MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 18/1985

Algebraische Gruppen

14.4. bis 20.4.1985

Die Tagung fand unter der Leitung von Herrn T.A. Springer (Utrecht) und J. Tits (Paris) statt. In den Vorträgen wurde berichtet über neuere Ergebnisse u.a. in den folgenden Themenkreisen:

Darstellungen von Hecke Algebren;
Algebraische Transformationsgruppen und Orbiträume;
Symmetrische Räume und deren Kompaktifizierungen;
Reduktive Gruppen über lokalen und globalen Körpern und Galois-Kohomologie;
Schubert-Mannigfaltigkeiten.

Vortragsauszüge

F. BRUHAT:

Schêmes d'immeubles des groupes classiques, sur un corps local

L'exposé vise à donner, dans le cas des groupes classiques, des interprêtations "concrètes" de l'immeuble d'un groupe réductif G défini sur un corps local K et des schémes en groupes $\underline{G}_x$ sur l'anneau des entiers de K associés aux points x de l'immeuble (suivant la théorie développée par J. Tits et l'auteur). Concrêtes signifie liées de manière simple à la représentation linéaire "naturelle" de G.
On a d'abord traité le cas de $G = GL_n(D)$, ou D est un corps gauche de centre K. L'immeuble I de G, on plus exactement l'immeuble élargi, s'identifie alors canoniquement à l'ensemble des normes sur $X = D^n$. A une telle norme $\alpha \in I$ est associé un ordre M_α de $M_n(D)$, qui ne dépend que de la facette de I contenant α, et on obtient ainsi un isomorphisme du complexe simplicial abstrait des facettes de I sur l'ensemble des ordres héréditaires de $M_n(D)$, organisé en complexe simplicial par la relation d'inclusion. Le schêma $\underline{G}_\alpha$ associé à α n'est autre que le groupe multiplicatif de l'ordre M_α.
Dans le cas des groupes unitaires l'immeuble de G s'identifie à l'ensemble des normes "maximinorantes", i.e. maximales parmi les normes minorant (f,q): si q est la forme pseudoquadratique et f la forme sesquilinéaire associée définissant G, on dit que α minore (f,q) si $\omega(f(x,y)) \geqslant \alpha(x) + \alpha(y)$ et $\omega(q(x)) \geqslant 2\alpha(x)$. Le schême $\underline{G}_\alpha$ est l'adhérence schématique de G dans le groupe multiplicatif de M_α.

J. CARRELL:

Regular orbits of the Weyl group and a result of DeConcini and Procesi

Let G be a semi-simple complex Lie group and O_n the G-orbit of a nilpotent element n in $\underline{g} = \mathrm{Lie}(G)$. In the case $G = SL_n(\mathbb{C})$, De Concini and Procesi have identified the coordinate ring $A(\overline{O}_n \cap \underline{h})$ with the cohomology ring of the variety of flags fixed by a

"dual" nilpotent n. Here $\overline{O}_n \cap \underline{h}$ is the unreduced variety obtained as the scheme theoretic intersection of $\overline{O}_n$ and a Cartan subalgebra $\underline{h}$ of $\underline{g}$, and the identification is valid for both the W-module and $\mathbb{C}$-algebra structures.
In this talk we will give a generalization of this result to arbitrary semi-simple G, but one must restrict the nilpotents under consideration. In particular we only consider nilpotents n which are Richardson elements for some parabolic P and for which $\overline{O}_n$ is normal in $\underline{g}$. The result is that if L is a Levi subgroup of P and u is regular in L, then there exists a surjective W-equivariant $\mathbb{C}$-algebra homomorphism $A(\overline{O}_n \cap \underline{h}) \to \mathrm{Im}(H^{\cdot}(G/B) \to H^{\cdot}((G/B)^u))$, $(G/B)^u$ being the flags fixed by u. The key to proving this result lies in studying the graded ring GrA(W.s) associated to the ring A(W.s) of functions on an orbit $W.s \subset \underline{h}$ of the Weyl group, where s is a regular element in $Z(\underline{l})$. In principle, this theorem is an application of some recent results on torus actions on singular varieties.

