MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 3/1985

Boolesche Algebren

13.1. bis 19.1.1985

Die Tagung wurde von S. Koppelberg (Berlin) und J.D. Monk (Boulder/Colorado) geleitet. Als besonders vorteilhaft erwies sich, daß gleichzeitig die Tagung über Mengenlehre stattfand und die Programme beider Tagungen zeitlich weitgehend aufeinander abgestimmt waren; neun Vortragsstunden wurden gemeinsam organisiert.
Innerhalb der Tagung „Boolesche Algebren" waren vier Teilnehmer um einstündige Übersichtsvorträge gebeten worden: R. S. Pierce (Ketonens Satz über abzählbare Boolesche Algebren), S. Shelah (Die Anzahl der Ideale in einer Booleschen Algebra), B. Balcar (Ultrafilter und topologische Dynamik). Da J. Remmel seine Teilnahme kurzfristig absagte, fiel der Übersichtsvortrag über rekursive Boolesche Algebren aus.
Insgesamt ergaben die Vorträge ein breites Spektrum von Fragestellungen über Boolesche Algebren, die vom rein Algebraischen bis zum stark Mengentheoretischen reichten. Als rein algebraisch im Sinne von universeller Algebra bzw. Logik bezeichnen kann man die Vorträge von H. Andrèka und I. Németi (Zylinderalgebren), H. Dobbertin (Vaughtsche Maße) und G. Hansoul (Strukturdiagramme für abzählbare Boolesche Algebren). Nicht explizit von speziellen Axiomen der Mengenlehre abhängig waren die Ergebnisse der Vorträge von F. J. Freniche (Maße auf Booleschen Algebren), M. Rubin (Rekonstruktion von Booleschen Algebren aus ihrer Automorphismengruppe), M. Bekkali und R. Bonnet (Algebren, für die jeder Quotient Faktor ist), A. Błaszczyk (Verallgemeinerung des Satzes von Pierce über Kardinalzahlen vollständiger Boolescher Algebren auf Verbände), G. Brenner (aus Halbordnungen konstruierte Boolesche Algebren), S. Koppelberg (Projektive Boolesche Algebren).
Mehr bzw. stark mengentheoretisch orientierte Vorträge waren die von W. Just (Quotienten von P(ω) und Isomorphismen zwischen ihnen), P. Vojtáš (Spiele auf Booleschen Algebren) und J. Raitman (Superatomare Boolesche Algebren); der letztgenannte Vortrag bestand vorwiegend aus Konsistenz-Resultaten.

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Schließlich gab es einen rein topologisch-funktionalanalytischen Vortrag von S. Argyros (Eberlein-kompakte Räume) und drei rein mengentheoretische von F. Franek (Saturierte Ideale), Th. Jech (Große Kardinalzahlen), A. Krawczyk (Kombinatorische Äquivalenzen zu Aussagen über Maß und Kategorie).

Die Tagung bot eine willkommene Gelegenheit, die Organisation des geplanten Handbuchs über Boolesche Algebren (Herausgeber: J. D. Monk) zu besprechen.

Vortragsauszüge

H. ANDRÉKA:

Proving a conjecture of Henkin-Monk-Tarski for varieties of Boolean algebras with operators

By Bo_α we mean a Boolean algebra with α-many operators.
PROPOSITION 1: No element of a free Bo_α is a fixed point of infinitely many of the operators. This result carries over to many varieteis of Bo_α's e.g. to cylindric algebras (with obvious modifications). Under rather mild conditions the result becomes true for all varieties:
THEOREM 2: Let $K \subseteq Bo_\alpha$ be quasi-variety. Assume that (in K) to any two operators c_i and c_j ($i,j \in \alpha$) there is a constant d_{ij} which is a fixed point of all operators except c_i and c_j , and that $c_i(d_{ij})=c_i(-d_{ij})=1$ holds in K. Assume that the complement of an i-fixed point is i-fixed again (i.e. $c_i x=x \rightarrow c_i -x=-x$). Then Proposition 1 carries over to K. I.e.: no nontrivial element of a K-free algebra is a fixed point of indinitely many of the c_i's. By a nontrivial element we mean one not generated by the constants. THEOREM 3: All conditions are needed in Theorem 2. The $K=CA_\alpha$ case solves Problem 2.10 of Hinkin-Monk-Tarski: Cylindric Algebras (North-Holland) affirmatively.

