x, y \in X$ there exists $\alpha \in Q$ such that $F(x,\alpha) = F(y, 1)$.

Theorem. A function $F : X \times Q \rightarrow X$ is a transitive solution of the translation equation iff there exists a subloop H of the loop Q, a function $f : X \rightarrow X$ and a bijection $g : f(X) \rightarrow \{\alpha H : \alpha \in Q\}$ such that

a) $f(x) = x$ for $x \in f(X)$,

b) $\alpha(\beta(\gamma H)) = (\alpha\beta)(\gamma H)$ for $\alpha, \beta, \gamma \in H$,

c) $F(x,\alpha) = g^{-1}(\alpha\cdot g(f(x))$ for $x \in X, \alpha \in Q$.

Thus the solution of the translation equation on the loop is very similar to the solution of this equation on the group.

F. NEUMAN: Functional and differential equations.

It was shown that under very general conditions any differential system with one (retarded or advanced) deviating argument can be transformed into a differential system with a constant deviation. If a system is linear, then the transformed system is also linear.

Connections with the translation equation were explained.

J. TABOR: Rational iteration groups

Let I be a closed interval in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $f : I \longrightarrow I$ a strictly increasing continuous function such that $f(I) = I$.

Definition. A family of continuous functions $\{f^t : I \rightarrow I, t \in \mathbb{R} (t \in \mathbb{Q})\}$ is called a real (rational) iteration group of f whenever $f^t \circ f^s = f^{t+s}$ for $t, s \in \mathbb{R} (t, s \in \mathbb{Q})$, $f^1 = f$. If, for every $x_0 \in I$, the mapping $f^t(x_0)$ is continuous, a real (rational) iteration group is called continuous.

In the paper a construction of a rational iteration group of f is presented. A necessary and sufficient condition for this group to be continuous is also

given. It is shown, by applying these results that every continuous rational iteration group can be extended to a continuous real iteration group. Finally Zdun's problem is investigated: can every real iteration group f^t, $t \in \mathbb{R}$ be written in the form $f^t = \tilde{f}^{\varphi(t)}$, where $\tilde{f}^t$, $t \in \mathbb{R}$ is a continuous real iteration group and $\varphi : \mathbb{R} \to \mathbb{R}$ is an additive function ? The answer is "no". A necessary and sufficient condition for this to be possible is also given.

R. LIEDL: Zentrale Ähnlichkeit auf Gruppen und die Translationsgleichung

Es wird der Begriff der zentralen Ähnlichkeit auf Gruppen mit differentialgeometrischen Methoden eingeführt.

Mit Hilfe dieses Begriffes wird eine Lösung der Translationsgleichung angestrebt.

L. REICH: Einbettung von Wurzeln formal-biholomorpher Abbildungen in kontinuierliche Iterationen

Es sei $F : x \mapsto Ax + \mathcal{P}(x)$ ein ordnungserhaltender, den Grundkörper $\mathbb{C}$ elementweise festlassender Automorphismus des Potenzreihenringes $\mathbb{C}[x_1, \ldots, x_n]$, d.h. A eine nichtsinguläre (n,n)-Matrix, $x = {}^T(x_1, \ldots, x_n)$, $\mathcal{P}(x) = {}^T(\mathcal{P}_1(x), \ldots, \mathcal{P}_n(x))$ eine Spalte formaler Potenzreihen mit $\operatorname{ord} \mathcal{P}_i(x) \geq 2$. $\rho_1, \ldots, \rho_n$ seien die Eigenwerte von A, $\Lambda = (\ln \rho_1, \ldots, \ln \rho_n)$ eine feste Wahl von Logarithmen der Eigenwerte. Es wird folgender Satz diskutiert:

F habe eine unendliche Folge konsekutiver Wurzeln,

$$F^{1/r_1}, \quad (F^{1/r_1})^{1/r_2} =: F^{1/r_1 r_2}, \ldots,$$

mit folgender Eigenschaft: Die Linearteile dieser Wurzeln gehören sämtlich einer zu Λ gehörigen analytischen Iteration A(t) von A an. Dann existiert eine zu Λ gehörige analytische Iteration $(F_t)_{t \in \mathbb{C}}$ von F mit

$$F_{1/r_1 \ldots r_j} = F^{1/r_1 \ldots r_j}, \quad j \geq 1 .$$

J. SCHWAIGER: Iterierbarkeit von Potenzen formaler Reihen

Es wird das folgende Problem untersucht: Gegeben eine formale Reihe $F = AX + \mathfrak{p}(x)$ mit $\det A \neq 0$. Gibt es dann eine ganzzahlige Iterierte $F^k = F \circ \ldots \circ F$, so daß F^k analytisch iterierbar ist ? Das Problem wurde schon 1939 von D. C. Lewis positiv gelöst. Hier wird ein neuer Beweis gegeben, der auf den Methoden fußt, die von L. Reich und dem Verfasser zur Untersuchung der Frage analytischer Iterierbarkeit formaler Potenzreihen angegeben wurden.

