MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 15/1987

Reelle Algebraische Geometrie

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This conference, directed by E. Becker (Dortmund), L. Bröcker (Münster) and M. Knebusch (Regensburg), was the second one on this subject held at Oberwolfach. Some of the major topics treated in its course have been algebraic geometry over the reals, semialgebraic geometry and the real spectrum, subanalytic geometry, quadratic forms and orderings of higher level, logical and model-theoretical methods and relations to other branches of mathematics such as Teichmüller theory. A feature of particularly growing interest is the search for algorithms computing geometrical invariants of real algebraic varieties. It became evident in the course of this week that the topics centered around the subject "real algebraic geometry" are intimately related to each other and therefore cannot be isolated. The large number of participants, most of them coming from abroad, showed that the interest in this area of mathematics is still increasing.

Vortragsauszüge

L. VAN DEN DRIES:

P-ADIC AND REAL SUBANALYTIC SETS

Question (Serre, ±1980): Let $f \in \mathbb{Z}_p\{X_1,\dots,X_m\}$ be a power series over $\mathbb{Z}_p$ converging on $\mathbb{Z}_p^m$, let $Z(f) \subset \mathbb{Z}_p^m$ be its zero set, put $N_k = \#$ image of $Z(f)$ under $\mathbb{Z}_p^m \to (\mathbb{Z}/p^k)^m$, $k = 0,1,2,\dots$. Is then the Poincaré series $\sum_{k=0}^{\infty} N_k t^k$ rational in t?

Denef (Invent. 1984): Yes, if f is a polynomial.
Denef, v.d. Dries (1986, to app. Ann. Math.): Yes, always.
For polynomials Denef used Macintyre's p-adic analogue of Tarski-Seidenberg. The general problem was solved by constructing a p-adic analogue of the theory of subanalytic sets. Since the proof of the real "fiber cutting lemma" has no p-adic analogue we invented a new technique that bypasses fibercutting completely, and that also works in the classical real case. We prove our central result (both in the p-adic and in the real case) by an elimination of quantifiers, which adds to the flexibility in working with subanalytic sets. In the real case our theorem can be stated as follows. Equip $I = [-1,1]$ with operations $f : I^m \to I$, one for each power series $f \in \mathbb{R}[[X_1,\dots,X_m]]$ that converges on a neighborhood of I^m and maps I^m into I, and also with the operation $D : I^2 \to I$ given by $D(x,y) = \begin{cases} x/y & \text{if } |x| \le |y|, y \neq 0 \\ 0 & \text{otherwise,} \end{cases}$
and finally with the binary relation <.
Formulas in the language with these basic operations and

relation are called D-analytic formulas.

We now have:

Theorem. Each D-analytic formula is equivalent to one without quantifiers.

As immediate consequences one obtains that the definable subsets of I^m are exactly the subsets of I^m that are subanalytic in $\mathbb{R}^m$, hence that the complement and the interior of a subanalytic set (in any manifold) are subanalytic, etc.

C. ANDRADAS:

ON THE STABILITY INDEX OF EXCELLENT RINGS

We develop a general theory of stability index for excellent rings. Let A be an excellent ring, X_A its real spectrum and $U \subset X_A$ a basic open constructible subset. We define st U = $\min\{k \in \mathbb{N} : U = \{x \in X_A : f_1(x) > 0, \ldots, f_k(x) > 0\}\}$ and st A = $\sup\{\text{st } U\}$. Our approach follows closely the works of Bröcker and Mahé.

(A) Theorem 1: The following are equivalent: (1) st A $< \infty$; (2) st $k(\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} A$.

In fact, only a finite number of $\mathfrak{p}$'s is needed in (2): the zero divisors of A, then the zero divisors of the regular locus of A, and so forth.

Corollary 2: Let B be a f.g. A-algebra or an excellent algebraic extension of A. If st A $<\infty$ then st B $<\infty$.

(B) Next assume that A is henselian, with residue field k and total rings of fractions K. Then

Theorem 3: $\dim A \leq \mathrm{st}\, K \leq \dim A + \mathrm{st}\, k + 1$.
Therefore from Theorem 1 we get st A $<\infty$ iff st k $<\infty$.

(C) Finally we apply the above results to global semianalytic subsets of a complete analytic real manifold M of dimension d.

Theorem 4: Any basic open global semianalytic subset S of M can be written as $S = \{x \in M: f_1(x) > 0, \ldots, f_s(x) > 0\}$ where $f_1, \ldots, f_s$ are analytic on M and

$$s \leq \begin{cases} d(d+2)/4 & \text{if } d \text{ is even} \\ (d+1)^2/4 & \text{if } d \text{ is odd} . \end{cases}$$

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H.W. SCHÜLTING:

HOMOLOGICALLY AND ALGEBRAICALLY TRIVIAL CYCLES

Let X be a projective, nonsingular, n-dimensional variety over $\mathbb{R}$, and let $Z_k(X)$ be the free abelian group generated by the k-dimensional subvarieties of X. There is a canonical homomorphism $cl = cl_k : Z_k(X) \to H_k(X(\mathbb{R}), \mathbb{Z}/2)$. We want to compute the kernel of cl_k. Define $Z_k^{\emptyset}(X) = \{z \in Z_k(X),$ z rational equivalent to a cycle $\Sigma n_i Z_i$, $Z_i(\mathbb{R}) = \emptyset\}$, $Z_k^{th}(X) = \{z \in Z_k(X)$, z rational equivalent to a cycle $\Sigma n_i Z_i$, $\dim Z_i(\mathbb{R}) < k\}$.