C. DE CONCINI:

Complete symmetric varieties and their cohomology

The starting motivation of this work was an attempt to find rigorous deductions of some numbers in enumerative geometry such as the number of quadrics in $\mathbb{P}^3$ tangent to nine quadrics in general position originally computed by H. Schubert.
We work over $\mathbb{C}$. Given a connected affine algebraic group G and a closed subgroup H we try to define a "ring of conditions $C^*(G/H)$. In such a generality it is easy to see that this is not possible.
We construct this ring in two cases:
a) G is a torus, $H = \{e\}$; b) G is a semisimple adjoint group, $\sigma : G \to G$ is an order two isomorphism, $H = G^{\sigma}$. In this case $C^*(G/H)$ is constructed as follows. One considers the set S of G-equivariant compactifications of G/H as a poset under the relation given by $X \geq Y$ if there exists a G-equivariant map $f : X \to Y$ extending the identity on G/H. Then $C^*(G/H) = \lim\limits_{\overrightarrow{Y \in S}} H^*(Y)$
In both cases a) and b) this is proved by a detailed study of the set of compactifications and their geometric properties.

A.G. ELASHVILI

Stationary subalgebras of points in general position for Borel subgroups of simple linear Lie groupes

Let G be a connected linear complex Lie group on a finite dimensional complex vector space V and let m_G be the maximal dimension of an orbit. We have classified all the simple linear Lie groups for which $m_B < \dim B$. We have also shown the existence of a Zariski open set U on which all Lie algebra stabilizers of points are B conjugate to a fixed sub-algebra $H \subset L(B)$. It is interesting to note that the sub-algebras $H \subset L(B)$ are Borel sub-algebras of regular sub-algebras of the simple Lie algebra L(G).

F.D. GROSSHANS

Hilbert's fourteenth problem for non-reductive groups

Let k be an algebraically closed field and let G be a semi-simple algebraic group over k. Let X be an affine variety over k on which G acts regularly. For H a subgroup of G, let $k[X]^H$ denote the algebra of regular functions an X which are fixed by H. In this talk, a large class of subgroups H was described for which $k[X]^H$ is a finitely generated k-algebra. For example, let B = TU be a Borel subgroup of G and let $H \subset U$ be a connected group which is normalized by T. Let $\{\alpha_1,\ldots,\alpha_r\}$ be the roots of U which are not in H. If $\{\alpha_1,\ldots,\alpha_2\}$ is linearly independent over $\mathbb{Q}$, then $k[X]^H$ will always be finitely generated. This holds, e.g., if $H \supset (U,U)$ or if $\dim (U/H) \leqslant 2$.

W.J. HABOUSH

The Linearization Conjecture for Reductive Actions

It is conjectured that given any action of a connected reductive group over $\mathbb{C}$ on affine n-space $\mathbb{A}^n_{\mathbb{C}}$, there is a polynomial automorphism of $\mathbb{A}^n$ which will conjugate this action to a linear representation (Kambayashi). Early progress was made by Bialynicki for a $(\mathbb{C}^*)^n$ acting faithfully first on $\mathbb{A}^n$ and in a later work on $\mathbb{A}^{n+1}$. Then Kambayashi-Russel established it for $(\mathbb{C}^*)^n$ acting with only positive characters on a space of any dimension. The conjecture implies the truth of cancellation and has other important consequences. It is easy to show that the conjecture implies that the Grothendieck group of equivariant bundles on $\mathbb{A}^n$ is isomorphic via pull back to the representation ring of the group over $\mathbb{C}$. Kraft and Popov proved the conjecture for any semisimple group acting on $\mathbb{A}^3$ and Panyushev established it for $\mathbb{A}^4$. It may also be proven by studying the fibre over the generic point of the quotient mapping $\mathbb{A}^n \to \mathbb{A}^n/G$. If this fibre is an affine space then the action is linear. This observation establishes linearization for $\mathbb{A}^3$ and for most actions on $\mathbb{A}^4$.