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S. A. ARGYROS:

Non uniform Eberlein compact sets

A compact set K is said to be Eberlein compact (uniform Eberlein compact) if it is homeomorphic to a weakly compact subset of a Banach space (Hilbert space). The first example of an E.C. set non U.E.C. was given by Y. Benyamini - T. Starbird (1976). In our talk we give a new proof of the non uniform property of the above space and we give a new example of topological weight ω_1. The last space is defined by using an almost disjoint family of subsets of the set of natural numbers.

B. BALCAR:

Ultrafilters and topological dynamics

A dynamical system (X,T) is given by a compact space X and a continuous mapping $T: X \longrightarrow X$. We deal with connections between different notions of recurrency and combinatorial properties of sets of time returns (subsets of ω). Th. There is no 2-multiply reccurent point in the ultrafilter dynamical system $(\beta\omega, T)$, where T is the shift to the right.
The family of equations on $\beta\omega - \omega$ $p+q=p$, $p+2q=p$ has no solution.
Let F denote the Hindman filter on $P(\omega)$, i.e. $F = \{A \subseteq \omega : \omega - A$ is not a sum set$\}$ Th. If (X,T) is minimal, then for every nonempty open set $V \subseteq X$ and for every $l \in \omega$ the following holds:
$\{d: V \cap T^{-d}[V] \cap \ldots \cap T^{-ld}[V] \neq 0\} \in F$.

M. BEKKALI and R. BONNET:

Spaces in which every closed subset is homeomorphic to a clopen subset

A topological space Y has the property (P) iff every closed subspace of Y is homeomorphic to a clopen subspace of Y.
Let X be a topological space whose topology is the order topology of a Dedekind complete linear ordering.
Theorem (M. Bekkali, R. Bonnet, M. Rubin) X has the property (P) iff X

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is homeomorphic to $\alpha + \sum_{i<n} (\beta_i + 1 + \gamma_i^*)$ where $n < \omega$, α, β_i, γ_i are ordinals and β_i, γ_i have only finitely many cofinal types.

Theorem (M. Bekkali, R. Bonnet). The following are equivalent:

(i) every clopen subset of X has the property (P)

(ii) X is homeomorphic to an ordinal.

A. BŁASZCZYK:

On ω-powers of lattices

Let B be an infinite Boolean algebra, B^c a completion of B and L an upward σ-complete sublattice of B^c containing B.

Theorem 1. Assume GCH. Then $|L|^\omega = |L|$.

Theorem 2. Assume $2^{\aleph_n} = \aleph_{\omega+n+1}$ for every $n < \omega$. Then there exists a B and an L such that $|L|^\omega > |L|$.

Theorem 3. Assume for every $u \in L$ there exists a sequence $\{u_n: n < \omega\} \subset L$ such that $\wedge \{u \wedge u_n: n < \omega\} = \emptyset$ and $u_{n+1} \ll u_n$ and $u_n \vee u = \mathbb{1}$ for every $n < \omega$. Then (in ZFC) $|L|^\omega = |L|$.

Theorem 3 answers a question by van Douwen and Zhou.

G. BRENNER:

Tailalgebras and Irredundance

For any poset P we define tailalg(P) as the closure under finite unions and complements relative to set P of $\{S_p : p \in P\}$ where for each $p \in P$ $S_p = \{q \in P : q \geq p\}$. If S is a subset of BA B we say S is irredundant iff $\forall s \in S$ $s \notin$ Subalgebra B generated by $S \setminus \{s\}$.

Some facts about tailalgebras:

1. If B has an irredundant set of generators then B is isomorphic to a tailalgebra (hence free BA's are tailalgebras).

2. (McKenzie) All BA's have a dense irredundant set (so all BA's embed a tail algebra densely).

3. The class of tailalgebras generated from trees is closed under homomorphic image but not under substructure.

Questions: 1. Is it consistent that every BA is isomorphic to a tailalgebra? In particular is P(ω)? (Note: Blass & Koppelberg answered the second question negatively during the conference).

2. Do all tailalgebras have irredundant generating sets?

3. Consider subclasses such as the classes generated by trees and pseudo-

trees. What are their closure properties under substructure, homomorphic image and product?