1) The solution is well known in the cases k = 0, k = n-1:

$$\text{kern}(cl_K) = Z_k^{\emptyset}(X) \text{ for } k = 0,\ k = n-1$$

2) The following example is constructed:

$X = \text{proj}\ \mathbb{R}[X_0,\dots,X_5]/(X_1^2 + \dots + X_5^2 - X_0^2)$, hence, $X(\mathbb{R}) = S^4$.

There exists a cycle z with $n \cdot z \notin Z_k^{\emptyset}(X)$ for $n > 1$ (but $cl(z) = 0$). It will follow from 4) that $z \in Z_2(X)$.

3) Using a theorem of Ischebeck we can prove: $\text{kern}(cl_k) = Z_k^{th}(X)$ for every k. The main tool is the classification of non-oriented bordism.

4) Using Hironaka's methods on "smoothing" of cylces, we prove $\text{kern}(cl_k) = Z_k^{\emptyset}$ if $k \le \min(3, \frac{n-1}{2})$.

Question: Can the 3 in the above bound be dropped?

G. STENGLE:

A MEASURE OF COMPLEXITY FOR COMPLEX POLYNOMIALS WITH APPLICATIONS TO THE ANGULAR DISTRIBUTION OF ZEROS

Let $P(z)$ be a complex polynomial of degree n. Let $M(P) = \{m_1,\dots,m_k\}$ be the set of nonzero exponents actually appearing in P. The following gives an internal measure of the additive complexity of the set $M(P)$.

Definition. Let M,G be subsets of a commutative semigroup. Let $d(M,G)$ (diameter of M with respect to G) be ∞ if for no k one has $M \subset G \cup (G+G) \cup \dots \cup (G+\dots+G)$ (k summands), and the minimum such k otherwise. Let $\gamma_M(s) = \min\{d(M,G) : |G| = s\}$.

Theorem. There exists a constant C such that if

$$\kappa(P) = \min_{j>0} [(j+1)\log_2 \gamma_{M(P)}(j) + j^2],$$

then any open sector of aperture $\pi/\deg P$ contains no more than $C^{\kappa(P)}$ zeros of P.

The point of this estimate is its independence of the coefficients and degree of P. The proof is a simple application of Khovansky estimates.

Corollary 1. The number of zeros of P in any sector of aperture π/n is no more than C^{k^2}.

Corollary 2. If the set $M(P)$ is an arithmetic progression then the bound of the previous Corollary is polynomial in k.

N.L. ALLING:

FOUNDATIONS OF ANALYSIS OVER SURCOMPLEX NUMBER FIELDS

Over the field of complex numbers $\mathbb{C}$ the following hold.

(A) Locally, the simple roots of $\varphi\hat{}(X) \in \mathbb{C}[X]$ are analytic functions of its coefficients.

(B) Let $\varphi\hat{}(X,Y) \in \mathbb{C}[X,Y]$ and let $(x_o,y_o) \in \mathbb{C}^2$ with $(d\varphi\hat{}/dY)(x_o,y_o) \neq 0$. Let $\varphi\hat{}(x,y) = 0$. Locally about (x_o,y_o), y is an analytic function of x.

(C) Let $\hat{\varphi}_1,\ldots,\hat{\varphi}_n \in \mathbb{C}[X_1,\ldots,X_n]$ define a map from $\mathbb{C}^n$ to $\mathbb{C}^n$, taking $\vec{0}$ to $\vec{0}$, that is non-singular at $\vec{0} \equiv (0,\ldots,0) \in \mathbb{C}^n$. Then it has an analytic inverse, defined in some neighborhood of $\vec{0}$, in the range space.

All of these classical theorems admit generalizations over the surcomplex number fields, and thus - a bit further restricted - over the surreal number fields.

T. RECIO:

THE WIDTH OF A SEMIALGEBRAIC SET AND THE COST OF AN ALGEBRAIC DECISION TREE

Lower bounds for the complexity of the membership problem (to a semialgebraic set $S \subseteq \mathbb{R}^n$) in the model of computation of algebraic decision trees have been systematically obtained either by considering the number of connected components of S or through the notion of "width of a complete proof" as introduced for semilinear sets by Rabin. Several attempts to extend this notion for the non-linear case have failed to produce non-linear lower bounds, as remarked by Ben-Or. On the other hand it has been shown that some problems arising in computational geometry can not be formulated with semilinear tasks. Therefore we have presented here a general definition of the width of a semialgebraic closed set $S \subseteq \mathbb{R}^n$, $w(S) = \min\{r \in \mathbb{N} \mid S = \bigcup_{i \in I} \{x \in \mathbb{R}^n \mid f_{i_1}(x) \geq 0, \ldots, f_{i_r}(x) \geq 0\}$, $f_{ij} \in \mathbb{R}[x]\}$ (a similar definition holds for open sets).
In the general case the width of the congruence class of a semialgebraic set S is defined as $w_{con}(S) = \min\{w(A) \mid A$ closed, s.a.set, $A \Delta S$ of cod. $\geq 1\}$. Then we have the following results:

i) The width of the congruence class of a sa.set S is a lower bound for the cost of any algebraic decision tree solving the membership problem for S.