M. C. ZDUN: Convex iteration semigroups

Let $\{f^t, t > 0\}$ be an iteration semigroup of continuous functions in a closed interval I. An iteration semigroup is said to be convex if all functions f^t are convex. Let $x_1 := \inf \bigcup_{t>0} f^t[I]$, $x_2 := \sup \bigcup_{t>0} f^t[I]$. We have the following characterization of convex iteration semigroups.

Theorem 1. An iteration semigroup $\{f^t, t > 0\}$ is convex iff for every $x \in I$ the mappings $t \to f^t(x)$ are continuous and there exists the derivative $\frac{\partial f^t}{\partial t}\Big|_{t=0} =: g$ in $(x_1, x_2) \cup \{x \in I : f^1(x) = x\}$, furthermore g and $f^0 := \lim_{t \to 0} f^t$ are convex.

We consider also the problem of the existence and uniqueness of convex iteration semigroups of a given function f (i.e. such semigroups $\{f^t, t > 0\}$ that $f^1 = f$).

Theorem 2. Let a convex function $f : I \to I$ be non-constant in neighbourhoods of its fixed points. If f has exactly one fixed point x_0 and $f'_+(x_0) \neq 0$, $f'_-(x_0) \neq 0$ or f has more than one fixed point, then f may possess at most one convex iteration semigroup.

Theorem 3. Let $f : \langle a, b\rangle \to \langle a, b\rangle$ be a convex function and $f(b) = b$, $f(x) > x$ for $x \in \langle a, b)$. If f'_+ is concave then f has exactly one convex iteration semigroup.

U. BURKART: A new proof of a theorem of Šarkovskii

We consider a continuous function $f : \mathbb{R} \to \mathbb{R}$ with a k-periodic point and its digraph as constructed by Straffin [1]. We prove the following

Theorem: 1. If $k \neq 2^j$, $j \in \mathbb{N}_o$, the digraph contains non-repetitive (nr-) cycles of all lengths 2^m, $m \in \mathbb{N}_o$.

2. If $k \geq 2$, the digraph contains nr-cycles of all lengths 2^m, $2^m \leq k$.

3. If $k \geq 3$, odd, the digraph contains nr-cycles of all lengths m, $m \geq k-1$ (this was already proven by Straffin in [1]) and all lengths m, m even.

4. If $k = 2^j \cdot n$, $n \geq 3$, odd, the digraph contains nr-cycles of all lengths $2^j \cdot v$, $v \geq n$, v odd and $2^{j+1} \cdot w$, $w \geq 3$, odd.

Combining our theorem with a theorem of Straffin [1], we construct a new, shorter and easier proof of the well-known Šarkovskii theorem [2] on periodic points of continuous functions.

References:

[1] Ph.D. Straffin, Jr., Periodic points of continuous functions, Math. Magazine, 51 (1978), 99 - 105

[2] A.N. Šarkovskii, Co-existence of cycles of a continuous mapping of the line into itself, Ukrain.Mat. Ž. 16 (1964), 61 - 71 (Russian)

R. GRAW: The orbit structure of discrete limit sets of iterative sequences

Let f be a continuous self-mapping of a Hausdorff space X satisfying the first axiom of countability. A limit set L(x), $x \in X$, is the set of all cluster points of the iteration splinter $\{f^n(x)\}$. Our purpose is to show that in infinite dimensional Banach spaces limit sets may have orbit structures which cannot occur in the finite dimensional case.

In metric spaces every discrete limit set is countable. For locally compact X we have two theorems. If a discrete limit set L(x) contains a point of an orbit with cycle Z then L(x) = Z and x is asymptotically periodic. In [1] it was shown that a finite limit set is a cycle. From [2] we know that a countably

infinite discrete limit set can have every possible orbit structure without cycles. In a nowhere locally compact metric space a countable discrete limit set (finite or infinite) can have all possible orbit structures (including cycles). Moreover, in this case asymptotic periodicity of x is not necessary for L(x) to be cyclic.

References:

[1] Kuczma, M. On a theorem of B. Barna, to appear in Aequationes Math.

[2] Graw, R., Kuczma, M. Infinite discrete sets of cluster points of iterative sequencesof continuous functions, to appear in Aequationes Math.