4. Are all subalgebras of algebras generated from trees isomorphic to algebras generated by pseudo-trees?

H. DOBBERTIN:

Vaught measures and applications in lattice theory

We gave a short survey about results appearing in the study of certain measures ("Vaught measures") on Boolean algebras with values in refinement monoids. As a particulas application we showed that a theorem of E. T. Schmidt has the following consequence: If L is a distributive algebraic lattice such that the sup-semilattice of all compact elements of L is locally countabel then L is isomorphic to the congruence lattice of some lattice. (It is a long-standing conjecture in lattice theory that every distributive algebraic lattice can be represented as the congruence lattice of a lattice.)

F. FRANEK:

Saturated ideals

Using (not necessarily well-founded) generic ultrapowers the following theorem is proved: There is no λ-complete $\aleph_\alpha$-saturated uniform ideal over κ, where $\alpha < \lambda$, λ an infinite cardinal, $\aleph_{\omega_\alpha} < \kappa < \aleph_{\tilde{\lambda}}$.
($I^+ = \{X \subseteq \kappa : X \notin I\}$ together with $\subseteq^*$, where $X \subseteq^* Y$ iff $(X-Y) \in I$, is considered as a forcing notion, and if G is $\langle I^+, \subseteq^* \rangle$ - generic over M, then one can form a generic ultrapower $M^* \cap M/G$ in $M[G]$. It follows that I cannot exist).
This generalises Taylor's theorem, and it also gives a simpler and shorter proof.

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F. T. FRENIDE:

Some classes of Boolean algebras related to the Vitali-Hahn-Saks and and Nikodym theorems

We introduce a new separation property on the pairwise disjoint sequences of a given Boolean algebra: the weak subsequential interpolation property. Its definition is as follows: for every pairwise disjoint sequence (a_n) in the Boolean algebra A and for every infinite subset M of ω, there exists $a \in A$ and an infinite subset N of M such that $a_n \leq a$ if $n \in N$ and $a_n \cdot a = 0$ if $n \notin M$.
Then we prove that the weak subsequential interpolation property implies the Grothendieck property on sequences of measures. As far as we know, this is the first general theorem proving the Grothendieck property but not the Vitali-Hahn-Saks property at the same time.

G. HANSOUL:

Boolean Algebras and Q.O.systems

A Q.O.system is a set eqiupped with a transitive binary relation $\lhd$.
We compare the Q.O.systems associated to a primitive Boolean algebra B by Hanf (denoted by S(B)) and by Pierce (denoted by P(B)). The result is that P(B) is isomorphic to the Q.O.system of all filters in S(B). Consequently P(B) and S(B) cincide if and only if B is a well-founded primitve Boolean algebra (i.e. if S(B) is well founded). Another consequence is that the topological Boolean algebra of all fully invariant subsets of a primitive space X (where fully invariant means invariant under any homeomorphism of X) is the least complete subalgebra of the topological Boolean algebra of all subsets of X together with topological derivation.

W. JUST:

Boolean algebras $P(\omega)/I$

Results on Boolean algebras of the form $P(\omega)/I$ are presented, where I is an ideal containing Fin. In the first part conditions for the ideal I are given, ensuring that $P(\omega)/I$ is $\aleph_1$-saturated.
In the second part the ideals

$I_1 = \{a \subset \omega: \limsup_{n \to \infty} \frac{|a \cap n|}{n} = 0\}$ - the ideal of sets of density 0 and

$I_2 = \{a \subset \omega: \limsup_{n \to \infty} \frac{\sum_{m \in a \cap n} \frac{1}{m+1}}{\log n} = 0\}$ - the ideal of sets of logarithmic density 0 are considered. A proof of the following theorem solving a problem of Erdös and Ulam is scetched:

Theorem: CH $\Longrightarrow P(\omega)/I_1 \simeq P(\omega)/I_2$.

The results were obtained jointly with A. Krawczyk.

S. KOPPELBERG:

Projective Boolean algebras

A Boolean algebra is projective iff it is a retract of a free one. The following results were presented:

Proposition ($\aleph_\omega < 2^\omega$) There is a rigid projective Boolean algebra of cardinality $\aleph_\omega$.

Theorem 1 If $\varkappa > \mathrm{cf}\,\varkappa$, then there are, up to isomorphism, exactly $2^\varkappa$ projective Boolean algebras of cardinality $\varkappa$.

