MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 40/1981

Topologie (Spezialtagung): Dynamical Systems

13.9. bis 19.9.1981

The meeting took place under direction of F. Takens (Groningen, Holland) and E. Vogt (Berlin). Many of the new results, on which the participants reported, were related to bifurcation theory and its applications. There were also reports of work in the direction of structural stability, ergodicity, interval mappings, and chaos.

Vortragsauszüge

A. CHENCINER:

Homoclinic points in degenerate Hopf bifurcations of diffeomorphisms of $\mathbb{R}^2$.

Let $H_{\mu,a}$ be a "generic" two-parameters unfolding of a local diffeomorphism $H_{o,o}$ of $(\mathbb{R}^2,o)$ whose derivative $DH_{o,o}(o)$ has eigenvalues $e^{\pm 2\pi i\omega_o}$, ω_o not a rational of small denominator, and whose first non linear term in a normal form has real part equal to O. Given a neighborhood O of O in $\mathbb{R}^2$, there exist values (μ,a) near (o,o) such that $H_{\mu,a}$ possesses in O a transversal homoclinic periodic point. The proof is analogous to Zehnder's proof in the area preserving case, the equation of the hamiltonian being replaced by the equation of a pendulum with friction, and the intersection property being replaced by the existence of parameters (Reference: chapter 5 of my "les Houches" course, July 1981).

F.DUMORTIER:

Embedding germs of planar diffeomorphisms in flows.

In the talk is described a method to study germs of C^∞ planar diffeomorphisms from a C^0 and C^∞ point of view. Emphasis is put on those diffeo's having as linear part I or $I + N$ (N nilpotent). By a theorem of Lewis such diffeo's can be formally embedded in the flow of some vector field. If that vector field satisfies some Łojasiewicz condition (always the case in finite codimension) then by means of a finite succession of blowing-ups the vector field can be desingularised. If moreover there is a characteristic orbit (this can be detected in the desingularisation) then the topological type of the V.F. is determined by some finite jet. If we now also suppose that in the desingularisation we do f.i. not have partially hyperbolic singularities then the diffeo C^∞-embeds in the flow at least on the union of hyperbolic and parabolic sectors, a modulus for C^∞ conjugacy exists in the elliptic sectors. If there are no restrictions on the desingularisation(besides the existence of a characteristic orbit) then the diffeo weakly C^0-embeds in the flow and in general the hyperbolic sectors give troubles for genuine C^0-embedding. The results in this talk were obtained in a joint work with R. Roussarie and P. Rodrigues.

D. FRIED:

Applications of Twisted Cohomology to Dynamical Systems.

We show that the classical Lefshetz fixed point formula can be generalized using twisted cohomology to give a more careful count of periodic points of a map. This results in a criterion for a map to have infinitely many periodic points that depends only on homology. When applied to flows with cross-section, one obtains an identity for the homology classes of the periodic orbits. For axiom A-No cycles flows satisfying a finiteness property, one has an identity in the group ring $\mathbb{Z}[H_1(M;\mathbb{Z})]$

$$\prod_{\gamma}(1 \pm [\gamma]^{\pm 1}) = \tau(M)$$

where $[\gamma]$ represents the homology class of the closed orbit γ and the $\pm$ signs depend on the unstable manifold of γ. $\tau(M)$ is a manifold invariant generalizing the Alexander polynomial of a link, and this formula extends one of Franks.
There is also a Twisted Entropy Conjecture (which can be verified in the same cases as Shub's untwisted version) that when the fundamental group acts isometrically on a vector space, the size of a cohomology class grows no faster than the topological entropy.

J. HARRISON:

A C^2 counterexample to the Seifert Conjecture.

We give an outline of a C^2 non-singular flow on the three-sphere S^3 with no closed orbit. The construction is based on a theorem which embeds a Denjoy counterexample in a C^2 diffeomorphism of $\mathbb{R}^2$. The invariant curve is semi-stable and necessarily non-rectifiable.

U. HELMKE:

A cellular decomposition of the Moduli Space for linear control Systems.