ii) $\forall S \subseteq \mathbb{R}^n$, s.a., $w_{con}(S) \leq n$. Therefore, taking n as a parameter, at best linear lower bounds can be obtained.

iii) The width of a basic closed s.a. set is in general smaller (and not always equal) to the t-invariant of its complement in $\mathbb{R}^n$, and smaller than the $\bar{s}$-invariant of the given set.

iv) Under certain conditions on $p_1,\dots,p_m \in \mathbb{R}[x]$, it can be shown that $w\{p_1 \geq 0,\dots,p_m \geq 0\} = m$ and that $w_{con}\{p_1 \geq 0,\dots p_m \geq 0\} = m$.

C. SCHEIDERER:

QUOTIENTS OF SEMIALGEBRAIC SPACES

We consider affine semialgebraic (s.a.) spaces over a real closed base field R and s.a. maps between them. For M such a space and $E \subseteq M \times M$ a closed s.a. equivalence relation on it we say that the (geometrical) quotient M/E exists if there is an identifying map $f : M \to N$ to some space N such that $E = M \times_N M$ (that is, the set of equivalence classes, equipped with the quotient topology, carries a (unique) s.a. structure). This notion coincides with that of effective epimorphisms (in the affine category) provided that M is locally complete. G.W. Brumfiel has shown M/E to exist if $p_1, p_2 : E \rightrightarrows M$ are proper. Assuming M locally complete we show that M/E exists if and only if there is some subspace K of M such that $p_1 | p_2^{-1}(K) : p_2^{-1}(K) \to M$ is proper and onto.

(In fact, $E_K := E \cap (K \times K)$ is proper over K in this case, and M/E just "is" K/E_K.) On the other hand, M/E does always exist if $E \rightrightarrows M$ are open (again M locally complete). Examples show that the local completeness hypotheses cannot be dropped. The proofs are given within the set-up of real spectrum. They make essential use of the fact that the theory of real closed fields with compatible non-trivial valuation admits elimination of quantifiers.

K. KURDYKA:

ARCWISE SYMMETRIC SEMI-ALGEBRAIC SETS

A semi-algebraic set E in $\mathbb{R}^n$ is said to be arcwise symmetric iff for each $\gamma : (-1,1) \to \mathbb{R}^n$, analytic arc, $\gamma(-1,0) \subset E$ implies $\gamma(-1,1) \subset E$. The class of all arcwise symmetric sets in $\mathbb{R}^n$ forms a class of closed sets for a noetherian topology. We call it AR-topology. This topology is between Zariski topology and strong topology. We claim that

1) If V is algebraic, $\dim V = k$, there is a 1-1-correspondence between AR-irreducible components of V of dim k and connected components of a resolution of singularities of V.
2) AR-irreducible immerged components correspond to Nash sheets of V.

3) If E is AR-closed, $f : E \to \mathbb{R}^m$ regular injective and proper, then $f(E)$ is AR-closed.

We define a ring of semi-algebraic functions on $\mathbb{R}^n$, which satisfies the following condition: If $\gamma : (0,1) \to \mathbb{R}^n$ is an analytic arc then $f \circ \gamma$ is analytic. We call this ring $A_a(\mathbb{R}^n)$.

4) For each E, AR-closed in $\mathbb{R}^n$, there exists $f \in A_a(\mathbb{R}^n)$ such that $f^{-1}(0) = E$.

5) codim sing(f) ≥ 2.

6) The ring $A_a(\mathbb{R}^n)$ is an integral domain, not noetherian, nor factorial, but Spec A_a is a noetherian space.

7) The Nullstellensatz holds true for $A_a(\mathbb{R}^n)$ (as in complex case).

W. PAWLUCKI:

SETS WITH POLYNOMIAL CUSPS IN APPROXIMATION THEORY

A subset E of $\mathbb{R}^n$ is said to be *uniformly polynomially cuspidal* (UPC), if there are three positive constants M,m,d such that for each point $x \in \bar{E}$ there exists a polynomial map $h_x : \mathbb{R} \to \mathbb{R}^n$ of deg $\leq d$, and such that:

i) $h_x(]0,1]) \subset E$, $h_x(0) = x$, and

ii) $\mathrm{dist}\,(h_x(t), \mathbb{R}^n \setminus E) \geq Mt^n$, for each $x \in \bar{E}$ and $t \in [0,1]$.

