MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 19/1984

Exponentialsummen

(Kloosterman'sche Summen)

22.4. bis 28.4.1984

Die Tagung fand unter der Leitung von Herrn Professor G.Kolesnik (Austin) und Herrn Professor Dr. H.-E.Richert (Ulm) statt. Im Mittelpunkt des Interesses standen Fragen , die Abschätzungen von Exponentialsummen betreffen. Es wurden sowohl Methoden zur Abschätzung solcher Exponentialsummen behandelt als auch deren Anwendungen bei zahlentheoretischen Problemen . In den Seminar-Sitzungen am Nachmittag wurde die von G.Kolesnik begründete Methode der Abschätzung von mehrdimensionalen Exponentialsummen eingehend behandelt . Die niedrige Teilnehmerzahl machte es möglich , daß erfreulich viel Zeit für anregende und ausführliche Diskussionen blieb .

Gefördert durch DFG Deutsche Forschungsgemeinschaft

Vortragsauszüge

R.C.BAKER :

Trigonometric sums over primes

In joint work with G.Kolesnik (to appear) we study

$$D(N) = \sup_{0\leqq\gamma\leqq 1} \left| \sum_{\substack{p\leqq N \\ \{p^\alpha\}\leqq\gamma}} 1 - \pi(N)\gamma \right|$$

i.e. the discrepancy (mod 1) of the sequence p^α (p prime) . Here $\alpha>1$ is not an integer . We give $D(N) \ll N^{157/168}$ for the case $\alpha=1.5$ but in this talk a result of interest for large α is given . Vinogradov 'essentially' showed

$$D(N) < N^{1-\rho} \quad \text{for} \quad N > C(\alpha)$$

where

$$\rho=(34\ 000\ 000\cdot\alpha^2)^{-1} .$$

We improve the constant 34 000 000 to 15 000 .

The techniques used are Vinogradov's mean value theorem (actually there is some saving in taking a sharper form of Arhipov and Karacuba) and the multidimensional large sieve applied to

$$\sum_{\underline{n}} \left| \sum_{x=1}^{B} e(A_1 n_1 x + \ldots + A_k n_k x^k) \right|^{2r}$$

(where r is about $5\alpha^2$) with $\underline{n}$ running over the integer points in the box $\prod_{j=1}^{k} [-jB^j , jB^j]$ and $A_1 , \ldots , A_k$ are the first k derivatives of a function of the form Rx^α , R between 1 and Z . Here k is about 3.8α ; a simpler technique can be used for the less interesting case $1<\alpha<10^{10}$.

H.DABOUSSI:

On a multiplicative exponential sum

Let $y \geqq 2$, v_y and u_y multiplicative functions defined by

$$u_y(p^r) = \begin{cases} 0, & p \leq y \\ 1, & p > y \end{cases} , \quad v_y(p^r) = \begin{cases} 1, & p \leq y \\ 0, & p > y \end{cases} .$$

Then $u_y * v_y = 1$ and $f = fv_y * fu_y$, this and the fact that

$$\frac{1}{x} \sum_{n \leq x} u_y(n) e^{2\pi i n \alpha} \to 0$$

give that for all multiplicative functions f with $|f(n)| \leq 1$ then

$$\frac{1}{x} \sum_{n \leq x} f(n) \, e^{2\pi i n \alpha} \to 0$$

for every irrational α.

J.-M. DESHOUILLERS:

Mean value of divisor functions

Application of the majorization of Kloosterman sums on average to expressions of the type $\sum_{n \leq x} w(\frac{n}{x}) \, f(n) \, g(n+1)$ is discussed, among others $\sum_{n \leq x} d(n) \, d_3(n+1)$.

E.FOUVRY:

On the Titchmarsh divisor problem

Let $T(x,a) = \sum_{a<p\leq x} d(p-a)$, where a is an integer , p a prime number and $d(n)$ the divisor function of the integer n . We prove an asymptotic formula for $T(x,a)$ with an error term $O_{a,A}(x(\log x)^{-A})$. This result is based on dispersion method and Kloosterman sums on average .

S.W.GRAHAM :

Two dimensional exponential sums

Grigori Kolesnik and I conjecture that if f is a function of two variables which satisfies

(1) $f(x,y) \sim Ax^{-\alpha}y^{-\beta}$ ($\alpha>0$, $\beta>0$)

(2) for fixed y , $f(x,y)$ is analytic in the domain $\{ z : |z-x| < cX$ for some x , $X\leq x\leq 2X \}$

(3) for fixed x , $f(x,y)$ is analytic in the domain $\{ z : |z-y| < cY$ for some y , $Y\leq y\leq 2Y \}$

then the exponential sum estimate

$$\sum_{(m,n)\in\mathfrak{D}} e(f(m,n)) << (\tfrac{F}{X})^k X^l (\tfrac{F}{Y})^k Y^l$$

holds for any one-dimensional exponent pair (k,l) .(Here , F denotes $AX^{-\alpha}Y^{-\beta}$ and $\mathfrak{D}$ is a suitable domain inside the rectangle $[X,2X] \times [Y,2Y]$.)