Let $\tilde{\Sigma}_{n,m} = \{(A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} | rk(B,AB,\ldots,A^{n-1}B) = n\}$ denote the open subset of all controllable linear systems

$$(*) \qquad \dot{x} = Ax + Bu \; .$$

Linear change of basis in state space $\mathbb{R}^n$ of $(*)$ induces the $GL_n(\mathbb{R})$ - action $(S,A,B) \to (SAS^{-1},SB)$ on $\tilde{\Sigma}_{n,m}$, called the similarity action. The corresponding orbit space $\Sigma_{n,m} := \tilde{\Sigma}_{n,m}/GL_n(\mathbb{R})$ - also called the 'moduli space' for linear systems (*) - is known to be a real analytic manifold of dimension nm. (M. Hazewinkel, R. Kalman, C. Byrnes). We are dealing with the problem of calculating mod 2 singular homology of $\Sigma_{n,m}$. It turns out that $\Sigma_{n,m}$ has the same mod 2 homology as the Grassmann $(n, n+m-1)$.

J. HOFBAUER:

Dynamical Systems on the Simplex: Some Exclusion and Cooperation Results.

The differential equation $\dot{x}_i = x_i\ [G_i(x_1,\ldots,x_n)-\Phi]$, $i=1,\ldots,n$ leaves the simplex $S_n = \{x \in \mathbb{R}^n : x_i \geq 0,\ \Sigma x_i = 1\}$ and all its faces invariant if $\Phi = \Sigma x_i G_i(x_1,\ldots,x_n)$. (For the biological background confer the talk of K. Sigmund).
Such a system is called cooperative, if the boundary of S_n is a repellor.

Some results:

a) If G_i are linear and there is no interior equilibrium, then all orbits converge to the boundary ($\Rightarrow$ no periodic orbits in interior).
b) If G_i is a decreasing function of x_i only, then there is a globally stable equilibrium.

$$\left.\begin{array}{l} \text{c) } G_i = x_{i-1}F_i(x_1,\ldots,x_n),\ F_i > 0 \text{ on } S_n \\ \text{d) } G_i = k_i x_{i-1} + q_i \text{ and an interior equilibrium exists} \end{array}\right\} \Rightarrow \text{cooperation}$$

References:

Competition and cooperation ..., J. Math. Biol. 11 (1981) 155-168
A General Cooperation Theorem ..., Monatsh. Math. 91 (1981) 233-240.

F. KLOK:

Broken extremals of variational problems.

A variational problem on $\mathbb{R}^n$, which comes from a nonconvex Lagrangian on the tangent bundle, can possess broken extremals. These extremals correspond with trajectories of a discontinuous Hamiltonian vectorfield on the cotangent boundle. A local, topological classification was given of these possible discontinuities in the cases $n = 1$, and $n = 2$ with an assumption of homogeneity of the Lagrangian.

K. KRZYŻEWSKI:

On convergence of certain series related to Axiom A diffeomorphisms.

Let f be a C^1-Axiom A diffeomorphism and Ω be a basic set for f such that $g = f|\Omega$ is topologically mixing. It is proved that if μ is a Gibbs measure for g and a Hölder function on Ω, then the series $\sum_{n=1}^{\infty} a_n \left(u \circ g^n - \int_{\Omega} u \, d\mu\right)$ is convergent μ-almost everywhere, where (a_n) is a sequence of real numbers with $\sum_{n=1}^{\infty} a_n^2 < +\infty$ and u is a Hölder function on Ω.

M. RYCHLIK:

Invariant measures and a variational principle for Lozi mappings.

(Ergodic properties of maps: $L(x,y) = (1-a|x|+by,x)$).

As it was shown by M. Misiurewicz, there exist strange attractors for Lozi maps $L(x,y) = (1-a|x|+by,x)$, where a and b are real parameters from a certain domain. We prove the existence of invariant measures with special properties and a variational principle for these attractors. For some parameters the resulting dynamical systems have the Bernoulli property. So, we develop for these strange attractors "statistical mechanics" as it has been done for hyperbolic attractors. They are stochastic attractors, i.e. the images of Lebesgue measures with supports contained in the basins of these attractors converge.

M. MISIUREWICZ:

Periodic points for maps of an interval and a circle.

