MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 1/1985

Metaplektische Gruppen

1.1. bis 5.1.1985

Die Tagung fand unter der Leitung von Herrn S.J. Patterson (Göttingen) und Herrn I.I. Piatetski-Shapiro statt. Diese Tagung wurde geplant, um zum ersten Mal denjenigen Mathematikern, die mit metaplektischen Gruppen arbeiten, die Möglichkeit eines Ideenaustausches anzubieten. Einige Teilnehmer wurden vorher gebeten, Referate über zentrale Gebiete der Theorie der metaplektischen Gruppen zu halten; Auswahlkriterium war, daß der Vortragende nicht über seine eigene Arbeit sprechen sollte. Diese Vorträge waren zweistündig, die übrigen einstündig. Die Themen dieser eingeladenen Vorträge waren "Waldspurger's Work" (Henniart), "The Howe theory of dual reductive pairs" (Kudla), "L-functions associated with classical groups" (Vignéras) und "General metaplectic groups" (Waldspurger); alles Themen, die sehr aktuell sind. Die anderen Vorträge waren naheliegenden Bereichen gewidmet.

Die Qualität aller Vorträge war sehr hoch und das Zusammentreffen, das in einer wunderschönen Schneelandschaft zu Neujahr stattfand, war fruchtbar und hat seine Aufgabe eines gegenseitigen Austausches

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völlig erfüllt. Die Teilnehmer drückten zum Schluß ihre Zufriedenheit mit der Tagung und der sehr angenehmen Atmosphäre des Forschungsinstituts aus.

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J.L. Waldspurger, Paris

F. Wielonski, Bonn

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Vortragsauszüge

G. Henniart: Waldspurger's work on the Shimura and Shintani correspondences

The purpose of this work is to study the automorphic representations of the metaplectic degree 2 covering $\tilde{S}$ of SL(2), over the ring of adeles A of a number field k. Let $\tilde{A}_{oo}$ be the space of genuine cuspidal automorphic functions on $\tilde{S}(A)$ which are orthogonal to theta series coming from a one dimensional quadratic form. If two irreducible components of $\tilde{A}_{oo}$ are equivalent almost everywhere and have the same central character, then they are equal. In particular the the weak multiplicity one theorem holds in $\tilde{A}_{oo}$. However the strong multiplicity are theorem generally fails, but the extent to which it fails can be described exactly.

All this is proved using the (Weil) oscillator representation relative to the dual reductive pairs $(\tilde{S}, G')$, where G' is the multiplicative group of a quaternion algebra over k, divided by its center (a special orthogonal group $SO(2,1)$. For example let $G' = PGL_2$ and A_o the space of cuspidal automorphic functions on $G'(A)$. Let ψ be a non-trivial character of k A. An explicit correspondence, involving θ-series can be defined between irreducible components π of A_o such that $L(\pi, 1/2) \neq 0$ and irreducible components τ of $\tilde{A}_{oo}$ having a non zero ψ-th Fourier coefficient. Each subspace τ of $\tilde{A}_{oo}$ appear for some choice of π and ψ. The correspondences for different ψ's can be compared; this yields that for many components π of A_o $L(\chi\,\pi, 1/2) \neq 0$ for some quadratic ideale class character χ of F. If G' is not PGL_2, one gets other correspondences $\pi' \longleftrightarrow \tau$. For a given ψ, the correspondences $\pi \longleftrightarrow \tau$ (for PGL_2) and $\pi' \longleftrightarrow \tau$ (for $G' \neq PGL_2$) are not compatible with the Jacquet-Langlands correspondence. A precise (local and global) study of this phenomenon yields a description of how the strong multiplicity one theorem fails.

S. Kudla: <u>Howe's duality theorem for the polynomial Fock model</u>

Let $(G,G') \subset Sp(W)$ be a dual reductive pair in the symplectic group of $W,<,>$. One considers the representation ω of the Lie algebras $(\mathfrak{g},K)$, $(\mathfrak{g}',K')$ in the polynomial Fock space $S \cong \mathbb{C}[z_1,\dots,z_n]$. (in case W $\mathbb{R}^{2n}$, $<,> \sim \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, so $Sp(W) \cong Sp(n,\mathbb{R})$. Let $R(\mathfrak{g},K,\omega)$ be the set of equivalence classes of irreducible admissible $(\mathfrak{g},K)$ modules π which occur as quotients of S, ie. for which $Hom_{(\mathfrak{g},K)}(S,\pi) \neq 0$. Given $\pi \in R(\mathfrak{g},K,\omega)$ let $N_\pi = \bigcap_\lambda \ker\lambda$ where λ runs over $Hom_{(\mathfrak{g},K)}(S,\pi)$, and write $S/N_\pi \simeq \pi \times \pi_1'$.

where π_1' is a $(\mathfrak{g}', K')$ - module. Howe then proves:

Theorem: (i) π_1' is finitely generated, admissible and quasi-simple.