R. RICE: The relationship between iterative roots of f, g, $f \circ g$ and $g \circ f$

Definition: If f and g are functions from a set S into itself and if, for some integer $n \geq 2$,

$$f = h^n \equiv \underbrace{h \circ h \circ \ldots \circ h}_{n}$$

then h is called an (iterative) root of f of order n.

Theorem: Let M, N, L and K be non-negative integers. There is a set S and functions f and g mapping S into itself such that for any $n \geq 2$

(1) f has a root of order n if and only if $n | M$,

(2) g has a root of order n if and only if $n | N$,

(3) $f \circ g$ has a root of order n if and only if $n | L$,

(4) $g \circ f$ has a root of order n if and only if $n | K$.

W. SCHEMPP: Splinefunktionen als Lösungen von inhomogenen linearen geometrischen Differenzengleichungen erster Ordnung

Es bezeichne h_o eine feste reelle Zahl > 1. Für die inhomogene lineare geometrische Differenzengleichung erster Ordnung

$$F(h_o x) - F(x) = 1$$

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werden als Lösungen auf der offenen rechten Halbgeraden $\mathbb{R}_+^{\times}$ neben $(f_o : x \longmapsto (\log x)/(\log h_o)$ die f_o in der geometrischen Knotenfolge $(h_o^n)_{n \in \mathbb{Z}}$ interpolierenden kardinalen logarithmischen Splinefunktionen S_m vom Grad $m \geq 1$ betrachtet. Zur Diskussion des Grenzübergangs $\lim_{m \to \infty} S_m(x)$ für $x \in \mathbb{R}_+^{\times}$ sollen zwei Methoden vorgestellt werden:

I. Reelle Transformationsmethode (Einseitige Laplace-Transformation; verschärfter Abel-Tauber-Satz).

II. Komplexe Transformationsmethode (Pincherle-Identität; Residuensatz).

Beide Methoden führen zum Nachweis des Newman-Schoenberg-Phänomens: Die Folge $(S_m(x))_{m \geq 1}$ konvergiert genau dann gegen $f_o(x)$, falls der Aufpunkt $x \in \mathbb{R}_+^{\times}$ mit einem der Interpolationsknoten $(h_o^n)_{n \in \mathbb{Z}}$ zusammenfällt.

I. FENYÖ: On a functional integral equation

The Goursat problem concerning a linear hyperbolic differential equation can be reduced to the following functional integral equation:

$$(\ast) \qquad x(s) + V(s)\, x(\alpha s) + \int_{\alpha s}^{s} P(s,t)\, x(t)\, dt = f(s) \qquad (s \in \mathbb{R})$$

where $|\alpha| < 1$ holds. If we solve $(\ast)$ by the method of successive approximation, then, in every step, we get a functional equation of the form

$$z(s) + z(\alpha s) = v(s).$$

If P and f are bounded functions, then it is proved that each of the functional equations has exactly one solution satisfying a necessary initial-value condition. This sequence of successive approximations converges uniformly in every compact interval to the unique solution of $(\ast)$.

S. MIDURA: Sur les solutions de l'équation fonctionnelle $f[x + 3f^2(y) - 3f(x)f(y) - y] = f(x) - f(y)$

L'équation fonctionnelle

(1) $$f[x + 3f^2(y) - 3f(x)f(y) - y] = f(x) - f(y),$$

où $f: R \longrightarrow R$ est la fonction inconnue a été signalée dans [1]. Nous allons trouver la solution générale de l'équation (1) dans la classe des fonctions impaires.

Théorème. La solution générale de l'équation (1) dans la classe des fonctions impaires est l'ensemble des fonctions qui satisfont à l'équation de Cauchy:

(2) $$f(x + y) = f(x) + f(y)$$

et

(3) $$f(f(x)f(y)) = 0$$

On présente ici une construction de la solution générale de l'équation (1) dans la famille des fonctions impaires, analogue à celle de la solution générale de l'équation de Cauchy.

[1] S. Midura, Sur la détermination de certains sousgroupes du groupe L_s^1 à l'aide d'équations fonctionnelles, Dissertationes Math. 105 (1973)

P. SCHROTH: A characterization of a family of functions

Special solutions of the system of functional equations

$$\forall p \in \mathbb{N}_+ \; \forall (x,y) \in D: f(x,y) = \frac{1}{p} \sum_{k=0}^{p-1} f(x + ky, py)$$

are the functions $(x,y) \longrightarrow y^m B_m(\frac{x}{y})$, $m \in \mathbb{N}$, B_m meaning the m-th Bernoulli-polynomial, $(x,y) \mapsto y\, e^{\alpha x}(e^{\alpha y} - 1)$, $\alpha \in \mathbb{R} \setminus \{0\}$, and $(x,y) \longrightarrow \psi(\frac{x}{y}) + \log y$ where in the first two cases $D = \mathbb{R} \times \mathbb{R}_+$ in the third $D = \mathbb{R}_+ \times \mathbb{R}_+$. We introduce a characterization of these functions as continuous solutions of the system noted above by means of certain separation types.