The theorem of Šarkovskiǐ caracterizes the possible sets $P(f)$ of periods of all periodic points of a continuous map f of an interval into itself. It says that if f has a periodic point of period n and k stands to the right of n in the following ordering $3,5,7,9,\ldots$ $\ldots,2\cdot3,2\cdot5,2\cdot7,2\cdot9,\ldots,2^2\cdot3,2^2\cdot5,2^2\cdot7,2^2\cdot9,\ldots$ $\ldots,2^4,2^3,2^2,2,1,$ then f has also a periodic point of period k.

One of generalizations of this theorem allows us to deduce the existence of periodic points of certain periods from the behaviour of a piece of non-periodic trajectory. Namely, if $f^n x \le x < fx$ (or $f^n x \ge x > fx$) and if there is no a such that all $f^j x$ with j even lie on one side of a and all $f^j x$ with j odd - on the other side $(0 \le j \le n)$, then f has a periodic point of an odd$(\neq 1)$ period.

Another generalization is to the continuous maps of a circle into itself. Then the results depend on the degree of f. The most complicated case is $\deg f = +1$. Then we get the following result: $P(f) = M \cup S_1 \cup S_2$ where:

$M = \{n : \exists k \text{ with } \frac{k}{n} \in \operatorname{int} L\}$ for some closed interval L, possibly degenerated to a point (rotation interval for f),

$$S_1 = \begin{cases} \emptyset & \text{if } \inf L \text{ is irrational} \\ n\cdot\tilde{S}_1 & \text{if } \inf L = \frac{k}{n},\ k \text{ and } n \text{ coprime,} \end{cases}$$

where $\tilde{S}_1$ is some set of Šarkovskiǐ type; and similarily for S_2 with $\sup L$ instead of $\inf L$. For every set

A of the above form, we can find a continuous map $f : S^1 \to S^1$ with $P(f) = A$.

Z. NITECKI:

Global structural stability on open surfaces (Joint work with M. Krych and J. Kotus).

We formulate sufficient conditions for the global structural stability of flows on open (i.e. non-compact) surfaces. For flows on the plane these conditions are also necessary. If one looks in the case of open surfaces for a homeomorphism relating the initial flow with a perturbation of it one not only needs the well-known hypotheses of Peixoto, but one also has to consider the behaviour of the flow at infinity. The most important additional hypothesis is the condition that the closure of all stable separatrices intersects the closure of the unstable separatrices in saddle points only.

H. PETERS:

Complex dynamical behaviour of retarded differential equations

It was conjectured by R.D. Nussbaum that a differential-delay equation of the type

$$(0) \qquad \dot{x}(t) = -\alpha \cdot f(x(t-1))$$

behaves chaotically, if f is a nonlinearity with a "hump", e.g. $f(x) = x(1+x^8)^{-1}$. We prove that this conjecture is true for the model-equation

$$(1) \qquad \dot{x}(t) = -f_\alpha(x(t-1)) \quad \text{with} \quad f_\alpha(x) = \operatorname{sign}(x)\begin{cases} \alpha \ ; & |x| < 1 \\ 1 \ ; & |x| \geq 1 \end{cases}$$

if $\alpha > \alpha^* \approx 2.884$. More precisely, the piecewise constancy of the nonlinearity f_α forces the solutions of equation (1)

into a finite dimensional phase space, such that the periodicity problem for equation (1) can be described by a one-dimensional "shift"-map $S_\alpha : [0,1] \to [0,1]$ (fixed points of S_α^j, $j \in N$, correspond to periodic solutions of equation (1)). For $\alpha > \alpha^*$ the one-dimensional map S_α has a point of period 3 which implies chaos in the sense of Li and Yorke. (C.R. Acad. Sci. Paris 290, 1980, 1119-1122).
If the nonlinearity f in equation (0) is chosen to be $f(x) = \mathrm{sign}(\sin \pi x)$ we prove a transition (depending on α) from periodic solutions to periodic solutions of the second kind (i.e. $x(t+p) = x(t)+C$) and "multiple" chaos caused by the "humps" of the nonlinearity and the additional equilibria of the equation.
(Springer Lect. Notes 1981, Proc. of a conference (Bremen 1980)).