(ii) π_1' has a unique irreducible quotient: π_1' .

(iii) The map $\pi \to \pi'$ determines a bijection:

$$R(\mathfrak{g}, K, \omega) \longrightarrow R(\mathfrak{g}', K', \omega) .$$

The proof depends on an analysis of the K and K' types in S. Specifically

$$S = \bigoplus_{\sigma \in \hat{K}} I_\sigma = \bigoplus_{\sigma' \in \hat{K}'} I_{\sigma'}$$

Then one considers centralizers M of K' in Sp(W) and M' of K in Sp(W) and obtains the diagram:

M	M'	There are Harish-Chandra decompositions of $\mathfrak{m} = \mathrm{Lie}(M)_{\mathbb{C}}$ ...
$\vert$	$\vert$	
G	G'	$\mathfrak{m} = \mathfrak{m}^{(2,0)} + \mathfrak{m}^{(1,1)} + \mathfrak{m}^{(0,2)}$
$\vert$	$\vert$	
K	K'	$\mathfrak{m}' = \mathfrak{m}'^{(2,0)} + \mathfrak{m}'^{(1,1)} + \mathfrak{m}'^{(0,2)}$.

Then one defines harmonics

$$\mathcal{H}(K) = \{\varphi \in S \mid \varphi \text{ is annihilated by } \mathfrak{m}'^{(0,2)}\}$$
$$\mathcal{H}(K') = \{\varphi \in S \mid \varphi \text{ is annihilated by } \mathfrak{m}^{(0,2)}\}$$

and finally

$$\mathcal{H} = \mathcal{H}(K) \cap \mathcal{H}(K').$$

Proposition: (i) $S = U(\mathfrak{g})U(\mathfrak{g}')\mathcal{H}$

(ii) $\mathcal{H} = \bigoplus_{\sigma \in \hat{K}, \sigma' \in \hat{K}'} H_{\sigma,\sigma'}$, $H_{\sigma,\sigma'} \cong \sigma \otimes \sigma'$ and

σ, σ' determine each other uniquely.

From this it follows if σ occurs in π and has lowest $\deg(\sigma) = \deg \mathcal{H}_{\sigma,\sigma'}$, then σ' must occur in π_1'. Further analysis leads to the theorem.

M.F. Vignéras: L-functions for classical groups (d'après Rallis and Piatetski-Shapiro).

Rallis and Piatetski-Shapiro have defined ζ-functions for all classical groups G which are obtained as isometry groups of a hermitian or skew-hermitian space over a global field k. This construction which arised in the context of the Weil representation, consists in a generalization of the Rankin-Selberg integral.

If π is an automorphic cuspidal irreducible representation of G, $\phi \in \pi$, $\check{\phi} \in \check{\pi}$, (,) the natural pairing between π and $\check{\pi}$.

If H = reductive algebraic group containing G x G, $P \subset H$ a parabolic, such that the action of G x G on P\H has a single main orbit, w a character of $P_k \backslash P_A$, $w|_{(G\times G \cap P)_A} = 1$, $G \times G \cap P = \{(g,g),\ g \in G\}$ and $f \in \mathrm{Ind}_{P_A}^{H_A} w$, the basic starting point is to consider the integral

$$\int_{(G\times G)_k \backslash (G\times G)_A} E_f(g_1,g_2)\, \phi(g_1)\, \check{\phi}(g_2)\, dg_1\, dg_2 = \int_{G_A} f(g,1)\, (\pi(g)\phi, \check{\phi})\, dg$$

where

$$E_f(h) = \sum_{\gamma \in P_k \backslash H_k} f(\gamma h) \ , \ h \in H_A$$

is an Eisenstein series.

