MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Algebraische K-Theorie

5.6. bis 11.6.1988

Die Tagung fand unter der Leitung von Herrn R.K. Dennis (Ithaca) und Herrn U. Rehmann (Bielefeld) statt.

Es wurden Vorträge aus unterschiedlichen Gebieten der algebraischen K-Theorie gehalten und neue Entwicklungen des jeweiligen Teilgebietes vorgestellt.

Vortragsauszüge

H. HURRELBRINK: On $K_2(\mathcal{O}_F)$

For long time, we have known the structure of the K-groups $K_1(\mathcal{O}_F)$ for rings $\mathcal{O}_F$ of integers of every number field F (Dirichlet). Recently we also learned the structure of the K-groups $K_3(\mathcal{O}_F)$ for every number field F (Merkurjev, Suslin). As of today, the information about the structure of the finite abelian K-groups $K_2(\mathcal{O}_F)$ is still limited.

We proposed the study of the structure of $K_2(\mathcal{O}_F)$ modulo the knowledge of the structure of related S-class groups, and exhibited 4-rank formulas for $K_2(\mathcal{O}_F)$. This led to a characterization of all number fields F with a wild kernel (Hilbert kernel) of odd order, and the determination of infinite families of number fields F for which the structure of the 2-primary subgroup of $K_2(\mathcal{O}_F)$ can be determined.

M. ROST. Hilbert's Satz 90 in Milnor K-Theory

For a quadratic extension $L = F(\sqrt{a})$, Char $F \neq 2$, Hilbert's Satz 90 states that the following sequence is exact.

$$K_n^M L \xrightarrow{1-\sigma} K_n^M L \xrightarrow{N_{L/F}} K_n^M F$$

Here K_n^M denotes Milnor K-Theory and σ is the generator of Gal(L/F). Hilbert's Satz 90 for quadratic extensions is proved for $n \leq 4$. The method of proof is to use specialization arguments relating Hilbert's Satz 90 to certain homology groups of the localization sequence in Milnor K-Theory for quadrics defined by Pfisterforms. In computing these groups one is led to consider the complex

$$\bigoplus_{v \in X_{(1)}} K_{n+1}^M K(v) \xrightarrow{d} \bigoplus_{v \in X_{(0)}} K_n^M K(v) \xrightarrow{N} K_n^M F$$

for (projective) quadrics X (where d is given by the tame symbol and $N = \sum_v N_{K(v)\,F}$). The exactness of this complex is proved for $n \leq 1$ if the form defining X is of type $\psi \oplus c\psi' \oplus \langle d \rangle$, where $\psi = \psi' \oplus \psi''$ is a Pfisterform and for $n = 2$, dim $X \leq 2$ (which leads to a proof for Hilbert's Satz 90 for $n \leq 3$, $n = 4$ respectively) and for $n = 3$, dim $X = 1$.

B. OLIVER: Whitehead groups of finite groups

This talk was a summary of current knowledge of the groups $K_1(\mathbb{Z}G)$ and $Wh(G)$ for finite groups G. By results of Bass, they are finitely generated, and their ranks are known. Also, by a theorem of Wall, the torsion subgroup of $Wh(G)$ is precisely the group

$$SK_1(\mathbb{Z}G) = \mathrm{Ker}[K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G)] = \mathrm{Ker}[nr: K_1(\mathbb{Z}G) \to Z(\mathbb{Q}G)^*].$$

Localization sequences are needed to make systematic computations of the $SK_1(\mathbb{Z}G)$. One way to see these is to consider the relative K-theory exact sequences (for all $n > 1$):

$$K_2(\mathbb{Z}/n[G]) \to SK_1(\mathbb{Z}G, n\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \to K_1(\mathbb{Z}/n[G]).$$