F. PRZYTYCKI:

Linked twist mappings: Ergodicity.

Consider two strips /annuli/ in the torus T^2, a horizontal strip $P = \{(x,y): a_o \le y \le a_1\}$ and a vertical $Q = \{(x,y): b_o \le x \le b_1\}$. Linked twist mapping /l.t.m./ is the composition of a twist $F(x,y) = (x+f(y),y)$ in P /identity outside P/ with a twist $G(x,y) = (x,y+g(x))$ in Q / $f : \langle a_o,a_1\rangle \to R$ is a C^2-function, $\frac{df}{dy} \ne 0$, $f(a_o) = 0$, $f(a_1)$ = an integer; similar conditions about g /.
We prove that if the twists are strong enough /i.e. $|\frac{df}{dy}(y_o)|$, $|\frac{dg}{dx}(x_o)|$ are sufficiently large for every x_o, y_o/, then the l.t.m. is ergodic, hence by almost hyperbolicity/ result of Burton, Easton and Wojtkowski/, a Bernoulli system.
This result generalizes to compositions of families of "horizontal" twists with "vertical" twists in any surface assumed horizontal and vertical annuli intersect transversally. One can also compose horizontal and vertical twists alternately more than two times.

Every hyperbolic toral automorphism and some Thurston pseudo-Anosov homeomorphisms can be decomposed into l.t.m. . So we can easily smooth them. This gives by the way an easy construction of a Bernoulli diffeomorphism on the 2-disc/Katok's diffeomorphism/.

O.E. RÖSSLER:

Higher-order Chaos in a Constrained Differential Equation with an Explicit Cross-section.

Constrained differential equations were in the context of non-Hamiltonian chaotic behavior considered by Levinson (1949), Takens (1976) and Rössler (1976). An example with an explicit cross-section was indicated by Mira (1978). The following system is a higher-dimensional analogue:

$$\begin{aligned}
\dot{x} &= w(-y-c) + (1-w)(z+c) \\
\dot{y} &= w(x-1+a) + (1-w)\, y(z+c)/(x+b-1) \\
\dot{z} &= w\, z(-y-c)/(x-b) + (1-w)(a-x) \\
0 = \varepsilon\dot{w} &= w(1-w)(w-1+x) - \delta(w-0.5), \quad \delta \longrightarrow 0 .
\end{aligned} \tag{1}$$

Simple geometry shows there is a cross-section over a certain range obeying

$$\begin{aligned}
y_{n+1} &= -c + \sqrt{2a-1+(2y_n+c)^2} \\
z_{n+1} &= -2c + 2\sqrt{2a-1+(z_n+c)^2} ,
\end{aligned} \tag{2}$$

a pair of uncoupled, chaos-generating, single-humped smooth maps. Appropriate parameter values are $a = 0.52$, $b = 2$, and $c = 1.3$. Equation (2) is, as a non-invertible two-dimensional limiting case, embedded in a three-variable "folded towel" diffeomorphism, for which a close to the limit arbitrarily good approximation can be indicated. The potential importance of such "higher-orderchaos" for a qualitative understanding of turbulence was discussed.

References: Levinson, N., 1949, Ann. of Math. 50, 127-153; Mira, C., 1978, Proc. Equadiff 78, 25-37; Takens, F., 1976, LNM 535, 237-253; Rössler, O., Z. Naturforsch. 31a, 259.

K. SIGMUND:

The replicator equation in biomathematical models.

The differential equation $\dot{x}_i = x_i((Ax)_i - x. Ax)$ $i=1,\dots,n$ on the (invariant) unit simplex S_n (A is an $n\times n$ matrix) plays a prominent role in (at least) four seemingly unrelated parts of evolution theory.

1. natural selection in population genetics (the equation of FISHER-HALDANE-WRIGHT)
2. Theoretical ecology (the equation of GAUSE-LOTKA-VOLTERRA)
3. prebiotic evolution (the hypercycle model of EIGEN and SCHUSTER)
4. sociobiology (the notion of evolutionary stability, introduced by MAYNARD SMITH, and the corresponding game dynamics).

G. VEGTER:

Bifurcations of gradientvectorfields.