We can prove a slightly modified version of this conjecture for $(k,l) = A^qB(0,1)$.

Let

$$\Delta(x) = \sum_{n\leq x} d(n) - x\log x - (2\gamma-1)x \quad .$$

The truth of our conjecture for $(k,l) = A^3B(0,1)$ implies that

$$\Delta(x) << x^{12/37} \log^A x \; .$$

With $A^3BA^2B(0,1)$, one gets the exponent $35/108$. If the conjecture is true for any arbitary (k,l), then the optimal exponent pair is

$$A^3BA^2BA^1BA^2BA^2B\ldots\;;$$

the resulting exponent is

$$0.3239247504\ldots \quad .$$

F.GRUPP :

On a problem of Erdös and Szemerêdi

In joint work with G.Bantle (to appear) we improve upon the following theorem of P.Erdös :

Let $\mathfrak{B} = \{b_i : 1<b_1<b_2<b_3 \ \ldots\}$ be an infinite sequence of integers, such that $(b_i,b_j)=1$ for all $i\neq j$ and $\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty$.

Then there exists a constant $0<\alpha<1$ with the following property : for sufficiently large x the interval $(x-x^\alpha,x]$ contains integers, that are divisible by no element of $\mathfrak{B}$.

Szemerêdi showed that Erdös' theorem holds true for $\frac{1}{2}+\varepsilon$ and in our paper we prove, that $\frac{1}{2}+\varepsilon$ may be replaced by $\frac{9}{20}+\varepsilon$. The proof uses nontrivial estimates of exponential sums.

D.R.HEATH-BROWN :

Complete multiple exponential sums to modulus p^2

Let $F(\underline{x})=F(x_1,\ldots,x_n)\in\mathbb{Z}[\underline{x}]$ be a form of degree d and $E(\underline{x})\in\mathbb{Z}[\underline{x}]$ a polynomial of degree $<d$. Let $p\nmid d$ be a prime and suppose that $F(\underline{x})=0$ defines an absolute irreducible non-singular projective variety over $\mathbb{F}_p$. It was shown by Deligne - as a consequence of his Riemann Hypothesis - that

$$\Big|\sum_{\underline{x} \bmod p} e((F(\underline{x})+E(\underline{x}))/p)\Big| \leq (d-1)^n p^{n/2} .$$

For applications one would like sums to general moduli. It is shown, under the same hypotheses, that

$$\Big|\sum_{\underline{x} \bmod p^2} e((F(\underline{x})+E(\underline{x}))/p^2)\Big| \leq (d-1)^n p^n . \qquad (*)$$

The analogous result $(\bmod\ p^l)$, $l\geq 3$, is false. The proof of (*) uses elementary algebraic geometry.

A.IVIĆ :

Exponential sums and the Riemann zeta-function

Exponential sums

$$\sum_{a<n\leq b} e(f(n)) \quad \text{and} \quad \sum_{a<n\leq b} d(n)e(f(n))$$

play a prominent rôle in many parts of zeta-function theory. Latest results in several problems which depend on the estimation of the above sums will be discussed. These include

a) The estimation of $\gamma_{n+1}-\gamma_n$, where γ_n is the ordinate of the n-th zero of $\zeta(s)$ on the critical line in the upper half-plane;

b) The lower bound for the number of zeros of $\zeta(\frac{1}{2}+it)$ in "short" interval $[T-G,T+G]$;

c) The deduction of an upper bound for $\int_{T-G}^{T+G} |\zeta(\frac{1}{2}+it)|^2 dt$

(G=o(T)) from the approximative functional equation for $\zeta^2(s)$;

d) The estimation of the double exponential sum

$$S(N) = \sum_{r,s\leq R} \Big| \sum_{N<n\leq 2N} n^{-1/2+it_r-it_s} \Big|^2$$

$$(\ |t_r|\leq T, \quad |t_r-t_s| \geq 1 \quad \text{for} \quad r\neq s \ )$$

which has important applications in zero-density estimates.

M.JUTILA :

On exponential sums involving the divisor function

Generalizations are given for the following results :

a) Voronoi's identity for the sum $\sum_{n\leq x} d(n)$

b) its truncated version

c) Voronoi's summation formula

d) the approximate functional equation of Wilton for the exponential sum $\sum_{n\leq x} d(n)e(n\eta)$.