Theorem 2 If $\varkappa = \mathrm{cf}\varkappa > \omega$, then there are, up to isomorphism, exactly $2^{<\varkappa}$ projective Boolean algebras of cardinality $\varkappa$. Theorem 2 is proved by defining a class of invariants for projective algebras of size $\varkappa$, the invariants being pairs (C,J) where C is projective, $|C| < \varkappa$, and J is an ideal of C.

A. KRAWCYK:

Combinatorial properties of Lebesgue measure and Baire category.

New results mentioned at the expository talk:

Def: Call a set A " nice " iff there exsist sequences I_n, J_n such that $\{I_n\}_{n\in\omega}$ is a partition of ω , $|I_n| < \aleph_0$, $J_n \subset 2^{I_n}$,

$$\Sigma \frac{|J_n|}{2^{|I_n|}} < \infty \text{ and } A \subset \bigcap_n \bigcup_{n>m} \bigcup\{[s]: s \in J_n\}.$$

Prop. 1 Any null set is a subset of union of two " nice" sets.

Prop. 2 Any set of cardinalitry less than $2^{\aleph_0}$ is " nice" .

Prop. 3 There exsists a null set which is not " nice" .

I. NÉMETI:

Free cylindiric and relation algebras are not atomic

Problem 4.14 of Henkin-Monk-Tarski: Cylindric Algebras (North-Holland) is settled.

Theorem 1: No free cylindric algebra (CA from now on) of dimension greater than 2 is atomic. In particular Fw_1CA_3 is not atomic.

Direct algebraic proof is available for dimension > 3. (For 3, indirect logical.) Definition 2: Relation algebras (RA's) are BA's enriched with an invertible monoid structure. (Think of the BA of all binary relations over a set, with relation composition as the monoid operation.) Corollary 3: No free RA is atomic. Neither are such the non-associative RA's. The proof of the CA_3 case goes by proving that Gödel's incompleteness theorem holds for first order logic with three variables ($\vdash^3$) but not for that with two variables ($\vdash^2$). The difficulty is that $\vdash^3$ is very weak (e.g. $\nvdash^3$ " relation composition is associative" or $\nvdash^3$ " the composition of functions is a function"). The key idea is a new way of associating a RA to " almost every" CA_3. This amounts to adding finitely many axioms (not schemes!) to $\vdash^3$ such that the above given two schemes (associativity ...) become provable (in a sense)

R. S. PIERCE:

Ketonen's Theorem

Ketonen proved the THEOREM: Every countable, commutative semigroup embeds in the semigroup of isomorphism types of countable Boolean algebras. Ketonen's proof used Baire category arguments carried out in a model of set theory that satisfies the constructability axiom. A new version of this proof is outlined. It uses only algebraic extension techniques in the context of naive set theory. The proof breaks into three parts: construction of pseudo-invariants for metrizable Boolean spaces; development of the Boolean hierarchy $\{\varkappa^{(\xi)} : \xi < \omega_1\}$ to classify uniform Boolean algebras; the embedding of a countable, commutative semiring in $\varkappa^{(3)}$. The details of this proof will appear in the countable Boolean algebra section of the Handbook of Boolean algebra.

J. ROITMAN:

A survey of cardinal invariants of superatomic Boolean algebras

A superatomic Boolean algebra B is one which is exhausted by the Cantor-Bendixson hierarchy: if $J_0 = \emptyset$, $J_{\alpha+1}$ is the ideal generated by $J_\alpha \cup \{x: x/J_\alpha$ is an atom of $B/J_\alpha\}$, and $J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$ for λ a limit, then $B = J_\alpha$ for some α. Let β be the least α with $B = J_\alpha$. The function $f: \beta \longrightarrow$ CARDS defined by $f(\alpha) = |\{$atoms of $B/J_\alpha\}|$ is the cardinal sequence of B. The question is: which functions can be cardinal sequences? This talk surveys the known results. For example, in ZFC we have: any $f: \alpha \longrightarrow \{\omega\}$, where $\alpha < \omega_2$, is a cardinal sequence (Juhasz-Weiss); any $f: \alpha \longrightarrow \{\omega, \omega_1\}$, where $\alpha < \omega_2$, is a cardinal sequence (Weese). Consistency results include: $f: \omega_2 \longrightarrow \{\omega\}$ is not a cardinal sequence (Just in the model obtained by adding ω_2 Cohen reals to a model of CH); neither is any $f: \omega_1+1 \longrightarrow \{\omega, \omega_1, \omega_2\}$ where $f(0) = \omega$, $f(\alpha) \leq \omega_1$ for $\alpha < \omega_1$, $f(\omega_1) = \omega_2$ (Just in the same model) although such f are cardinal sequences in an ad hoc model construcion (Roitman); $f: \omega_1+1 \longrightarrow \{\omega_1, \omega_2\}$ where $f(\alpha) = \omega_1$, for $\alpha < \omega_1$, $f(\omega_1) = \omega_2$ is a cardinal sequence (Weese from a Canadian tree, Roitman from MA $+\neg$ CH) and is not a cardinal sequence (Baumgartner in the Mitchell model). Open questions include: can