In the framework of Thom's theory of catastrophes one usually passes from bifurcations of gradient dynamical systems to the unfoldings of their potential functions.
This is justified if the coranks of the singularities of the function are at most 1.
However, in 1973 John Guckenheimer gave an example which shows that this procedure is not always justified if the corank of the singularity is greater than 1. He shows that a universal unfolding of the germ $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$ contains 3 parameters, while a universal unfolding of the

corresponding gradient vectorfield $x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$ (within the class of gradient vectorfields) is of dimension 4.
In the lecture it is shown that if one considers the Riemannian metric $g_z = dx \otimes dx + z(dx \otimes dy + dy \otimes dx) + dy \otimes dy$, then for $z \neq o$ the universal unfolding of the germ of f and that of the gradient vectorfield of f with respect to the metric g_z contain the same number of parameters. Moreover, for $z > o$ the catastrophe set of the gradient family is larger than that of the family of potential functions. This is due to the occurrence of saddle connections for some values of the parameters.

Yieh-Hei WAN:

Bifurcations in three dimensional elasticity.

Consider a hyperelastic material, occupying a region Ω in R^3 as its reference configuration. Suppose the reference configuration stress-free.
The traction problem: Describe the equilibrium solutions of this elastic material for a small traction field τ along the boundary of Ω and a small body force B on Ω.
A variational approach together with singularity theory turns out to be very successful. Indeed, we have
Th. Our traction problem for load λl_o, $l_o = (B,\tau)$, has at least 4 solutions, and one of them is stable.
In order to describe the solutions in more detail, the load l_o needs to be classified into type $0,1,\ldots,4$.
Th. Generically, there are (a) exactly 4 solutions for type 0 loads and at most (b) 6 solutions for type 1 loads (e) 14 solutions for type 2 loads (d) 8 solutions for type 3 loads (e) 40 solutions for type 4 loads.
For type 1 loads, a more precise bifurcation result can be obtained.

F. W. WILSON:

Computer Verification of Periodic Solutions to 3-dimensional Flows.

In numerical experiments with differential equations, one may be led to suspect that one has discovered a periodic solution. If the solution is not stable, it may be impossible to follow it for very many periods, and so doubts arise.
Since the periodic solution can be identified with a fixed point of its Poincaré Mapping, and since the (integer-valued) Poincaré-Hopf Index of this suspected fixed point can be viewed as the obtruction to its homotopy removability, a numerical computation which indicates that the Poincaré-Hopf Index is non-zero provides strong evidence that the periodic solution does exist.
An algorithm has been developed for performing this computation in dimension three. There are several numerical sensitivities which must be considered if the reliability of the computation is to be understood. Applications to several examples were discussed.

M. WOJTKOWSKI

A model problem with the coexistence of stochastic and integrable behaviour.

Let T^2 be a two dimensional torus with coordinates (φ_1,φ_2) mod 1. Let us consider the twist mapping $F_1 : T^2 \to T^2$, $F_1(\varphi_1,\varphi_2) = (\varphi_1+\varphi_2,\varphi_2)$ and the mapping $F_2 : T^2 \to T^2, F_2(\varphi_1,\varphi_2) = (\varphi_1,\varphi_2+\varepsilon f(\varphi_1))$ where $\varepsilon \in \mathbb{R}$ and f is the real function $f(t) = |t| - 1/4$, $|t| \leq 1/2$. The object of our study are ergodic properties of the family of transformations $F_{\varepsilon,N} = F_2 \circ F_1^N$. This family can be treated as a perturbation of the twist mapping F_1^N. We prove that for $A = \varepsilon N \geq 4$ $F_{\varepsilon,N}$ has strong mixing properties in the whole torus. For $A < 4$ there appear regions where $F_{\varepsilon,N}$ is a linear rotation. We prove that for $A = 2(\cos \pi/n + 1)$, $n = 2,3,\ldots$ $F_{\varepsilon,N}$ has strong mixing properties outside these regions. The case $A = 1$ is also treated. Thus we obtain examples with coexistence of stochastic and integrable behaviour.