Upon taking the inverse limit over all n, this gives an exact sequence

$$\prod_p K_2^c(\hat{\mathbb{Z}}_p G) \to \varprojlim_n SK_1(\mathbb{Z}G, n\,\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \xrightarrow{\ell} \prod_p SK_1(\hat{\mathbb{Z}}_p G) \to 1.$$

For any $\mathbb{Z}$-order $\mathcal{A}$ in a finite dimensional semisimple $\mathbb{Q}$-algebra A, $\varprojlim SK_1(\mathcal{A}, n\mathcal{A})$ vanishes iff the congruence subgroup problem holds for $\mathcal{A}$; i.e., iff any subgroup of finite index in $SL_r(\mathcal{A})$ $(r \geq 3)$ contains some congruence subgroup $SL_r(\mathcal{A}, n\mathcal{A})$. The group

$$C(A) = \varprojlim_n SK_1(\mathcal{A}, n\mathcal{A}) \simeq \mathrm{Coker}[K_2(A) \to \bigoplus_p K_2^c(\hat{A}_p)]$$

is independent of $\mathcal{A}$; and in many cases - including the case $A = \mathbb{Q}G$ - has been completely described in works of Bass, Milnor, Serre; Bak, Rehmann, Prasad, Raghanathan, and others.

The $SK_1(\mathbb{Z}G)$ are thus described by 2 exact sequences

$$1 \to Cl_1(\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \xrightarrow{\ell} \prod_p SK_1(\hat{\mathbb{Z}}_p G) \to 1 \qquad (Cl_1(\mathbb{Z}G) := \mathrm{Ker}(\ell))$$

and $\prod_p K_2^c(\hat{\mathbb{Z}}_p G) \to C(\mathbb{Q}G) \to Cl_1(\mathbb{Z}G) \to 1.$

The $SK_1(\mathbb{Z}_p G)$ can be described precisely, for any finite G, in terms of the functor $H_2(-)$ applied to subquotients of G. The map ℓ is naturally split in odd torsion. Formulas for the odd torsion in $Cl_1(\mathbb{Z}G)$ are known. For example, if G is a p-group for odd p, if $\mathbb{Q}G \simeq \prod_{i=1}^k A_i$, $A_i \simeq M_{r_i}(F_i)$, has irreducible representation V_i, and

$F_i = \mathbb{Q}(\mu_i)$ where μ_i is a group of p-power roots of unity, then:

$$Cl_1(\mathbb{Z}G) \simeq \left[\prod_{i=1}^{k} \mu_i\right] / \langle \psi(g \otimes h) : g,h \in G,\ gh = hg \rangle$$

where

$$\psi(g \otimes h) = (\det_{F_i}(g, V_i^h))_{i=1}^k. \qquad (V_i^h = \{x \in V_i \mid hx = x\})$$

A. BAK: The structure of classical groups below the stable range and nonabelian K-Theory

Let A denote an associative ring which is finite over a commutative ring with 1. Let $G_n(A)$, $n \geq 3$, denote a classical group over A, i.e., either $G_n(A) = GL_n(A)$ or $G_n(A)$ is the automorphism group of a nonsingular form of Witt index $\geq n$. Let $E_n(A)$ denote the elementary subgroup of $G_n(A)$. Algebraic K-theory treats the groups $G(A) = \lim\limits_{n} G_n(A)$ and via stability theory, one can apply K-theory to obtain information about certain subquotients of $G_n(A)$, for example $G_n(A)/E_n(A)$, providing $n > sr(A)$ = stable range of A. Until recently, almost nothing was known about $G_n(A)/E_n(A)$ when $n \leq sr(A)$, one reason being that there is no K-theory for these groups. The following results close these gaps.

THEOREM A. There is a filtration $G_n^{-1} = G_n \supset G_n^o \supset \dots G_n^j \supset \dots E_n$, functorial in A, satisfying:

(1) $G_n^j(A) \triangleleft G_n(A)$.

(2) If A is commutative and $G_n = GL_n$ then $G_n^o(A) = SL_n(A)$.