B. EBANKS: The branching property for measures of vector information

A measure of vector information is a sequence of maps $\mu_n : \bigtimes_{j=1}^{m} S_{jn} \to G$ $(n = 2, 3, \ldots)$, where $(G, +)$ is a commutative, divisible group, and each S_{jn} $(j = 1, 2, \ldots, m)$ is either Γ_n (the set of complete probability distributions of length n) or S_j^n for some commutative monoid (S_j, j, e_j) in a certain class. For notational convenience, we write the argument of μ_n as an element of $(\bigtimes_{j=1}^{m} S_j)^n$. μ_n has the branching property if

$$\mu_n(\underline{v}_1, \ldots, \underline{v}_n) = \mu_n(\underline{v}_1, \ldots, \underline{v}_{i-1}, \underline{v}_i \circ \underline{v}_{i+1}, \underline{e}, \underline{v}_{i+2}, \ldots, \underline{v}_n) + \Delta_{ni}(\underline{v}_i, \underline{v}_{i+1})$$

for some maps Δ_{ni} $(i = 1, 2, \ldots, n-1)$, where $\underline{e} = (e_1, \ldots, e_n)$ and where $\underline{v} \circ \underline{w} = (v_1 \, 1 \, w_1, \ldots, v_n \, n \, w_n)$. It is found that μ_n is branching and symmetric, if and only if

$$\mu_n(\underline{v}_1, \ldots, \underline{v}_n) = \sum_{i=1}^{n} \varphi_{ni}(\underline{v}_i) + \varphi_{n0}(\underline{v}_1 \circ \ldots \circ \underline{v}_n)$$

for some maps φ_{ni} $(i = 0, 1, \ldots, n)$.

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Problemstellungen und Bemerkungen

1. Remark (to Kannappan's question at the 1978 meeting).
Given $f: \mathbb{R} \longrightarrow \mathbb{R}$, satisfying Cauchy's equation such that $f(x)\, f(\frac{1}{x}) > 0$ holds for all $x \in \mathbb{R},\ x \neq 0$. Does this imply $f \in C(\mathbb{R})$?

We have no answer to this question but we give an example of a function $f: \mathbb{R} \longrightarrow \mathbb{R}$, satisfying Cauchy's equation, such that $f \notin C(\mathbb{R})$ and such that $1 \geqslant f(x)\, f(\frac{1}{x}) > 0$ holds true for all $x \in \mathbb{R},\ x \neq 0$.

Consider a noncontinuous automorphism φ of $\mathbb{C}$ and put $f(x) := \operatorname{Re} \varphi(x)$. Then $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f \notin C(\mathbb{R})$. Moreover

$$f(\frac{1}{x}) = \frac{f(x)}{f(x)^2 + g(x)^2} \quad \text{for all } x \in \mathbb{R},\ x \neq 0,$$

by putting $g(x) := \operatorname{Im} \varphi(x)$ and using $\varphi(x) \cdot \varphi(\frac{1}{x}) = 1$.

Hence $1 \geqslant f(x)\, f(\frac{1}{x}) > 0$ for all $x \in \mathbb{R},\ x \neq 0$.

If there exists a noncontinuous automorphism φ of $\mathbb{C}$ such that $\varphi(x) \notin i\mathbb{R}$ for all $x \in \mathbb{R},\ x \neq 0$, then this automorphism would lead to a counterexample $1 \geqslant f(x)\, f(\frac{1}{x}) > 0$ for all $x \in \mathbb{R},\ x \neq 0$.

W. Benz

2. Problem. In connection with investigations of J. Aczél and coworkers, S. C. Martin, A. Grząślewicz and P. Sikorski on extensions of homomorphisms from semigroups onto groups generated by them in a special manner, I have the following question: Does anyone know of an example of a group G and its subsemigroup S such that $G = SS^{-1}S$, but at the same time $G \neq SS^{-1}$?

More generally $G = SS^{-1}S \ldots S^{\varepsilon}$ (where $\varepsilon = +1$ or $\varepsilon = -1$ according as the number of factors is odd or even), but no smaller number of factors is sufficient?