Berichterstatter: E. Vogt

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Tagungsteilnehmer

Prof. Dr. H. Broer
Rijksuniversiteit Te Groningen
Mathematisch Instituut
Postbus 800

Groningen

Prof. Dr. J. Harrison
Mathematics Institute
24-29 St. Giles

Oxford OX; 3LB

England

Prof. Dr. A. Chenciner (Université Paris VII)

Adress:
11, quai Bourbon
75004 Paris
France

Prof. Dr. S. Heimsath
Universität Bremen
Studienbereich 4 Mathematik
Kufsteinerstr.

2800 Bremen 33

Prof. Dr. D. Chillingworth
Faculty of Mathematical Studies
University of Southampton

Southampton S 09 5NH
England

Prof. Dr. U. Helmke
Universität Bremen
Studienbereich 4 Mathematik
Kufsteinerstr.

2800 Bremen 33

Prof. Dr. F. Dumortier
Institut de Mathématique
Université de Brussels

1050 Brussels

Belgium

Prof. Dr. J. Hofbauer
Mathematisches Institut
der Universität Wien
Strudelhofgasse

A-1090 Wien

Austria

Prof. Dr. D. Erle
Institut für Mathematik
Postfach 500 500

4600 Dortmund 50

Prof. Dr. A. Hummel
Rijksuniversiteit Te Groningen
Mathematisch Instituut
Postbus 800

Groningen

Prof. Dr. D. Fried
Department of Mathematics
University of California
at Santa Cruz

Santa Cruz, California
U.S.A.

Prof. Dr. F. Klok
Rijksuniversiteit Te Groningen
Mathematisch Instituut
Postbus 800

Groningen

Prof. Dr. K. Krzyzewski
Institut of Mathematics
University of Warsaw
PKIN, IX, p.

00-901 Warszawa

Polen

Prof. Dr. W. Manthey
Universität Bremen
Studienbereich 4 Mathematik
Kufsteinerstr.

2800 Bremen 33

Prof. Dr. M. Misiurewicz
Institut of Mathematics
University of Warsaw
PKIN, IX, p

00-901 Warszawa
Polen

Prof. Dr. Z. Nitecki
Department of Mathematics
Tufts University

Medford, Mass. 02155

U.S.A.

Prof. Dr. H.-O. Peitgen
Universität Bremen
Studienbereich 4 Mathematik
Kufsteinerstr.

2800 Bremen 33

Prof. Dr. H. Peters
Universität Bremen
Studienbereich 4 Mathematik
Kufsteinerstr.

2800 Bremen 33

Prof. Dr. F. Przytycki
Mathematics Institute
Polish Academy of Sciences
ul. Sniadeckich 8

00-950 Warszawa
Polen

Prof. Dr. O.E. Rössler
Institut für Physikalische und
Theoretische Chemie
Universität Tübingen

7400 Tübingen 1

Prof. Dr. M. Rychlik
Institut of Mathematics
University of Warsaw
PKIN, IX, p.

00-901 Warszawa

Polen

Prof. Dr. I. Sanzot
Université de Namur
Département de Mathématique
Rempart de la vierge, 8

B-5000 Namur
Belgium

Prof. Dr. H.W. Siegbert
Universität Bremen
Studienbereich 4 Mathematik
Kufsteinerstr.

2800 Bremen 33

Prof. Dr. K. Sigmund
Mathematisches Institut
der Universität Wien
Strudelhofgasse

A-1090 Wien

Austria

Prof. Dr. F. Takens
Rijksuniversiteit Te Groningen
Mathematisch Instituut
Postbus 800

Groningen

Prof. Dr. G. Vegter
Rijksuniversiteit Te Groningen
Mathematisch Instituut
Postbus 800

Groningen

Prof. Dr. E. Vogt
Freie Universität Berlin
Institut für Mathematik I
Arnimallee 2-6
1000 Berlin 33

Prof. Dr. F.W. Wilson
Department of Mathematics
University of Colorado
Boulder, Colorado 80302
U.S.A.

Prof. Dr. Y.H. Wan
Department of Mathematics
SUNY at Buffalo
Buffalo, New York 14214
U.S.A.

Prof. Dr. M. Wojtkowski
Institut of Mathematics
University of Warsaw
PKIN, IX, p.
00-901 Warszawa
Polen