(3) $G_n^{-1}(A)/G_n^o(A)$ is abelian.

(4) $G_n^o(A) \supset G_n^1(A) \supset \dots \supset G_n^j(A) \quad \dots$ is a descending central series.

THEOREM B. If $sr(A)$ is finite then $G_n^i(A) = E_n(A)$ whenever $i \geq sr(A)$.

Theorem B says that $G_n^o(A)/E_n(A)$ is nilpotent of class $\leq sr(A)$. This result can be improved to the following: if $z \in \mathbb{Z}$, let $[z] = z$ if $z \geq o$, and o if $z \leq o$.

THEOREM C. If $sr(A)$ is finite then $G_n^o(A)/E_n(A)$ is nilpotent of class $\leq 1 + [sr(A) + 2-n]$.

The results above are proved by introducing 'nonabelian K-theory'.

For each functor G_n^i above an algebraic K-theory with K-theory groups $K_jG_n^i$ ($j \geq 1$) is defined such that $K_1G_n^i(A) = G_n^i(A)/E_n(A)$. Whereas, K_j for $j \geq 2$ is always abelian, K_1 is not necessarily abelian, hence the rubric 'nonabelian K-theory'. The main theorems are deduced with the help of certain exact Mayer-Vietoris sequences for the K-theory above, in particular the M.-V.-sequence associated to a localization-completion square.

L. VASERSTEIN: Structure of gauge groups

Let $G = G(\mathbb{R})$ be a simple Lie group. E. Cartan and van der Waerden proved that $G(\mathbb{R})^o$/center is simple as an abstract group. Let A be a ring of continuous functions $X \to \mathbb{R}$ on a topological space X. Assume that $A \supset \mathbb{R}$ and GL_1A is open in A. We define $G(A)$ as a subgroup in the group of continuous maps $X \to G$. When $X = S^1$, these groups are known as loop groups. In general, they appear in mathematical physics as gauge groups. Assume that G is of classical type or splits (e.g. G is complex) (this condition probably is not necessary) and that there are N roots of 1 for elements of A or $A[i]$ close to 1 (where N is a certain number depending on G). Then a subgroup H of $G(A)$ is normalized by $G(A)^o$ iff $G(B)^o \subset H \subset G(B)$ for an ideal B of A.
When $X = \{\text{point}\}$, this is the Cartan - van der Waerden result. When $X = S^1$, the maximal normal subgroups of $G(A)^o$ were described by de la Harpe and (for some G) Segal-Pressley (they use $G(B)$ with maximal ideals B of A).

W. RASKIND: Some Remarks on $H^1(X,\underline{\underline{K}}_2)$ of Curves

Let X be a smooth, projective, geometrically connected curve over a number field k and set

$$V(X) =: \operatorname{Ker}(H^1(X,\underline{\underline{K}}_2) \xrightarrow{N} k^*).$$

A conjecture of Bloch and a more general conjecture of Vaserstein say that $V(X)$ should be a torsion group. Let now $\bar{k}$ be an algebraic closure of k and $\bar{X} = X\times_k\bar{k}$. Then one can easily show that $V(X)$ is torsion if and only if

$$V(\bar{X})^{\mathrm{Gal}(\bar{k}/k)} = 0.$$

In this lecture I stated and outlined the proof of the following

THEOREM: Let X be as above with $X(k) \neq \emptyset$. Then the natural map

$$V(X) \longrightarrow V(\bar{X})^{Gal(\bar{k}/k)}$$

is surjective.

Since $V(\bar{X})^{Gal(\bar{k}/k)}$ is uniquely divisible, the theorem states that either $V(X)$ is a torsion group or it is quite large.

The proof of the theorem uses results of Saito to prove the corresponding local statement and then a recent theorem of Jannsen to pass from the local to the global.