M. Kuczma

3. Remark. C. Wagner (Univ. of Tenn. and Center for Advanced Study in the Behav. Sci., Stanford, Calif.) has conjectured the following:

There exist constants $a_1, \dots, a_n \in \mathbb{R}$ with $\sum_{j=1}^{n} a_j = 1$ such that $f(x_1, \dots, x_n) = \sum_{j=1}^{n} a_j x_j$ for all $x_1, \dots, x_n \in \mathbb{R}$ if and only if $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous and $\sum_{i=1}^{k} x_{ij} = s$ $(j = 1, \dots, n)$ imply $\sum_{i=1}^{k} f(x_{i1}, \dots, x_{in}) = s$.

I have proved this even if the last implication is supposed only for $k = 2$ and 3 and if f is assumed only continuous at a point or bounded from one side on a set of positive measure.

J. Aczél

4. Remark (Answer to M. Kuczma's question 2.)

We find a group G and a subsemigroup S such that

(i) $G = SS^{-1}S$, (ii) $SS^{-1} \neq G \neq S^{-1}S$.

Define $G := \{(a,b,c,d) \mid a,b,c,d \in \mathbb{Q} \text{ and } ac \neq 0\}$ with the multiplication

$(a,b,c,d) \cdot (\alpha, \beta, \gamma, \delta) := (a\alpha, a\beta + b, c\gamma, d\gamma + \delta)$

and define $S := \{(a,b,c,d) \in G \mid a,b,c,d \in \mathbb{Z}\}$.

Now $(1, \frac{1}{2}, 1, 0) \notin SS^{-1}$, $(1, 0, 1, \frac{1}{2}) \notin S^{-1}S$, which proves (ii).

Given $(r_1, r_2, r_3, r_4) \in G$ write $r_1 = \frac{x}{z}$, $r_2 = \frac{y}{z}$, $r_3 = \frac{u}{w}$, $r_4 = \frac{v}{w}$.

Then $(1, 0, u, v) \cdot (\frac{1}{z}, 0, \frac{1}{w}, 0) \cdot (x, y, 1, 0) = (r_1, r_2, r_3, r_4)$.

Hence (i) holds also.

W. Benz

5. Remark (D. Zupnik's characterization of Cayley sets).

A Cayley set is the set of left inner translations of a semigroup.

D. Zupnik has recently characterized all Cayley sets, as follows:

Let S be a non-empty set and F a set of functions each mapping S into S. Then F is a Cayley set if and only if there exists a subset A of S such that:

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1) if f_1, f_2 are in F and the restrictions of f_1 and f_2 to A coincide, then $f_1 = f_2$;

2) for any z in A there is a function g_z mapping S into S such that g_z commutes with each function in F, and Ran $g_z = \{ f(z) | f \text{ in } F \}$;

3) for every x in S, y, z in A, and f in F, we have

$$g_y(x) = f(y) \text{ if and only if } g_z(x) = f(z).$$

When A is a unit set, then this characterization reduces to that announced in Aequationes Math. 17 (1978), 375-376, Remark 7.

A. Sklar

6. Problem. Solve the functional equation,

$$f(xy) + f(x+y) = f(xy + x) + f(y), \tag{1}$$

where the domain and range of f is a (commutative) field.

Equation (1) is obtained from the following equation by setting z = 1

$$f(xy) + f(xz + yz) = f(xy + xz) + f(yz). \tag{2}$$

I can prove that (2) is equivalent to

$$f(x + y) + f(0) = f(x) + f(y) \tag{3}$$

for all fields with at least 4 elements. From this fact I would guess that (1) is equivalent to (3).

T. M. K. Davison

7. Problem. Consider the functional equation:

$$(*) \quad f(x)\, f(y) = f(xy)\, f\left(\frac{x+y}{2}\right). \qquad (x, y \in \mathbb{R})$$

If A is a set of reals such that:

$$x, y \in A \Longrightarrow xy \in A \text{ and } \frac{x+y}{2} \in A,$$

then the characteristic function χ_A of A fulfills (*).

Is it true, that every solution of (*) is the characteristic function of a set of type A ?

I. Fenyö

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8. Remark (Contribution to T. M. K. Davison's question).

Given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous on $\mathbb{R}$ such that

$$f(xy) + f(x + y) = f(xy + x) + f(y)$$

holds for all $x, y \in \mathbb{R}$. Then $f(x) = ax + b$, where a, b are real constants.

Proof: 1. $f(x + x^n y) - f(x^n y) = f(x + y) - f(y)$ holds for all $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, $x \neq 0$: by induction via

$f(x+x^{n+1}y) - f(x^{n+1}y) = f(x+x \cdot x^n y) - f(x \cdot x^n y) = f(x+x^n y) - f(x^n y)$

2. Consider $0 \neq |x| < 1$. Then $n \longrightarrow \infty$ yields (see 1.)

$$f(x) - f(0) = f(x + y) - f(y).$$

This holds also in the case $|x| > 1$ considering 1. for $n \longrightarrow -\infty$. Hence

3. $f(x) - f(0) = f(x + y) - f(y)$ for all $y \in \mathbb{R}$ and all $x \in \mathbb{R}$, $x \neq 0, 1, -1$.