A.G. HELMINCK

Groups with involutions

We shall present a fairly simple method to classify (locally) affine symmetric spaces and their fine structure, like the restricted root systems with their Weyl groups, multiplicities, signatures, etc. To do so, we show first that there is a bijection between (locally) affine symmetric spaces and pairs of commuting involutitive automorphisms of a reductive algebraic group G, defined over an algebraically closed field of characteristic not 2, and we classify these.

R. KOTTWITZ

Galois cohomology and the stable trace formula

Let F be a local field and let $\Gamma = \mathrm{Gal}(\overline{F}/F)$.
Then for any connected reductive group G over F there exists a canonical map

$$H^1(F,G) \to \pi_0(Z(\hat{G})^\Gamma)^D,$$

where $Z(\hat{G})$ denotes the center of the connected Langlands dual group $\hat{G}$, $(\ )$ denotes Γ-invariants, $\pi_o(\ )$ denotes group of connected components, and $(\ )^D$ denotes the dual finite abelian group. If F is p-adic, the map is a bijection.
Now let F be a number field and let $\bar{\mathbb{A}}$ denote the adele ring of $\bar{F}$. Then there exists a canonical map

$$H^1(F, G(\bar{\mathbb{A}})/Z_G(\bar{F}) \to \pi_o(Z(\hat{G})^\Gamma)^D,$$

where Z_G denotes the center of G.
These results can be applied to the stabilization of the trace formula for connected reductive groups over number fields.

V. LAKSHMIBAI

Geometry of G/P-VI

Let G be a semi-simple, simply connected Chevalley group defined over a field k. Let T be a maximal (split) torus, B a Borel subgroup, $B \supset T$. Let P be a maximal parabolic subgroup. Let W (resp W_P) denote the Weyl group of G (resp. P) and (,), a W-invariant scalar product on $\mathrm{Hom}(T, \mathbb{G}_m)$. When P is of classical type, i.e., the associated fundamental weight satisfies the condition $2\frac{|(\omega,\alpha)|}{(\alpha\ \alpha)} \leqslant 2$, for every root α, a basis for $H^\circ(G/P,L)$ was constructed in G/PI-V as a generalization of the classical Hodge-Young theory (here, L is the ample generator of Pic(G/P). In G/P-VI, we describe this basis in a very explicit way for the case of classical groups and using this we obtain results on singular loci of Schubert varieties and also results on the behaviour of the Schubert varieties (in the "classical" flag varieties) under the canonical involution on the general linear group.

D. LUNA

Espaces homogènes sphériques

Soit G un groupe réductif connexe /$\mathbb{C}$, et soit H un sous-groupe algébrique de G (non nécessairement réductif ni connexe). Les conditions suivantes sont équivalentes:

(1) quel que soit le G-fibré en droites L sur G/H, les composantes isotypiques du G-module $H^{\circ}(G/H,L)$ sont toutes de longueur 1;

(2) un (et par conséquent tout) sous-groupe de Borel de G a une orbite ouverte dans G/H;

(2') un (et par conséquent tout) sous-groupe de Borel de G n'a qu'une nombre fini d'orbites dans G/H;

(3) quelle que soit la G-variété algebrique Z et quel que soit $z \in {}^{H}Z$, G n'a qu'un nombre fini d'orbites dans $\overline{G.z}$.