J. URBANOWICZ: Connections between $|K_2\mathcal{O}_F|$ for real quadratic fields F and class numbers of appropriate imaginary quadratic fields

I gave some connections between the order of the group $K_2\mathcal{O}_F$ for general quadratic fields F and class numbers of appropriate imaginary quadratic fields. I applied an old series (see the paper of M. Lerch in Acta Mathematica, 1905). From the obtained formulas we got some congruences for $|K_2\mathcal{O}_F|$ modulo powers of 2. These congruences are more general and modulo larger powers of 2 ones of Gras (see Manuscripta Math. 57(1987), 373-415). We got the exact divisibilities of $|K_2\mathcal{O}_F|$ by powers of 2 which then answered questions (conjectures) of Candiotti (Acta Arithm., to appear).

R.W. THOMASON: Higher Algebraic K-theory of schemes and of derived categories (joint work with Thomas F. Trobaugh (†))

Let X be a quasiseparated and quasicompact scheme. Recall from SGA 6 Grothendieck's notion of a perfect complex on X. This is a complex of $\mathcal{O}_X$-modules which is locally quasi-isomorphic to a bounded complex of algebraic vector bundles. Using quasi-isomorphisms as the weak equivalences this is a category with cofibrations and weak equivalences in the sense of Waldhausen. His work then defines a K-theory spectrum $K(X)$. When X has an ample family of line bundles, for example when X is quasiprojective over an affine scheme or is regular noetherian, then this K(X) is homotopy equivalent to Quillen's K(X).

KEY LEMMA: Let U be a quasicompact open in X. A perfect complex $F^{\cdot}$ on U is the restriction of some perfect complex on X up to quasi-isomorphism iff the class $[F^{\cdot}] \in K_o(U)$ is the image of $K_o(X)$.

Using this, and techniques of Waldhausen K-theory, we prove:

THEOREM 1: (Bass Fundamental Thm.) There is a functorial spectrum $K^B(X)$ such that

a) $K_n^B(X) = K_n(X)$ for all integers $n \geq o$.

b) there is an exact sequence for all $n \in \mathbb{Z}$:

$$0 \to K_n^B(X) \to K_n^B(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[T]) \oplus K_n^B(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[T^{-1}]) \to K_n^B(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[T,T^{-1}]) \xrightarrow{\partial} K_{n-1}^B(X) \to 0$$

with ∂ naturally split by multiplication by $T \in K_1(\mathbb{Z}[T,T^{-1}])$.

THEOREM 2: (Quillen Projective Space Thm.) If ε is a rank r vector bundle over X, there is a homotopy equivalence

$$K^B(\mathbb{P}\varepsilon_X) \simeq \prod_1^r K^B(X).$$

For $Y \subseteq X$ closed, define $K(X \text{ on } Y)$ as the K-theory of the category of those perfect complexes on X which are acyclic on $X - Y$. There is a $K^B(X \text{ on } Y)$ satisfying the analog of the "Bass fundamental theorem", Thm 1.

THEOREM 3: (Localization) For $U \subseteq X$ quasicompact open, there is a homotopy fibre sequence

$$K^B(X \text{ on } X-U) \to K^B(X) \to K^B(U)$$

Hence there is a long exact sequence

$$\cdots \to K_n^B(X \text{ on } X-U) \to K_n^B(X) \to K_n^B(U) \xrightarrow{\partial} K_{n-1}^B(X \text{ on } X-U) \to \cdots$$

THEOREM 4: (Excision) If $i : Y \to X$ is a finitely presented closed immersion and $f : X' \to X$ is a map such that

1) $\mathcal{O}_{X',y'}$ is flat over $\mathcal{O}_{X,y}$ if $f(y') = y \in i(Y)$

2) f induces an isomorphism $f^{-1}(Y) \xrightarrow{\simeq} Y$

then $f^* : K^B(X \text{ on } Y) \xrightarrow{\sim} K^B(X' \text{ on } Y')$ is a homotopy equivalence.