But 3. ist trivial for $x = 0$ and turns out to be true for $x = 1, -1$ by using again $f \in C(\mathbb{R})$ and $x \longrightarrow 1$ (or $x \longrightarrow -1$) in 3.

4. Hence $g(x) := f(x) - f(0)$ is continuous on $\mathbb{R}$ and satisfies Cauchy's equation. Thus $g(x) = ax$, i. e. $f(x) = ax + f(0)$.

W. Benz

9. Remark (on Fenyö's problem).

1) Let $L = \{ x \,|\, f(x) \neq 0 \}$. Then one deduces immediately that

$$(*) \qquad x \in L \wedge y \in L \iff xy \in L \wedge \frac{x+y}{2} \in L.$$

Now let L be an arbitrary subset of $\mathbb{R}$ satisfying $(*)$. Then the function which is 1 on L and 0 on the complement of L satisfies Fenyö's equation.

2) The property $(*)$ can be reformulated as:

$$x \in L \iff \exists k, m \in L \text{ such that } x^2 - 2kx + m = 0.$$

Using this, one can show that, except for the trivial cases $L = \{0\}$, $L = \{1\}$, L must be dense in $(0, \infty)$.

T. M. K. Davison

10. Problem and Remark. Let F and G be functions from $[0,1] \times [0,1]$ into $[0,1]$ such that

(i) F and G are associative, non-decreasing in each place and continuous;

(ii) $F(x,0) = F(0,x) = 0$, $G(x,0) = G(0,x) = x$;

(iii) $F(x,1) = F(1,x) = x$, $G(x,1) = G(1,x) = 1$.

Prove or disprove that the unique solution of the functional equation

$$F(x,y) \cdot G(x,y) = x \cdot y, \quad \text{for all } x,y \in [0,1],$$

is F = Minimum and G = Maximum.

Remark. 1. If $G(x,y) = 1 - F(1-x, 1-y)$, for all $x,y \in [0,1]$, the conjecture is true.

2. The equation $F(x,y) + G(x,y) = x + y$, for all $x,y \in [0,1]$ has been solved by M. J. Frank (Aequationes Math., 18 (1978), 266-267).

This equation is unsolved in infinite intervals like $[0, +\infty]$ or $[-\infty, +\infty]$.

C. Alsina

11. Problem. What is the general solution of the functional equation

$$f(z) = \inf \{ f(x) + f(y) \mid x + y = z, \quad x, y > 0 \},$$

where f is a function from $\mathbb{R}^+$ into $\mathbb{R}^+$. Note that the Dirichlet function is a solution.

C. Alsina and J. Dhombres

12. Problem. Under what conditions on $\xi_i : I \longrightarrow \mathbb{R}$, $i = 1, \ldots, k$, $I \subset \mathbb{R}$, does there exist a solution $\varphi \in C^n$, $\varphi' \neq 0$, of the system of functional equations

$$\varphi \xi_i(t) = \varphi(t) + c_i,$$

c_i being constants.

The problem is connected with the transformation equation $F(F(t,u),v) = F(t, u+v)$, where we look for F satisfying $F(t, c_i) = \xi_i(t)$.

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either by extending ξ_i $(i = 1, \ldots, k)$ into a one-parameter group of transformations, or with aid of continuous iteration semigroups.

For some necessary conditions and for connections of the problem with transformations of differential systems with deviating arguments into differential systems with constant deviations, see F. Neuman: On transformations of differential equations and systems with deviating argument, to appear in Czechoslovak. Math. J.

F. Neuman

13. Remark and Problem (This is a report on a joint work by J. Matkowski and W. Ogińska: Note on iteration of some entire functions).
Let f be entire or rational function. Consider the sequence of iterations
$f_o(z) = z$, $f_{n+1}(z) = f[f_n(z)]$, $n = 0, 1, \ldots$.
In the iteration theory an important part is played by the set F(f) of those points of the complex plane $\mathbb{C}$ where $\{f_n\}$ is not a normal family in the sense of Montel. It is well known that the set F(f) is nonempty, perfect and completely invariant with respect to f.

In 1918 Latte constructed a rational function for which $F(f) = \mathbb{C}$. In 1970 I. N. Baker proved that there is a $k > e^2$ such that $F(kze^z) = \mathbb{C}$. Matkowski and Ogińska have proved the following

Theorem: $F(2k\pi ie^z) = \mathbb{C}$ for $k = \pm 1, \pm 2, \ldots$.