((1) ⇔ (2) est du à Vinberg-Kimel'fel'd,
(2) ⇒ (2') à Brion et Popov-Vinberg, (1) ⇒ (3) à Servedio,
(3) ⇒ (1) à Ahiezer)

Si ces conditions sont remplies, on dit que G/H est un espace homogène (et H un sous-groupe) sphérique.
Soit H un sous-groupe sphérique de G. Fixons un sous-groupe de Borel B de G tel que BH est ouvert dans G. Posons $P_{++} = \{f \in \mathbb{C}[G],\ f^{-1}(0) = G - BH\}$. Posons $P = \{s \in G,\ sBH = BH\}$; c'est un sous-groupe parabolique de G dont nous noterons P^u le radical unipotent. Pour $f \in \mathbb{C}[G]$, posons $L^f = G_{df(e)}$ et notons S^f le centre connexe de L^f (où $(df)(c) \in \mathfrak{g}'$ désigne l'élément de la représentation coadjointe de G obtenu en évaluant df en l'élément neutre de G).

Théorème (Brion-Vust-L.) Si $f \in P_{++}$,
A) L^f est un sous-groupe de Levi de P, on a $L^f \cap H = P \cap H$, ce groupe est réductif, et $(L^f, L^f) \subset H$ (d'où il suit en particulier que (L^f, L^f) ne dépend pas de f);
B) quelle que soit la G-variété algébrique Z et quel que soit $z \in {}^{H}Z$, $P^u.S^f.z$ contient un ouvert non vide de toute orbite de G dans $\overline{G.z}$.

G. LUSZTIG

Representations of Hecke algebras

Let G be a semisimple simply connected algebraic group over $\mathbb{C}$, let $u \in G$ be a unipotent element and let $\sigma \in G$ be a semisimple element such that $\sigma u \sigma^{-1} = u^q$ where q is a real number > 1.
Let H_q be the Hecke algebra of the affine Weyl group W.P with the parameter q, where W is the Weyl group of G and P is the

lattice of weights of a maximal torus of G.
Let $\mathcal{B}_u$ be the variety of Borel subgroups of G containing u, let M be a maximal compact subgroup of the closure of the group generated by $(\sigma,q) \in G \times \mathbb{C}^*$. Then M acts naturally on $\mathcal{B}_u$. Let R_M be the representation ring of M and let us regard $\mathbb{C}$ as an R_M-module via the homomorphism $R_M \to \mathbb{C}$ given by evaluation at (σ,q). Consider the equivariant K-theory $K_M(\mathcal{B}_u)$.
Jointly with Kazhdan we have constructed a natural action of H_q on $K_M(\mathcal{B}_u) \otimes_{R_M} \mathbb{C}$. (This was also shown by V. Ginsburg.)
Using this one gets a classification of irreducible representations of H_q thereby confirming a conjecture of Langlands and Deligne.

K. POMMERENING

The extension property of differential forms on arithmetic quotients of Hermitean symmetric spaces

Let D be a Hermitean symmetric space of noncompact type, hence isomorphic to a bounded symmetric domain. Let a discontinuous group Γ act on D such that Γ is arithmetically defined. That is, Aut(D) is isogeneous to $G(\mathbb{R})$ where G is a semisimple algebraic group over $\mathbb{Q}$, and Γ is an arithmetic subgroup of $G(\mathbb{Q})$. Let X be a nonsingular model of the field $K = \mathbb{C}(D/\Gamma)$, the field of Γ-automorphic functions. Then there is a natural injection $\Omega^p(X) \hookrightarrow \Omega^p(D)^\Gamma$. The main result of the talk is, that this is an isomorphism for $p < \dim D$ (I assume, for the sake of simplicity that G has no normal $\mathbb{Q}$-subgroup of dimension 3).
The main point in the proof is a problem on the fixed points of certain representations of certain algebraic groups.
This result means that the birational invariants $g_p = \dim_{\mathbb{C}} \Omega^p(X)$ of K are the dimensions of certain spaces of vector valued automorphic forms. For the Siegel case there already are useful applications: Recent results by Weissauer give strong vanishing theorems for g_p and, in the case where $g_p \neq 0$, dimension formulas that express g_p by the dimensions of spaces of harmonic forms, that are well-known representation spaces of the orthogonal group.