THEOREM 5: (Mayer-Vietoris) If U and V are quasicompact open in X, there is a homotpy cartesian Mayer-Vietoris square

$$\begin{array}{ccc} K^B(U \cup V) & \longrightarrow & K^B(U) \\ \downarrow & & \downarrow \\ K^B(V) & \longrightarrow & K^B(U \cap V) \end{array}$$

THEOREM 6: (Brown-Gersten) If X is noetherian of finite Krull dimension, there is cohomological descent for the Zariski and Nisnevich topologies

$$K^B(X) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{Zar}(X;K^B)$$

$$K^B(X) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{Nis}(X;K^B)$$

hence spectral sequences $H^p_{Zar}(X;\tilde{K}^B_q) \Rightarrow K^B_{q-p}(X)$.

The Nisnevich descent part of Thm 6 allows one to remove the hypothesis that X is regular in my old theorem that

$$K/\ell^\nu(X)[\beta^{-1}] \simeq K^{Top}/\ell^\nu(X).$$

S. SAITO: SK_1 of punctuated Spec of 2-dimensional local rings

Let A be a 2-dimensional normal local domain. Let $F = A/m_A$ its residue field. $K = Q(A)$ its quotient field, P the set of all prime ideals of height 1 in A and put

$$X = \mathrm{Spec}(A) - \{m_A\}.$$

Let

$$SK_1(X) \overset{def}{=\!=} \mathrm{Ker}(K_1(X) \longrightarrow A^*)$$

By the localization theory on X we know

$$SK_1(X) \simeq \mathrm{Coker}(K_2(K) \xrightarrow{\partial} \bigoplus_{p \in P} K(p)^*)$$

where ∂ is given by tame symbols. The localization sequence

$$K_2(K) \longrightarrow \bigoplus_{p \in P} K(p)^* \longrightarrow \mathbb{Z}$$

gives use to

$$\delta : SK_1(X) \longrightarrow \mathbb{Z}$$

and we put

$$SK_1(X)^o = \mathrm{Ker}(\delta).$$

Bloch proves

<u>THEOREM</u>: If A is regular δ is an isomorphism.

In this talk I give the following theorem which treats $SK_1(X)$ in general case but assuming F is <u>finite</u>.

<u>THEOREM</u>: Assume that F is finite.

(1) $SK_1(X)^o$ is <u>torsion</u>.

(2) Let $D(X) \subset SK_1(X)$ be the maximal divisible subgroup then $SK_1(X)^o/D(X)$ is <u>finite</u>.

(3) There exists a canonical isomorphism

$$SK_1(X)^o/D(X) \simeq \mathrm{Gal}(\hat{K}^{ur}/\hat{K})_{tor}.$$

Here $\hat{K}$ is the quotient field of the completion $\hat{A}$ of A. $\hat{K}^{ur}$ is the maximal abel extension of K which is unramified over any $p \in P$.
We conjecture $D(X) = 0$. Concerning this we have

<u>PROPOSITION</u>: Assume that A has rational singularity. Then the prime-to-ch(F) part of $D(X)$ is trivial.

As a corollary of Thm. and Prop. we get

<u>CON</u>. Let B be a 2-dimensional regular local ring with finite residue field F. Let G be a finite group acting on B such that

(1) for any $\sigma \in G - \{id\}$ $\mathrm{length}_B\, B/I_\sigma < \infty$ where $I_\sigma = \langle b^\sigma - b \mid b \in B \rangle$.

(2) any $\sigma \in G$ acts trivially on F.

Put $A = B^G$ which is a 2-dimensional normal local ring. Then we have

$(p = ch(F))$ $\quad SK_1(X)^o \simeq G^{ab} \oplus$ (p-primary torsion divisible group).