Every open bounded subanalytic subset of $\mathbb{R}^n$ is UPC; in particularly every open bounded semi-algebraic subset is UPC. Let E be a UPC subset of $\mathbb{R}^n$. We have the following version of Markov's inequality:
$\|D_p^{\alpha}\|_E \leq C\cdot k^{r|\alpha|}\|p\|_E$, where $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of deg $\leq k$, $\alpha \in \mathbb{N}^n$, $\|p\|_E = \sup\{|p(x)| : x \in E\}$ and positive constants C,r depend only on E. Markov's inequality is used in the proof of the following version of Bernstein's theorem: Assume that F is compact. Let $f : E \to \mathbb{R}$. Let P_k denote the space of polynomials of deg $\leq k$, on $\mathbb{R}^n$. Then the following conditions are equivalent:

1) there exists a C^{∞} extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ of f,
2) for every $r > 0$: $\lim_{k\to\infty} k^r\cdot \mathrm{dist}_E(f,P_k) = 0$ (where $\mathrm{dist}_E(f,P_k) \overset{df}{=} \inf\{\|f-p\|_E : p \in P_k\}$).

J. BOCHNAK:

ALGEBRAIC MAPPINGS INTO S^{2n}

Working jointly with W. Kucharz we got several results concerning the structure of real algebraic (i.e. regular) mappings between real algebraic sets. Here is a sample of results (all spaces are supposed to be compact and connected).

Th. 1. Let M be a C^∞ surface. Then the following conditions are equivalent:

i) M is homeomorphic to S^1, $\mathbb{P}^2(\mathbb{R})$ or the Klein bottle.

ii) For each nonsingular real algebraic set X, diffeomorphic to M, the set $R(X,S^1)$ is dense in $C^\infty(X,S^1)$ (in C^∞ topology).

Notation. Given a nonsingular real algebraic set, $R(X,S^n)$ (resp. $C^\infty(X,S_n)$) denotes the set of regular (resp. C^∞) mappings from X into S^n.

Th. 2. Let M be a C^∞ surface. Then the following conditions are equivalent:

i) M is nonorientable of odd genus;

ii) for each nonsingular real algebraic set X diffeomorphic to M, the set $R(X,S^2)$ is dense in $C^\infty(X,S^2)$.

Th. 3. Let M be a C^∞ surface. Assume that M is either orientable, or nonorientable of even genus. Then there is a nonsingular real algebraic set X diffeomorphic to M, such that each $f \in R(X,S^2)$ is homotopic to a constant.

Th. 4. Let M be an orientable 4-dimensional C^∞ manifold. The following are equivalent:

i) The signature $\sigma(M)$ of M is 0;

ii) there is a nonsingular real algebraic set X diffeomorphic to M, such that each $f \in R(X,S^4)$ is homotopic to a constant.

Th. 5, 6, 7, ... etc.

J.J. ETAYO:

N-GONAL CYCLIC REAL ALGEBRAIC CURVES

Let C be a projective irreducible non-singular algebraic curve defined over $\mathbb{R}$, whose real part $C(\mathbb{R})$ is Zariski-dense in C. We denote by p the genus of C and by k the number of connected components of $C(\mathbb{R})$. C is called N-gonal cyclic if there exists a birational isomorphism φ of C, of order N, such that C/φ is rational, and $\varphi(C(\mathbb{R})) \subset C(\mathbb{R})$ and $(C/\varphi)(\mathbb{R}) = C(\mathbb{R})/\varphi|_{C(\mathbb{R})}$.

We study here the existence of such a curve according to the values of the couple (p,k) distinguishing whether $C \setminus C(\mathbb{R})$ is connected or not. Moreover the characterization we obtain allows us to decide in an effective way the branching orders of the projection $C(\mathbb{R}) \rightarrow C(\mathbb{R})/\varphi$.

In a second step we calculate, given the number N, the minimum genus of an N-gonal cyclic curve, and determine the topological types of the curves achieving this bound.

The technique we use involves Klein surfaces and NEC groups.

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M. SEPPÄLÄ:

COMPLEX CURVES WITH REAL MODULI

The moduli space M^g of smooth complex algebraic curves of genus g, $g \geq 2$, is a quasiprojective variety defined over $\mathbb{Q}$. M^g can be embedded in a projective space $\mathbb{P}^N(\mathbb{C})$ in such a way that the complex conjugation in $\mathbb{P}^N(\mathbb{C})$, restricted to M^g, is the mapping $M^g \to M^g$, $[C] \to [\overline{C}]$. Here $\overline{C}$ is the complex conjugate of the curve C.

We show, applying the above observation, that the real part of M^g, $M^g(\mathbb{R})$ is <u>the moduli space of genus g converings of real algebraic curves</u>.

Using Teichmüller theory we can analyze the topological, analytic and algebraic structures of $M^g(\mathbb{R})$ and its various parts.

Use the notation

$$M^{p,n}_{\mathbb{R}} = \left\{ \begin{array}{l} \text{complex isomorphism classes of real algebraic} \\ \text{curves of genus } p \text{ with } n \text{ distinguished points} \end{array} \right\}.$$

<u>Thm.</u> $M^{g,0}_{\mathbb{R}}$ is the closure of the regular part of $M^g(\mathbb{R})$ provided that $g \geq 4$.