$f: \omega_2 \longrightarrow \{\omega\}$ consistently be a cardinal sequence?
What about $f: \varkappa^{++} \longrightarrow \varkappa$? And what of positive nontrivial results on $> \omega_1$ atoms?
(Many of the negative results generalize, but few of the positive ones do).

M. RUBIN:

<u>When two Boolean algebras have the same automorphism group</u>

Let B be a Boolean algebra Ult(B) be the Stone space of B, and G be a subgroup of the automorphism group of B. We regard G as a group of homeomorphisms of Ult(B). For $x \in \mathrm{Ult}(B)$ let $G(x) = \{g(x) \mid g \in G\}$. We say that $< B,G >$ is strongly closed if for every $g \in G$ and clopen $V \subseteq \mathrm{Ult}(B)$: if $g(V) = V$ then $g \upharpoonright V \cup \mathrm{Id} \upharpoonright (\mathrm{Ult}(B) - V) \in G$, and if $g(V) \cap V = \emptyset$ then $g \upharpoonright V \cup g^{-1} \upharpoonright g(V) \cup \mathrm{Id} \upharpoonright (\mathrm{Ult}(B) - V - g(V)) \in G$. Let
$K = \{< B,G > \mid$ (1) $< B,G >$ is stongly closed ; (2) the set $\{x \in \mathrm{Ult}(B) \mid |G(x)| \geq 3\}$ is dense in Ult(B) ; and (3) either (3.1) $(\forall\, x \in \mathrm{Ult}(B))\ (|G(x)| \geq 2)$ or (3.2) for every regular open non-clopen subset V of Ult(B) the boundary of V contains a point x such that $|G(x)| \geq 3\}$.
<u>Theorem:</u> If $< B_1,G_1 >, < B_2,G_2 > \in K$ and $G_1 \simeq G_2$, then $B_1 \simeq B_2$.

S. SHELAH:

<u>On the spread and the number of ideals</u>

We prove some abstract theorem of the following kind: Suppose X is a topological space, φ a function from the power set of X to class of cardinals, $\lambda = \sum_{i<\theta} \chi_i$, χ_i increasing $< \lambda$, $\mu = \mu^\theta$. Let
$\mathrm{Ch}_\varphi(x) = \mathrm{Min}\ \{\varphi(u) : u$ an open neighbourhood of $x\}$. We assume
(i) $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$
(ii) $|\{y \in X : \mathrm{Ch}_\varphi(y) \geq \chi_i\}| \geq \mu$
(iii) if $Y_\alpha \subseteq X$ for $\alpha < \mu$, and $\varphi(Y_\alpha) < \chi_i$ for each α then
$\varphi(\bigcup_{\alpha<\mu} Y_\alpha) < \chi_i$. We want to deduce that there are open u_i $(i < \theta)$ s.t. $\varphi(u_i - \bigcup_{j \neq i} u_j) \geq \chi_i$ for each i.

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We prove this e.g. for $\mu = \beth_5(\theta)^+$. Under additional conditions we get the result for smaller μ. This has applications for the possible values of the spread and the number of open sets.

P. VOJTÁŠ:

Boolean games and refining problems

We deal with the transfinite game on Boolean algebras introduced by T. Jech. We prove that if the density of a Boolean algebra B is 2^ω and $2^\omega <$ 1st w.i. cardinal number and if the player Black has a winning strategy, then B has a σ-closed dense subset. Moreover, we investigate properties of sets of ordinal numbers of the length of games for which Black, respectively White, has a winning strategy. The results suggest that the game is " cardinal-type" rather than " ordinal type". Moreover we survey some results on refining properties of Boolean algebras and give the current stage of problems. We mention connections in topology with ν-points and an application: the construction of a pure OUT-OUT space.

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