B. MAGURN: Absolute stable rank and Witt cancellation for noncommutative rings

In a ring A, a list $a_o,\ldots,a_n$ "can be shortened" if there are $t_i \in A$ with $a_o + t_o a_n,\ldots,a_{n-1} + t_{n-1}a_n$ lying in exactly those maximal left ideals containing $a_o,\ldots,a_n$; if every such list in A can be shortened, we say A has absolute stable rank $asr(A) \leq n$. This condition is designed to imply transitive action of $U(q)$ on all nonsingular vectors v (in a $(\Lambda,\varepsilon,\alpha)$-quadratic space (M,q)) of equal length. By a standard argument it implies (M,q) is cancellative when q has Witt index $\geq asr(A) + 2$ (or $asr(A) + 1$ provided the involution α on A is trivial). In general $asr(A) \geq sr(A)$ = the stable rank of A. By a recent theorem of J.T. Stafford, $asr(A) \leq Kdim(A/rad\ A) + 1$, where $Kdim(A)$ is the Krull dimension of a left noetherian ring. So Witt cancellation (for sufficiently large index) applies to quadratic spaces over $\mathbb{Z}G$ when G is a polycyclic-by-finite group .

B. KAHN: Trivializing Milnor's K-theory

Let F be a field. The talk defined two series of groups $\hat{K}_n(F)$, $\tilde{K}_n(F)$, "lifting" the Milnor K-groups $K_n^M(F)$. $\hat{K}_n(F)$ (resp. $\tilde{K}_n(F)$) is defined as $\mathbb{G}_m^{\otimes n}$ (resp. $\Lambda^n(\mathbb{G}_m)$) in the category of Mackey functors. So, loosely speaking, $\hat{K}_n(F)$ is defined by generators $Cor_{E/F}(x_1 \otimes x \ldots \otimes x_n)$, $[E:F] < +\infty$, $x_i \in E^*$, with relations given by the projection formula. Same thing for $\tilde{K}_n(F)$ with $x_1 \wedge \ldots \wedge x_n$. There are surjective homomorphisms:

$$\hat{K}_n(F) \twoheadrightarrow \tilde{K}_n(F) \twoheadrightarrow K_n^M(F),$$

and $\quad Ker(\hat{K}_n(F) \to K_n^M(F))$ and $Ker(\tilde{K}_n(F) \to K_n^M(F))$ are divisible.

Thus the Milnor-Kato conjecture may be phrased as follows: the natural maps $\hat{K}_n(F)/m \to H^n(F,\mathbb{Z}/m(n))$ (resp. $\tilde{K}_n(F)/m \to H^n(F,\mathbb{Z}/m(n))$) are isomorphisms.

CONJECTURE 1. There are canonical isomorphisms:

$$H^{n-1}(F,\mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \hat{K}_n(F)_{tors}$$
$$H^{n-1}(F,\mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\sim} \tilde{K}_n(F)_{tors}$$

I am able to construct such maps for $n = 2,3$ (at least, away from 2-torsion: for 2-torsion I have to assume that $\mathrm{Gal}(F(\mu_{2^\infty})/F)$ is torsion free).

Assume that F is perfect; define $\mathbb{Z}(1)$ as $\mathbb{G}_m[-1]$ (as a complex of $\mathrm{Gal}(\bar{F}/F)$-modules) and $\hat{\mathbb{Z}}(n)$ as $\mathbb{Z}(1)^{\overset{L}{\otimes} n}$ (in the corresponding derived category). Set $\hat{K}'_n(F) = \mathbb{H}^n(F,\hat{\mathbb{Z}}(n))$. Then cup-product induces a homomorphism

$$\hat{K}_n(F) \xrightarrow{\alpha} \hat{K}'_n(F),$$

and

CONJECTURE 2. α is an isomorphism.

The link between conjectures 1 and 2 is the following (easy) theorem.