<u>Cor.</u> $M^{g,0}_{\mathbb{R}}$ is semialgebraic, $g \geq 4$.

These results hold also for the moduli spaces $M^{p,n}$ provided that $p \geq 4$ or $p = 3$ and $n > 9$ or $p = 2$ and $n > 6$.

E. BIERSTONE:

UNIFORMIZATION AND RECTILINEARIZATION OF SUBANALYTIC SETS

Hironaka has used his desingularization and local flattening theorems to prove the following fundamental results: Let M be a real analytic manifold and let X be a subanalytic subset of M.

Uniformization theorem. Suppose X is closed. Then there is a real analytic manifold N (dim N = dim X) and a proper real analytic mapping $\varphi : N \to M$ such that $\varphi(N) = X$.

Rectilinearization theorem. Let $K \subset M$ be compact. Then there are finitely many real analytic mappings $\varphi_i : \mathbb{R}^m \to M$ ($m = \dim M$) such that:

(1) There are compact subsets K_i of $\mathbb{R}^m$, such that $\cup\varphi_i(K_i)$ is a neighborhood of K in M.

(2) For each i, $\varphi_i^{-1}(X)$ is a union of quadrants.

Elementary proofs of these results, using neither resolution of singularities nor local flattening are presented in this talk and the following one by P. Milman. Our approach stands in the same relation to local resolution of singularities of real or complex analytic spaces as Zariski's uniformization theorem does to desingularization of algebraic varieties.

P. MILMAN:

TRANSFORMING AN ANALYTIC FUNCTION TO NORMAL CROSSINGS BY BLOWING-UP

In this talk we presented an elementary proof of the following theorem (a variant of desingularization).

<u>Theorem.</u> Let M be an analytic manifold (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let $f \in \mathcal{O}(M)$. (Assume that f does not vanish identically on any component of M.) Then there is a countable collection of analytic mappings $\pi_j : W_j \to M$ such that:

(1) Each π_j is the composition of a finite sequence of local blowing-ups (with smooth centers).

(2) There is a locally finite open covering $\{V_j\}$ of M such that that $\pi_j(W_j) \subset V_j$, for all j.

(3) If K is a compact subset of M, then there are compact subsets $L_j \subset W_j$ such that $K = \bigcup_j \pi_j(L_j)$. (The union is finite by (2).)

(4) For each j, $f \circ \pi_j$ is locally normal crossings on W_j.

L. BRÖCKER:

SEPARATION OF SEMIALGEBRAIC SETS BY POLYNOMIALS AND NASH FUNCTIONS

Let V^n be a real algebraic affine variety of dimension n. Let $A,B \subset V(R)$ be disjoint basic closed semialgebraic subsets.

<u>Theorem.</u> a) If $n \leq 2$, then A and B can be separated by a polynomial.

b) If $n \geq 3$ there exist semialgebraic disjoint closed basic sets which cannot be separated by a polynomial.

Part b) is proved by an explicit counterexample. Part a) follows from the following result on abstract spaces of orderings:

<u>Theorem.</u> Let (X,G) be a space of orderings and let A,B be open and closed disjoint constructible subsets of X. Then A and B can be separated by an element $g \in G$ iff this holds for the restriction to all finite subspaces of X.

More generally, the Mostowski number m(A,B) is considered for any pair of disjoint semialgebraic subsets $A,B \subset V(R)$, A,B closed. This is the minimal number m such that $f \in R[V][\sqrt{q_1},\dots,\sqrt{q_m}]$ separates A and B, where the $q_i \in R[V]$ are strictly positive. Estimates are given for the Mostowski number $m(n) = \sup\{m(V): \dim V = n\}$ (where $m(V) = \sup\{m(A,B): A,B \subset V(\mathbb{R})\}$), namely $n-1 \leq m(n) \leq \bar{s}(n)\bar{t}(n)$ for $n \geq 2$. Here $\bar{s}(n)\bar{t}(n)$ is the minimal number of polynomials which describes an arbitrary closed semialgebraic set as a union of basic closed sets.

D. TROTMAN:

ON TRANSVERSALS TO SEMIALGEBRAIC/SUBANALYTIC SETS

The notion of transversal submanifold to an embedded singular space is in terms of transversality to the strata of a C^1 stratification.

Thm. 1. In $C^1(\mathbb{R}^n, \mathbb{R}^p)$ the set of transversal maps to a C^1 stratification Σ of a closed subset Z of $\mathbb{R}^p$ is open iff Σ is Whitney A-regular.

It is thus interesting to study transversals to A-regular stratifications.

Thm. 2. The topological type of the germ at 0 of the intersection $X \cap T$, for T a C^1 transversal at 0 to Y, where (X,Y) are C^1 A-regular strata, is independent of T.

Let $\gamma : E \to \mathbb{R}^n$ be the blowing up of $\mathbb{R}^n$ at the origin. If (X,Y) is Whitney A-regular in $\mathbb{R}^n$, $\gamma^{-1}(X)$ is Verdier W-regular over $\mathbb{P}_o^{n-1}$ = space of hyperplanes transverse to Y.
Thm. 2 follows.
Transversals to $\mathbb{P}_o^{n-1}$ of class C^k are sent by γ to transversals to Y of class C_+^k. We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is of class C_+^k if $f(x) = \sum_{i=1}^{n} x_i\, g_i(x)$ where g_i are of class C^k.