G. PRASAD

Topological central extensions

I sahll describe some new results on the topological central extensions of the group of rational points of an absolutely simple, simply connected group defined over a local field, as well as the corresponding results for the group of S-adelic points of an absolutely simple, simply connected group over a global field. Since for isotropic groups the results are fairly complete and in print (G. Prasad and M.S. Raghunathan: Annals of Math. Vol 119 (1984), Inventiones Math. Vol. 71 (1983)), I shall restrict myself to the case of anisotropic groups. It is known (Kneser, Bruhat-Tits) that over a local field k, an absolutely simple, simply conn. anisotropic group is of the form $SL_{1,D}$, D a central division algebra over k. In a joint work with Raghunathan, we have proved that $H^2_{ct}(SL_1(D), \mathbb{R}/\mathbb{Z})$ is trivial if k contains no nontrivial pth roots of unity (p = characteristic of the residue field). Also if for example, k is a tamely ramified extension of $\mathbb{Q}_p$, then $H^2_{ct}(SL_1(D), \mathbb{R}/\mathbb{Z}) \cong \hat{\mu}(k)_p$. Our expectation is that $H^2_{ct}(SL_1(D), \mathbb{R}/\mathbb{Z})$ is always isomorphic to $\hat{\mu}(k)_p$. Once this is proved, using a variant of a theorem of Moore in global class field theory, which I proved last year, one can compute the relative fundamental group of the group of S-adelic points of any abs. simple, simply conn. group of type other than E. This has obviuos implications for the congruence subgroup problem for groups for which the centrality of the congruence subgroup kernel is known.

C. PROCESI

Cohomology of equivariant compactifications

The classical enumerative geometry of Schubert is strongly related to the computation of the ordinary cohomology of equivariant compactifications of homogeneous spaces. I will present some results obtained with De Concini in this topic.

R.W. RICHARDSON

Conjugacy classes of n-tuples in Lie algebras and algebraic groups

Let G be a linear algebraic group over an algebraically closed field of characteristic zero and let $\underline{g} = L(G)$. We consider the adjoint action of G on G^n and $\underline{g}^n$: $g.(x_1,\ldots,x_n) = (g.x_1,\ldots,g.x_n)$. We generalize to this setting a number of standard results on conjugacy classes in G and $\underline{g}$. In particular we define semisimple n-tuples and prove that, if G is reductive, then the orbit $G.(x_1,\ldots,x_n)$ is closed iff $(x_1,\ldots,x_n)$ is a semisimple n-tuple. We also have an analogue of the Jordan decomposition for n-tuples.
If S is a linearly reductive group of automorphisms of G and $K = G^S$, then most of our results extend to the action of K on $\underline{g}^n$ and G^n.

J. SEKIGUCHI

A bijective correspondence of nilpotent orbits of a symmetric pair and its dual

Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its Cartan decomposition. Let $[N(\mathfrak{g})]$ be the set of real nilpotent orbits of g. On the other hand, let $[N(\mathfrak{p}_c)]$ be the set of K_c-orbits of $\mathfrak{p}_c$. The main purpose of this talk is to prove the existence of a bijective correspondence between $[N(\mathfrak{g})]$ and $[N(\mathfrak{p}_c)]$. This was conjectured by B. Kostant. The speaker formulates a generalization of this correspondence to the case of semisimple symmetric pairs. The proof of this case is similar to the above one.

D. SHELSTAD

Stable conjugacy and endoscopy

Let F be a field of characteristic zero, either local or global. We describe a family of connected reductive guasi-split groups over

F associated to a pair (G,θ), where G is a connected reductive group over F and θ is an F-automorphism of G. Let H be such a group. Then there is a $Gal(\overline{F}/F)$-map from semisimple conjugacy classes in H(F) to θ-conjugacy classes in G(E). This provides norm correspondences from G(F) to H(F). These correspondences are useful for matching orbital integrals.

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