THEOREM 1. a) There is a canonical isomorphism

$$H^{n-1}(F,\mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\simeq} \hat{K}'_n(F)_{tors}.$$

b) There is a canonical injection

$$\hat{K}'_n(F)/m \hookrightarrow H^n(F,\mathbb{Z}/m(n)).$$

If the Galois symbol in degree n is surjective, this injection is an isomorphism.

It is easy to see that $\mathrm{Ker}\,\alpha$ and $\mathrm{Coker}\,\alpha$ are torsion. On the other hand, there is the following result.

THEOREM 2. a) α is surjective iff the Galois symbol in degree n is surjective.

b) Assume $n = 2$ or 3. Then the restriction of α to $\hat{K}_n(F)_{tors}$ is split surjective, with divisible kernel.

T. GOODWILLIE: Traces and Fixed Points

The main point of the talk was to give a particular description of Dennis' trace map from the K-theory $K(A)$ of a ring A to the Hochschild homology $H(A)$. The description is as follows:

Define $K(A)$ by the Waldhausen method, so $K(A) = \Omega|BiS.\mathcal{C}|$ where

$\mathcal{C}$ = category of A-modules (finitely generated proj.)

$S_k\mathcal{C}$ = category of filtered objects in $\mathcal{C}$

$$0 = P_o \subset P_1 \subset \dots \subset P_k = P$$

$iS_k\mathcal{C}$ = category with these same objects, but only isomorphisms.

B = nerve

Then Dennis' map can be described as the composition

$$(*) \qquad \begin{array}{c} \Omega|BiS.\mathcal{C}| \xrightarrow[\alpha]{} \Omega|\Lambda iS.\mathcal{C}| \xrightarrow[\beta]{} \Omega|\Lambda S.\mathcal{C}| \xrightarrow[\gamma]{} \Omega|HS.\mathcal{C}| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \simeq \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H(A) \end{array}$$

Here Λ is "cyclic nerve" (whereas a p-simplex of $B\mathcal{C}$ is a diagram $P_o \xrightarrow{f_o} \dots \xrightarrow{f_{p-1}} P_p$ in $\mathcal{C}$, a p-simplex of $\Lambda\mathcal{C}$ is a diagram

$$\begin{tikzcd} & P_1 \arrow[r, "f_1"] & \cdots \arrow[d, dotted] \\ P_o \arrow[ur, "f_o"] & & \cdots \arrow[dl, dotted] \\ & P_p \arrow[ul, "f_p"] & \end{tikzcd}$$

).

The map α is based on the fact that $B\mathcal{C} \hookrightarrow \Lambda\mathcal{C}$ when every arrow in $\mathcal{C}$ is invertible

$$\underset{B}{\underset{\cap}{(f_o,\dots,f_{p-1})}} \longmapsto \underset{\Lambda}{\underset{\cap}{(f_1,\dots,f_p)}}, \qquad f_p \circ f_{p-1} \circ \dots \circ f_o = 1$$

The map β forgets the requirement that maps are invertible. The map γ takes products of Hom-sets to tensor products of Hom-groups. Its target is defined like its source except that in the forming cyclic nerves a p-simplex is an element of

$$\bigoplus_{P_o,\ldots,P_p} \mathrm{Hom}(P_o,P_1) \otimes \ldots \otimes \mathrm{Hom}(P_{p-1},P_p) \otimes \mathrm{Hom}(P_p,P_o)$$

rather than

$$\coprod_{P_o,\ldots,P_p} \mathrm{Hom}(P_o,P_1) \times \ldots \qquad \ldots \times \mathrm{Hom}(P_p,P_o)$$

The inclusion $H(A) \longrightarrow \Omega\Sigma H(A) \longrightarrow \Omega\Sigma HS_1 C \longrightarrow \Omega|HS.C|$

(analogous to inclusion $BGL_1(A) \longrightarrow K(A)$) is an equivalence, by a theorem of Randy McCarthy. ($H(A)$ here is the "tensor product cyclic nerve" of the one-object category A; it is isomorphic to the usual model for Hochschild homology.)