The question whether $F(e^z) = \mathbb{C}$ is still open.

M. Kuczma

14. Remark (presented by H. H. Kairies). The solutions $f: \mathbb{R} \longrightarrow \mathbb{R}$ of the difference equation

$$f(x + 1) f(x - 1) = f(x) + c, \quad c \in \mathbb{R},$$

have a remarkable periodicity property, depending on the parameter c. For example: If c = 0 resp. 1 then f has period 6 resp. 5.

A. Clausing

15. Problem. Let $f: \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function with the following properties:

(a) $f(a, o) = a$,

(b) $f(a, b) = f(b, a)$,

(c) $f(a, b) \geqslant f(c, b)$, whenever $a \geqslant c$,

(d) $f(a, f(b, c)) = f(f(a, b), c)$.

In connection with his study of the characterization of L_p-norms, F. Bohnenblust [Duke Math. J. 6 (1940), 627-640] has shown that if, in addition, $f(ta, tb) = tf(a, b)$ for all $t > 0$, then either $f(a, b) = (a^p + b^p)^{\frac{1}{p}}$, $p > 0$, or $f(a, b) = \mathrm{Max}\,(a, b)$. In order to define a well-behaved Cartesian product of a finite number of metric spaces, one needs a function f which satisfies (a) - (d) and

(e) $f(a_1 + b_1, a_2 + b_2) \leqslant f(a_1, a_2) + f(b_1, b_2)$.

Find a "nice" characterization of the functions f that satisfy (a) - (d) and (e).

B. Schweizer

16. Problem. The following is a paraphrase of a problem posed by Mourad Ismail of McMaster University. Determine the asymptotic behavior as $n \longrightarrow \infty$ of the function defined by

$$P_{n+1}(x) = x \cdot P_n(ax+b) + (1 - x) \cdot P_n(cx + d), \quad n = 1, 2, 3, \dots,$$

$0 < x < 1$ and a, b, c, d given constants. The initial conditions of interest are (i) $P_o(x) = x$ and (ii) $P_o(x) = 1 - x$. A solution is known when $a = c$ or $b = d = 0$.

T. M. K. Davison

17. Remark. Dr. J. Schwaiger has posed at this symposium the following problem. Let $r_1, r_2, \dots$, be a sequence of positive integers. Consider a sequence $f^{\frac{1}{r_1}}$, $(f^{\frac{1}{r_1}})^{\frac{1}{r_2}}$, ..., $(\dots(f^{\frac{1}{r_1}})\dots)^{\frac{1}{r_n}}$ of fractional iterates (here $(f^{\frac{1}{r_1}})^{\frac{1}{r_2}}$ is an iterative root of order r_2 of the $f^{\frac{1}{r_1}}$ etc.). Let us

introduce the following shorter notation

$$f^{\frac{1}{r_1 r_2 \cdots r_n}} := (\ldots(f^{\frac{1}{r_1}})^{\frac{1}{r_2}} \ldots)^{\frac{1}{r_n}} .$$

When does the sequence $f^{\frac{1}{r_1}}, f^{\frac{1}{r_1 r_2}}, \ldots, f^{\frac{1}{r_1 \cdots r_n}}$, generate (by composition) a rational iteration semigroup?

The answer to this question is given in the following

Theorem. The sequence $f^{\frac{1}{r_1}}, \ldots, f^{\frac{1}{r_1 \cdots r_n}}$ generates a rational iteration semigroup of f iff for every positive integer n there exists a positive integer i such that $n \mid r_1 \cdot r_2 \ldots r_i$.

Proof of this theorem will be published shortly.

J. Tabor

18. Problem. Let S be a non-empty set and f a function mapping S into itself. No regularity conditions whatever are imposed on f. If f has iterative roots of all orders (i.e., if for every positive integer n, there is a function $g_n : S \longrightarrow S$ such that $g_n^n = f$), can f be embedded into a rational iteration semigroup? I conjecture that the answer in general is no, but counterexamples seem to be difficult to construct.

A. Sklar

19. Problem. Die Translationsgleichung $F(F(x,u),v) = F(x,u+v)$ $\forall u, v \in \mathbb{R}$ hat nicht für alle $F(x,1) = G(x)$ eine Lösung. Unter welchen Bedingungen kann man aber zu gegebenem G weitere Funktionen $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$ finden, so daß $G = G^{(n)} \circ G^{(n-1)} \circ \ldots \circ G^{(1)}$ und die Translationsgleichung für $F(x,1) = G^{(k)}(x)$ $(k = 1,\ldots,n)$ lösbar ist?