We deduce by stratification arguments and by a 1985 (Topology) theorem that "transverse C_+^1 transversals $\to$ finitely many homeomorphic C_+^1 transversals" for subanalytic sets. This provides a partial confirmation of a conjecture of the author. We are also led to consider Lipschitz transversals, Hölderian C^α transversals in this semialgebraic context.

Z. SZAFRANIEC:

ON THE TOPOLOGICAL INVARIANTS OF GERMS OF ANALYTIC FUNCTIONS

There is given a definition of germs with property A_d (for example, each homogeneous polynomial of degree d has property A_d). Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ with property A_d, and let $f_{\mathbb{C}}$ be the complexification of f. Let us denote by $e(f_{\mathbb{C}})$ the Euler characteristic of the Milnor's fiber of $f_{\mathbb{C}}$. The main result is:

$$\frac{1}{2}\chi(L) \equiv \frac{1}{2}\chi(S^{n-1}) + \frac{1}{d}\, e(f_{\mathbb{C}}) \mod 2$$

where $L = f^{-1}(0) \cap S_r$, r small. This theorem is a generalization of a theorem which was proved by C.T.C. Wall (Topology, 1983).

Z. DENKOWSKA:

APPLICATIONS OF SUBANALYTIC SETS

The following (already classical)

Thm. 1. Let E be a subanalytic (semian.) rel. compact subset of $M \times N$, M,N real analytic manifolds, $\pi : M \times N \to N$ the projection. Then there is a uniform bound C for the number of of connected components of $E_y, y \in N$.

permits to obtain the following result related to Hilbert's 16th problem:

Thm. (Françoise, Pugh 1986) Fix $T > 0$ and $d \in \mathbb{N}$. Then the number of limit cycles of the dynamical system $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ with f,g polynomials of deg $\leq d$, having period $\leq T$, is uniformly bounded by a constant B(d,T).

We obtain this result by applying Thm. 1 to
$A = \{(t,\xi,v) \in [0,T] \times \mathbb{D} \times S : \varphi_v(t,\xi) = \xi\}$,
where $\mathbb{D}$ is the Poincaré compactification of $\mathbb{R}^2$, S is the unit sphere in the space of polynomial vector fields of deg $\leq d$, and φ_v is the flow of v.

But the estimate here is not explicit. Another subanalytic theorem that is likely to apply (lengths of orbits?) is

Thm. 2. Let E,M,N be as in Thm. 1. Then there exists a uniform bound C for the lengths of arcs in the fibres E_y.

R. SILHOL:

REAL CUBIC SURFACES (REVISITED?)

A classification of real cubic surfaces can be made in terms of the number of real lines and the number of connected components. It goes as follows:

type	F_1	F_2	F_3	F_4	F_5
# real line	27	15	7	3	3
# components	1	1	1	1	2

It is well known that surfaces of type F_1 to F_4 are all birationally isomorphic (over $\mathbb{R}$). But the fact that the surfaces of type F_5 are all birationally isomorphic seems to be unknown or at least forgotten. This is what we prove.

The idea of the proof is to relate such surfaces to surfaces, defined by $x^2+y^2 = g(t)$, g a function regular on $\mathbb{P}^1(\mathbb{R})$ with 4 real simple zeros. Modulo an elementary transformation such a surface is isomorphic (over $\mathbb{C}$) to $\mathbb{P}^2_{\mathbb{C}}$ blown up in 5 points. The real part has 2 components, and blowing up one more real point gives a real cubic surface of type F_5. All surfaces of type F_5 are obtained in this way. To see this let W be a surface of this type and D a real line in it. If P is a plane passing through D, $P \cap (W \setminus D)$ is a conic. Using classical theory one can prove that the fibration of $W \setminus D$ thus obtained is of the preceding type.
A priori the birational isomorphism class of the surface depends on the cross-ratio of the 4 zeros of g. Since we

have 3 real lines and hence 3 fibrations, the proof reduces to building a cubic surface such that 2 of the fibrations correspond to 2 given cross-ratios.

B. REZNICK:

IF $P(X_1,\dots,X_n)$ IS A POSITIVE SEMI-DEFINITE FORM, WHEN IS $P(X_1^k,\dots,X_n^k)$ A SUM OF SQUARES?