Generalization means here , that d(n) is replaced by d(n)e(nh/k), where (h,k)=1 in the respective sums , and analogues of a)-d) are found for these new sums.

G.KOLESNIK :

On the method of exponent pairs

The exponential sums of type

$$S= \sum_{(x,y)\in\mathfrak{D}} e(f(x,y)) ,$$

where $f(x,y) \sim Ax^{-\alpha}y^{-\beta}$ are estimated by using the operations A (H.Weyl-van der Corput Inequality) and operation B (Poisson-summation formula - method of stationary phase) . Using the method , we proved the following result :

Theorem 1: Let $1 \leq k_1, k_2 \leq 3$ be integers and let α, β be real numbers such that α, β are not integers, $\alpha+\beta+k_1 \neq 0$, $\alpha+\beta+k_1+1 \neq 0$, $\alpha+\beta+k_1+1+1/k_2 \neq 0$, $(\alpha+k_1+1)^2+\beta(\alpha+k_1) \neq 0$ and let $\mathcal{D} \subset \{(x,y) : X \leq x \leq 2X, Y \leq y \leq 2Y\}$ be a domain such that its boundary consists of a finite number of algebraic curves. Then we have

$$S \ll N^{1+\varepsilon}/\sqrt{Q}$$

where

$$Q \leq \min\Big((N^{2k_1K_1+4K_1-k_1k_2-2k_1-k_2-4} F^{2k_2+4-4K_1})^{a_1} ;$$

$$(N^{3k_1K_2+6k_2-2k_1-2} F^{4-6K_2})^{b_1} ;$$

some other expressions, usually large in applications$\Big)$

with

$$K_1 = 2^{k_1}, \quad K_2 = 2^{k_2},$$

$$a_1 = (4K_1K_2-2K_2-k_2K_1-3K_1+k_2+2)^{-1}, \quad b_1 = (6K_1K_2-2K_1-3K_2+2)^{-1}.$$

Using the above theorem, we can show that

$$|\zeta(\tfrac{1}{2}+it)| \ll |t|^{139/858+\varepsilon}, \qquad \Delta(x) \ll x^{139/429+\varepsilon},$$

and a similar estimate is obtained for the error term in the circle problem. The case when α, β are negative integers or $\alpha+\beta+k_1+1=0$ can also be estimated and the estimate is better than in the Theorem 1, since the Hessian is small and the sum S can be reduced to a simpler sum.

C.J.MOZZOCHI :

Siegel zeros and the Goldbach problem

It is generally known that under the generalized Riemann hypothesis one could establish the Goldbach conjecture by the circle method provided one could obtain a certain estimate for the integral of the representation function over the minor arcs. Here it is first shown that the generalized Riemann hypothesis in the above statement can be weakend to the assumption that Siegel zeros do not exist. The case when Siegel zeros do exist is then considered.

J.PINTZ:

On the distribution of squarefree integers

In the lecture several results - proved in collaboration with R.C.Baker and S.W.Graham - are reported concerning the error term

$$R_k(x) = \sum_{n\leq x} \sum_{d^k/n} \mu(d) - \frac{x}{\zeta(k)} .$$

Let

$$\Delta_k = \inf\{\theta : R_k(x)=O(x^\theta)\} .$$

Supposing the Riemann hypothesis (RH), the trivial estimate $\Delta_k\leq 1/k$ was improved by Axer in 1911 to $\Delta_k\leq 1/(k+1/2)$. Montgomery and Vaughan improved this in 1979 to $\Delta_k\leq 1/(k+1)$, $\Delta_2\leq 9/28$. S.W.Graham showed in 1981 $\Delta_2\leq 8/25$.

Theorem 1: $\Delta_2\leq 7/22$.

Various results may be proved for larger k, using the method of exponential pairs and Vinogradov's method. What concerns lower estimates of $|R_k(x)|$ the results

$$R_k(x) = \Omega\,(x^{1/(2k)}) \quad \text{and} \quad \int_1^Y \frac{|R_k(x)|}{x}\,dx > c_k Y^{0.18/k}$$

of Stark-Evelyn-Vaidia and Katai, respectively, can be improved to

Theorem 2: $$\frac{1}{Y}\int_1^Y |R_k(x)|\,dx > c_k Y^{1/(2k)} .$$

A.SARKÖZY:

Hybrid problems in additive number theory

Let $a_1<a_2<\ldots$, $b_1<b_2<\ldots$ "dense" sequenses of integers, and let $P(n)$ denote the greatest prime divisor of n. The solvability of the equations

$$a_x-a_y=z^2 \ (>0) \quad , \quad a_x-a_y=p-1$$

is studied. Furthermore it is shown that there exist a_x, b_y with any of the following properties :

(i) $P(a_x+b_y)$ is "small" ;

(ii) $P(a_x+b_y)$ is "large" ;

(iii) there exists a large prime p with $p^2/(a_x+b_y)$.