R. Liedl

20. Remark. In connection with a problem by Professor Liedl, I am reminded of an old problem by Professor Moszner (Aequationes Math. 1 (1968), 150) which has remained open for several years now. Let, for any

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positive integers n, k, D_k^n denote the class of those mappings $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which are of class C^k in $\mathbb{R}^n$ and have a positive Jacobian throughout the whole $\mathbb{R}^n$. Let Q_k^n denote the class of those $f \in D_k^n$ for which there exists a $\varphi \in D_k^n$ such that $\varphi^2 = f$ (where $\varphi^2 = \varphi \circ \varphi$ is the second iterate of φ). It is known that $D_k^n \neq Q_k^n$. The problem is as follows: Can every function $f \in D_k^n$ be represented as a superposition of a finite number of functions from Q_k^n ?

$$f = f_1 \circ \dots \circ f_p, \quad f_i \in Q_k^n, \quad i = 1, \dots p.$$

A positive answer to this question is known for n = 1 (M. Kuczma: On squares of differentiable functions, Ann. Polon. Math. 22 (1969), 229-237), where it has been proved that $p \leqslant 4$. But for general n the problem is still open.

M. Kuczma

21. Remark and Problem. Let the set S be fixed and $f \in R_n$ if f has an n-th iterative root. The theorem of Dr. Rice can be interpreted as follows. If $f \in R_n$, $g \in R_n$ there exist bijections α and β of S such that $\alpha^{-1} \circ f \circ \alpha \circ \beta^{-1} \circ g \circ \beta \in R_n$ and similarly for the other seven cases. Since membership in R_n depends on certain features of the orbit structure, but does not determine the structure entirely (theorems of R. Isaacs and G. Zimmermann, S. Lojasiewicz, M. Bajraktarević, A. Sklar) the following questions offer themselves (they may lead to results stronger than the Rice-theorem):

1) Under what conditions on $f, g, h: S \longrightarrow S$ do there exist bijections $\alpha, \beta, \gamma: S \longrightarrow S$ such that $\alpha^{-1} \circ f \circ \alpha \circ \beta^{-1} \circ g \circ \beta = \gamma^{-1} \circ h \circ \gamma$? In the class of functions where this is true, an arbitrary orbit structure can be produced composing suitable functions with given orbit structures.

2) Given two arbitrary orbit structures on S, do there exist mappings f, g: $S \longrightarrow S$ such that $f \circ g$ has one orbit structure and $g \circ f$ the other ?

Gy. Targonski

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22. Problem. Let A be an arbitrary set. Suppose that a function $f : A \longrightarrow A$ has iterative roots of all orders. Prove or disprove the following.

For every positive integer n there exists an iterative root of f of order n which has again iterative roots of all orders.

The problem 18. of A. Sklar can be reduced to this problem.

J. Tabor

23. Remark and Problem. Clelia Marchionna could prove several regularity theorems about generalized Cauchy equations, which give a partial solution to a problem, posed by myself on the 16th Symposium on functional equations 1978 in Graz. For simplicity we only consider a special case of her results, which will appear in Boll. Unione Math. Ital.

<u>Theorem</u>

(a) $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$

(b) $F, H : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

(c) for all $x, y \in \mathbb{R}$ the partial derivates F'_x, F'_y are continuous, different from 0, and F_x is surjective for all $x \in \mathbb{R}$ ($F_x(y) := F(x,y)$)

(d) $H \in \mathscr{X}(\mathbb{R}) \subsetneqq \mathscr{C}(\mathbb{R})$

(e) g is measurable or has the Baire-property

$\Longrightarrow$ f continuous.

The class $\mathscr{X}(\mathbb{R})$ contains, for instance, all continuous functions H, such that for all $y \in \mathbb{R}$ the partial derivate H'_y is continuous and different from 0.

Question: Is the conclusion of the above theorem also true for all $H \in \mathscr{C}(\mathbb{R})$?

W. Sander

24. Remark. Results reported in my lecture can be used to prove that all β-recursive inset entropies ($\beta \neq 0, 1$) which are symmetric have the form

$$I_n\begin{pmatrix} E_1, & E_2, \ldots, & E_n \\ p_1, & p_2, \ldots, & p_n \end{pmatrix} = \sum_{j=1}^{n} \gamma(E_j)p_j^{\beta} - \gamma\left(\bigcup_{j=1}^{n} E_j\right)$$

for some arbitrary map $\gamma : B \longrightarrow \mathbb{R}$, where $E_j \in B$, a ring of sets, for all $i = 1, 2, \ldots, n$, and $E_i \cap E_j = \Phi$ for all $i \neq j$. For $\beta = 2$, this proves a conjecture of Aczél and Kannappan.

This conjecture has also been proven in another way by Aczél.

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