Let $p = p(x_1,\dots,x_n)$ be a real psd form of degree m and $Y(p) = \{k\colon p(x_1^k,\dots,x_n^k)$ is sos (= sum of squares)$\}$. Clearly $rk \in Y(p)$ if $k \in Y(p)$. The only other information comes from explicit examples. An <u>agiform</u> is a psd form with shape $c(\lambda_1 \underline{x}^{\underline{u}_1} + \dots + \lambda_n \underline{x}^{\underline{u}_n} - \underline{x}^{\underline{w}})$ where $c > 0$, $\lambda_i \geq 0$, $\Sigma\lambda_i = 1$ and $\underline{w} = \Sigma\lambda_i \underline{u}_i$ is a lattice point. These occur in the literature from Hurwitz, Motzkin, Choi, Lam and Reznick. Using a purely geometric criterion for agiforms to be sos, one can show that there exist agiforms p_s, q_s such that $Y(p_s) = \{s,s+1,s+2,\dots\}$ and $Y(q_s) = \{2,4,\dots,2s-2,2s,2s+1,2s+2,\dots\}$. By contrast, $Y(H) = \emptyset$ for the psd (non-agiform) Horn form H. in 5 variables. Questions: Does there exist a form p in ≤ 4 variables for which $Y(p) = \emptyset$? If $Y(p) \neq \emptyset$, does $Y(p)$ contain all but finitely many k? etc. Conjecture: If p is psd then there exists an invertible linear transformation T such that for $q(\underline{x}) = p(T\underline{x})$ there is an odd $k \in Y(q)$. It would imply that every psd p can be written as sos of forms in the "fractional" variables $(\Sigma a_{ij}\ x_j)^{1/k}$, $1 \leq i \leq n$.

K. REICHARD:

SOME REMARKS ON REAL POLYNOMIAL MAPPINGS WITH CONSTANT DETERMINANT

A famous conjecture says: If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial mapping with $\det(\partial F/\partial x) \equiv 1$, then there exists a polynomial inverse mapping. This conjecture is not yet proven. But the following weaker special case is true: If $n = 2$ and F is as above, then F is injective. It follows from a theorem of Borel that F is bijective, and so has a real-analytic inverse mapping.

M. SHIOTA:

NASH MANIFOLDS AND NONSINGULAR REAL ALGEBRAIC VARIETIES

A C^ω affine Nash manifold is by definition a C^ω submanifold of $\mathbb{R}^n$ which is also semialgebraic. Let r be a natural number or ω. Then a C^r Nash map between affine C^ω Nash manifolds is by definition a C^r map with semialgebraic graph.

Theorem. An affine C^ω Nash manifold is C^ω Nash diffeomorphic to some nonsingular affine algebraic variety.

The compact case is equivalent to the well-known theorem of Tognoli. The noncompact case is pastly proved by Akbulut-King. Their famous theorem is that the interior of a compact C^ω manifold is C^ω diffeomorphic to some nonsingular affine algebraic variety. We see easily that this is equivalent to the statement that an affine C^ω Nash manifold which is non-compact is C^ω diffeomorphic to some nonsingular affine algebraic variety. However the theorem does not follow automatically from this because there exist two affine C^ω Nash manifolds which are C^ω diffeomorphic but not C^ω Nash diffeomorphic.

For the proof of theorem we need some facts about Nash manifolds and some topological methods. The most important fact is as follows. We can construct a topology on the set of C^r Nash maps between affine C^ω Nash manifolds so that a close approximation of a C^r Nash diffeomorphism is a diffeomorphism and that we can approximate a C^r Nash map by a C^ω one.

J.P. FRANCOISE:

REAL TORII AND JACOBIANS ASSOCIATED TO A.C.I. SYSTEMS

M. Adler and P. van Moerbeke introduced the notion of algebraically completely integrable Hamiltonian systems. For these systems one can compute the actions and relate the singularities of these actions to singularities of period mappings of families of Riemann surfaces. Some problems of real algebraic geometry nature arise to find the real invariant torii inside the complex torii (Jacobians or Prym variety) or to understand the monodromy of the actions. We gave a complete description of the three cases of integrability of the motion of a solid body about a fixed point as an illustration.

F. CUCKER:

COMPUTATION OF THE ANALYTIC STRUCTURE OF A REAL ALGEBRAIC CURVE

Given a polynomial $F(X,Y) \in \mathbb{Z}[X,Y]$ and its zero set $C \subseteq \mathbb{R}^2$, some algorithms have recently been given which compute the topology of C, i.e. they produce a planar graph homeomorphic to C.

In our work an improvement of this algorithm is given, which outputs the same graph with the edges numbered in such a way that they share the number iff they belong to the same global analytic component.

To do so we calculate the discriminant locus, the points of the curve lying over this set and for each of these points the half-branches of the curve centered at it. Once we have this information, edges of the same global component are collected together by just a pursuit of the component in the graph: At each critical point we follow a half-branch in the half-branch that gives us the whole branch.

The computation of the branches (and half-branches) is performed using

a) Duval's algorithm for rational Puiseux expansions, and

b) Coste and Roy's codification of real algebraic numbers.

E. BECKER:

SOME REMARKS ON THE REAL SPECTRUM UNDER REAL CLOSED FIELD EXTENSIONS

Let R be a real closed field and $V|R$ an affine real variety. Then V(R) can be naturally identified with a dense subspace of the compact space Max Sper R[V] =: $\hat{V}(R)$. In order to understand this compactification one may first study varieties over $\mathbb{R}$ and investigate thereafter the behaviour of $\hat{V}(R)$

under real closed field extensions.