W.SCHEMPP:

On exponential sums occurring in cardinal spline interpolation

Based on an inversion theorem for the Mellin transform , a complex contour integral representation (with non-compact path of integration) of cardinal logarithmic spline integration is established . An application of the Cauchy residue theorem then yields a representation of these spline functions involving a bi-infinite exponential sum . This representation combined with an appropriate uniform distribution argument proves the Newman-Schoenberg divergence phenomenon. In particular , it shows that the asymptotic behaviour of the cardinal logarithmic spline interpolants as their degrees tend to infinity is totally different from the asymptotic behaviour of the cardinal exponential spline interpolants.

P.SHIU:

The distribution of square-full integers in a short interval

An application of Roth's method reduces the problem of the title to that of the estimation of the two dimensional exponential sum

$$U(x,t) = \sum_{a^2b^3 \leq xt^{-6}} \psi\left(\left(\frac{x}{a^2b^3}\right)^{1/6}\right)$$

which is given the estimate

$$U(x,t) << x^{1/2}t^{-7/2}\log^2 x$$

uniformly in $\log x < t \leq x^{\beta}$, where $\beta = \frac{421}{3841}$.

The paper has already appeared in Glasgow Math.J. 25(1984)127-134.

J.SZMIDT :

Application of the Burgess inequality to automorphic function theory

It is discussed the connection between automorphic function theory and exponential sums . In the works of Kuznetsov , Deshouillers and Iwaniec automorphic functions (the sum formulae of Bruggeman-Kuznetsov type) were used to obtain the estimations of sums of Kloosterman sums like the following one :

$$\sum_{c\leqq x} \frac{S(m,n;c)}{c} \ll_{m,n,\varepsilon} x^{1/6+\varepsilon}$$

Deshouillers and Iwaniec applied these estimations to number theoretical problems .

It appears that some problems arising inside automorphic function theory are connected with estimations of Kloosterman sums and character sums . Let us list some of them :

(1) The Selberg eigenvalue conjecture: the first positive eigenvalue λ_1 of Laplace operator connected with congruence subgroups is $\geqq 1/4$. A.Selberg using A.Weil estimation for Kloosterman sums was able to prove that $\lambda_1 \geqq 3/16$.

(2) Using the Burgess inequality for character sums it is possible to obtain the following density theorem for exceptional eigenvalues of Hecke congruence groups $\Gamma_0(q)$.

Theorem (Iwaniec,Szmidt): For $\Gamma_0(q)$ with q being a prime number we have

$$\sum_{\substack{0<\lambda_j<1/4 \\ \lambda_j=1/4-t_j^2,\, t_j>0}} q^{\delta t_j} \ll_\varepsilon q^{1+\varepsilon} \quad ;$$

ε is any positive constant and $\delta=24/11$.

(3) Prime geodesic theorem for primitive classes of hyperbolic modular group. Using the Kuznetsov sum formula and and Burgess inequality H.Iwaniec was able to prove that

$$\pi(x) = \mathrm{li}\, x + O(x^{\Theta}) \quad , \text{ with } \quad \Theta=35/48 \quad .$$

R.C.VAUGHAN:

The estimation of complete exponential sums

Let $f \in \mathbb{Z}[x]$, $n = \deg f \geq 2$. Let e denote the maximal order of the complex roots of f' . For a given $F \in \mathbb{Z}[x]$ define $D(F)$ as follows . Let K denote the algebraic number field generated by the roots of F and let ord_p denote any extension to K of the additive p-adic valuation , normalized so that $\mathrm{ord}_p p = 1$. Let ξ denote a root of F and let e_ξ denote the order of F . Let D(F) be the intersection of the fractional ideals generated by the $F^{(e_\xi)}(\xi)/e_\xi!$ and let $\delta(F) = \mathrm{ord}_p D(F)$.

In joint work with J.H.Loxton the following theorem is established:

Theorem. Let

$$S(f;q) = \sum_{x \bmod q} e(f(x)/q) \quad .$$

Then

$$|S(f;q)| \leq C_n^{1/(e+1)} (n-1)^{\omega(q)} q^{e/(e+1)} (D(f'),q)^{1/(e+1)}$$

where

$$(D(f'),q) = \prod_p p^{\min(\mathrm{ord}_p D(f'), \mathrm{ord}_p q)} \quad .$$

This establishes a conjecture of Loxton and Smith . It is also shown to be best possible in many different and diverse situations.

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