On the left hand-side, the analytical properties of the integral result from those of the Eisenstein series.
On the right hand-side, one gets the zeta function

$$L(w,f,\phi,\check{\phi})$$

which has an Euler product when $f = \prod f_v$, $\phi = \prod \phi_v$, $\check{\phi} = \prod \check{\phi}_v$ are decomposable: $L(w,f,\phi,\check{\phi}) = \prod_v L(w_v,f_v,\phi_v,\check{\phi}_v)$. When $G = Sp(2n)$ or $SO(2n)$, they prove that in the unramified case, (almost every where) the local zeta functions are (for a good choice of w up to a translation of s) the classical L-functions of Langlands attached to the evident representation r of ${}^LG^o$, $L(s,\pi,\gamma)$. They prove that $L(s,\pi,\gamma)$ has a meromorphic continuation with finitely many poles.

J.L. Waldspurger: <u>Reprèsentations exceptionnelles des groups métaplectique gènèraux</u>.

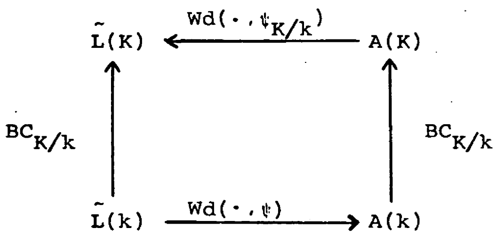
In this lecture the essential content of the following two papers was sketched:

D.A. Kazhdan, S.J. Patterson: Metaplectic forms, Publ. Math. IHES 59 (1984) 35 - 142

S.J. Patterson, I.I. Piatetski-Shapiro: A cubic analogue of the cuspidal theta representations. J. Math. pures et appl. 63 (1984) (to appear)

J.W. Cogdell: <u>Base change for $\widetilde{SL}_2$</u> .

Let k be a local field of char. 0 and K/k a cyclic extension of prime degree. Let $\tilde{L}(k)$ be the admissible irred. representation of $\widetilde{SL}_2(k)$ up to L-equivalence and A(k) the irred. adm. reprs. of $PGL_2(k)$. Then we define a base change map $\tilde{L}(k) \xrightarrow{BC_{K/k}} \tilde{L}(K)$ by "pulling back" the base change for PGL_2 via the Waldspurger correspondence, i.e., define $BC_{K/k}$ by



This definition is independent of ψ and defines a local base change for $\widetilde{SL}_2$.

Now let k,K be number fields, K/k cyclic of prime degree. We define $BC_{K/k} : \tilde{L}(k) \longrightarrow \tilde{L}(K)$ as the product of the local base changes (here $\tilde{L}(k)$ are the L-equiv. classes of genuine aut. irred. reprs).

It is possible that certain classes have no base change. However, if K/k is an odd cyclic ext. of prime degree then given $[\tilde{\sigma}] \in \tilde{L}(k)$, $[\sigma]_{K/k} \neq \emptyset$.

If $(K:k) = 2$ things are more interesting. Let $\tilde{B}(K/k) =$
$= \{[\tilde{\sigma}] \in \tilde{L}(k) \mid \tilde{\sigma}$ is cuspidal, not a θ-series from a 1-dm quadratic form, s.t.

$$\text{(i)} \quad \prod_{v \in \Sigma(\tilde{\sigma})} \chi_v(-1) = -1$$

$$\text{(ii)} \quad \Sigma[\tilde{\sigma}_{K/k}] = \emptyset \}$$

where χ is the quadratic char of K/k and $\Sigma[\tilde{\sigma}]$ is the set of places where $[\tilde{\sigma}]$ is not principal series.

Then the main theorem for $(K:k) = 2$ is that $BC_{K/k}[\tilde{\sigma}] = \emptyset$ iff $[\tilde{\sigma}] \in \tilde{B}(K/k)$.

If $[\tilde{\sigma}]_{K/k} = \emptyset$, then for any further odd degree base change, it does not reappear. However on any further base change of degree 2, then $[\tilde{\sigma}]_{K/k}$ is again automorphic and can never disappear again.

D.A. Kazhdan: <u>Algebraic geometry and representation theory.</u>

Ten years ago P. Deligne and G. Lusztig showed how to associate to any irreducible representation π of a reductive group G_{F_q} a semi-simple conjugacy class $s_\pi \in {}^L G_{F_q}$. Let G_1^V be the set of irreducible $\pi \in G_{F_q}^V$ such that $s_\pi = \{e\}$. G.Lusztig has associated to $\pi \in G_1^V$

a unipotent conjugacy class $u \subset G_{F_q}$. Fix $u \in G_F$. Let $G_u^v \subset G^v$ be the set corresponding to u. G. Lusztig has shown that $G_u^v \simeq \mathfrak{M}(\mathcal{G}_u)$ where $\mathcal{G}_u$ is a finite group associated with u and $\mathfrak{M}(\mathcal{G}_u)$ is its "cotangent space".