Prop. 1. Let $V/\mathbb{R}$ be an integral curve.

i) $\#(\hat{V}(\mathbb{R}) \setminus V(\mathbb{R})) = 2 \cdot \#(Y(\mathbb{R}) - \tilde{V}(\mathbb{R}))$, where Y is the smooth proj. model of $\mathbb{R}(V)$ and $\tilde{V}$ is the normalization of V;

ii) If $K \subset \mathbb{R}$, K real closed, then $\hat{V}(K)$ and $\hat{V}(\mathbb{R})$ are homeomorphic.

For a more general study let $K \subset R$ be an extension of real closed fields, V be a real variety $/K$, $A := K[V]$, $\tilde{V}$ the base extension of V to R, $B = R[\tilde{V}] = K[V] \otimes_K R$.

Prop. 2. i) B is also a real algebra;

ii) Every $\alpha \in \mathrm{Sper}\, A$ extends to $\beta \in \mathrm{Sper}\, B$ with $\dim \alpha = \dim \beta$;

iii) If $R|K$ is Archimedean then $\mathrm{res} : \mathrm{Sper}\, B \to \mathrm{Sper}\, A$ induces a surjective map $\mathrm{res} : \hat{\tilde{V}}(R) \to \hat{V}(K)$; if $R|K$ is even dense, $\mathrm{res}|_{\tilde{V}(R)} : \tilde{V}(R) \to \hat{V}(K)$ is injective;

iv) If $R = \mathbb{R}$ we have a surjective map $\mathrm{res} : \hat{\tilde{V}}(\mathbb{R}) \to \hat{V}(K)$, injective on $\tilde{V}(R)$. If in addition V is a curve or $\tilde{V}(\mathbb{R})$ is compact then this map is a homeomorphism;

v) If $K \subsetneq \mathbb{R}$ then $\mathrm{res} : \mathrm{Max\,Sper}\, \mathbb{R}[X,Y] \twoheadrightarrow \mathrm{Max\,Sper}\, K[X,Y]$ is not injective.

D. GONDARD

FIRST ORDER AXIOMS FOR CHAIN CLOSED FIELDS AND 17TH HILBERT'S PROBLEM AT LEVEL N

Let $\mathcal{L}$ be the language of fields, and α an additional constant symbol.

In the first part we give axioms in the language $\mathcal{L}(\alpha)$ for a field K to be the chain closed. Also an axiomatization of the theory of chain closed fields in $\mathcal{L}$ is given. These and other results on the model theory of chain closed fields will appear in C.R.A.S. 1987. The 2^{nd} part is joint work with F. Delon (Paris 7). Some results:

Thm. Let K be chain closed, such there is only one henselian valuation with real closed residue field on K. For $f \in K(x)$, the following are equivalent:

i) $f \in \Sigma(K(x))^{2^n}$;

ii) for every real algebraic extension L/K, f satisfies $f(x) \in L^{2^n}$ $\forall x \in L$.

Thm. For chain closed fields $K \subset L$, the following are equivalent:

i) $K \cap L^2 = K^2$

ii) K is relatively algebraically closed in L

iii) $K \prec L$.

G. BRUMFIEL:

I. A FIXED POINT THEOREM FOR SEMI-ALGEBRAIC SETS

Theorem. Let $X \subset \mathbb{R}^n$ be any semi-algebraic set, $f : X \to X$ a continuous semi-algebraic map, $tr(f_*) = \Sigma(-1)^i$ trace $(H_i(X) \to H_i(X))$. Let $\tilde{X} \subset \mathrm{Spec}_R\, \mathbb{R}[x_1,\ldots,x_n]$ be the constructible associated to X, and $\tilde{f} : \tilde{X} \to \tilde{X}$ the map associated to f. Then if $tr(f_*) \neq 0$, $\tilde{f}$ has a fixed point.

II. THE TREE OF A NON-ARCHIMEDEAN HYPERBOLIC PLANE

If $\Lambda \leq (\mathbb{R},+)$ is an ordered group, a Λ-tree (more precisely, its set of vertices) is a metric space $d : T \times T \to \Lambda$ which satisfies certain axioms such as (1) each pair of points of T are endpoints of a unique segment (= subspace isometric to an interval in Λ); (2) if s_1, s_2 are segments and $s_1 \cap s_2$ = point then $s_1 \cup s_2$ is a segment; (3) if two segments s_i have an endpoint of one in common, then $s_1 \cap s_2$ is a segment.

Let F be a non-Archimedean ordered field, and $\mathbb{H}F^2$ the hyperbolic plane with cross-ratio distance $D(A,B) \geq 1$ between points of $\mathbb{H}F^2$. Define $d(A,B) = \log D(A,B) \in \mathbb{R}$, where $\log : F^*_+ \to \mathbb{R}$ is log with base a big element $b > 0$ (b big means $\forall a \in F$, $a \leq b^n$, some n). Then $\log|.| : F^* \to \mathbb{R}$ is a valuation with value group $\Lambda \subset \mathbb{R}$.

Theorem. $\mathbb{H}F^2/\sim$ with metric d is a Λ-tree, where $A \sim B$ means $d(A,B) = 0$.

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