S. Rallis: <u>L-functions and metaplectic groups</u>.

This talk was a continuation of that of M.F. Vignéras; the primary objective was to describe the local theory of the L-functions introduced by Piatetski-Shapiro and the speaker. The central questions considered were the rationality of the L-functions and the location of their poles and zeros. The rationality followed from an inductive argument on the rank and a representation of the intertwining operator as an integral operator with kernel $|\det(h-I)|^s$. Once these properties were established the intertwining operators could be so normalized that one could also prove inductive properties of the L-function, analogous to those of Godement-Jacquet. The normalization involves the detailed analysis of certain generalized Whittaker functionals.

D. Zagier: <u>Heegner points and modular forms of half-integral weight</u>.

Let $X_o(N)$ $(N \in \mathbb{N})$ be the modular curve parametrizing pairs of elliptic curves (E,E') together with a cyclic N-isogeny $E \longrightarrow E'$. Let $k = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. A point $x \in X_o(N)$ is called a <u>Heegner-point</u> for k if E and E' both admit complex multiplication by the ring of integers of k. Over $\mathbb{C}$, $X_o(N) =$

$= \Gamma_0(N)\backslash\mathfrak{H} \cup \{\text{cusps}\}$ ($\mathfrak{H}$ = upper half-plane) and Heegner points correspond to $\tau \in \mathfrak{H}$ satisfying a quadratic equation $a\tau^2 + b\tau + c = 0$ $(a,b,c \in \mathbb{Z})$ with $b^2 - 4ac = -d$, N a. The number of such points is $h(k)\cdot 2^t$ (h = class number, t = number of p N), they are defined over the Hilbert class field of k, and are generated by the Galois group of this field over Q. Hence

$$P_d = \sum_{x = \text{Heegner point for } k} (x)-(\infty) \in \mathcal{J} = \mathrm{Jac}(X_0(N)) \text{ is defined}$$

over $\mathbb{Q}$.

On $J(\mathbb{Q})$ there is a positive definite bilinear pairing $<\ ,\ > : J(\mathbb{Q}) \times J(\mathbb{Q}) \longrightarrow \mathbb{R}$ (canonical height pairing). In 1982 B. Gross and I proved a formula for the height $< P_d, P_d >$. If $f \in S_2(\Gamma_0(N))$ is a Hecke eigenform of weight 2, and $P_d^{(f)}$ the f-eigencomponent of $P_d \in J(\mathbb{Q}) \otimes \mathbb{R}$, then this formula says $< P_d^{(f)}, P_d^{(f)} > = \frac{d}{\|f\|^2} L'(f,1) \cdot L(f,(\frac{-d}{\cdot}),1)$ if $f \in S_2^+$, 0 if $f \in S_2^-$, where f is the norm of f in the Petersson metric $(f \in S_2^{\pm} \Leftrightarrow f(-\frac{1}{N\tau}) = \pm N_\tau^2 f(\tau) \Leftrightarrow$ functional equation of $L(f,s)$ has a $(\mp)$-sign).

On the other hand, a theorem of Waldspurger says that the value $L(f,(\frac{-d}{\cdot}),1)$ of the twisted L-Series of f at s=1 equals $\lambda\cdot d^{1/2}c(d)^2$, where c(d) is the d^{th} Fourier coefficient of the modular form g of weight 3/2 associated to f by the Shimura-Shintani correspondence and λ a constant independent of d (essentially $\lambda = \frac{\|f\|^2}{\|g\|^2}$). Hence $< P_f^{(d)}, P_f^{(d)} >$ is proportional to $c(d)^2$. This suggests:

Theorem: The Heegner points $P_d^{(f)} \in (J(\mathbb{Q}) \otimes \mathbb{R})$ lie in a one-dimensional space and are proportional to the Fourier coefficients c(d). This was proved by Kohnen, Gross and myself. The lecture described part of the proof, which consists in calculating the scalar product of f with a restriction of an Eisenstein series for a real quadratic field. A generalization involving the theory of "Jacobi Forms" (the subject of a forthcoming book of M. Eichler and myself) was also discussed.

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