One point of the construction is that the circle group acts on the diagram (*) because cyclic nerves are cyclic objects in the sense of Connes. The intermediate terms can be identified as follows:

(1) $\Omega|\Lambda S.iC| = \Omega|B.i\,\mathrm{Aut}_C|$, the K-theory of A-modules-with-automorphisms.

(2) $\Omega|\Lambda S.C|$ seems to be equivalent to the K-theory of A-modules-with-endomorphism, minus $K(A)$, that is

$$K(\mathrm{End}_C| = \Omega|B.i\,\mathrm{End}_C| \simeq K(A) \times \Omega|\Lambda S.C|$$

(The idea of proving (2) only came up after the talk, in response to a question of Thomason. With a little help from Grayson it now looks like it can be proved.)

S. GELLER: <u>Is the KABI conjecture true?</u>
(This is joint work with Chuck Weibel)

<u>KABI CONJECTURE</u>: Let A and B be rings, I an ideal of A, and $f : A \to B$ such that $f(I)$ is an ideal of I and $I \simeq f(I)$. Then for all $n \geq 1$

$$K_n(A,B,I) \otimes \mathbb{Q} \simeq HC_{n-1}(A,B,I) \otimes \mathbb{Q}.$$

Previously, the conjecture was known to be true for

a) $n = 1$ (Geller-Weibel)

b) I nilpotent (Goodwillie)

c) $B = A/J$ (Ogle-Weibel)

Also, it is sufficient to prove that $K_n(A,B,I) \simeq HC_{n-1}(A,B,I)$ for $\mathbb{Q}$-algebras $A \subseteq B$ with I an ideal of both rings.

In this talk, for $\mathbb{Q} \subseteq A \subseteq B$ and I an ideal of both rings, triple relative groups $K_n(A,B,I,J)$, J an ideal of A, were defined, a module structure over the ring of Witt vectors $W(\mathbb{Q})$ was discussed and the following results were announced with some proofs given.

For $\mathbb{Q} \subseteq A \subseteq B$ and I an ideal of both A & B:

1) KABI conjecture $\Leftrightarrow NK_n(A,B,I) \simeq NHC_{n-1}(A,B,I) \quad \forall\, n \geq 1$

2) KABI conjecture $\Leftrightarrow K_n(A[t],B[t],I[t],t^k) \hookrightarrow$

$HC_{n-1}(A[t],B[t],I[t],t^k) \quad \forall\, n \geq 1$

3) KABI conjecture $\Leftrightarrow$ the weight s summand of $K_n(A[t],B[t],t^kI[t])$ is zero for $s < k$ and $\forall\, n \geq 1$
(hence, if the weight s summand of $K_n(A[t],t^kI[t]) = 0$ for $n \geq 2$, then the KABI conjecture is true).

4) $K_2(A,B,I) \to HC_1(A,B,I)$ is onto.
Hence, for A,B,I as in the conjecture
$K_2(A,B,I) \otimes \mathbb{Q} \to HC_1(A,B,I) \otimes \mathbb{Q}$ is onto.

B. DAYTON: <u>Naturality of Pic, SK_0 and SK_1</u>

This talk reports on joint work with C.A. Weibel. Transfer maps are constructed for SK_0 and SK_1. From these it follows that if $A = \bigoplus_{i \geq 0} A_i$ is a graded commutative ring with $A_+ = \bigoplus_{i > 0} A_i$ and $A_0 = R$ then $SK_0(A,A_+)$, $SK_1(A,A_+)$, $Pic(A,A_+)$, $NSK_0(R)$, $NSK_1(R)$, $NPic(R)$ are all modules over the ring $W(R)$ of Witt vectors over R. Various consequences of these module structures are discussed. In particular we consider the case where $A = \bigoplus_{i \geq 0} A_i$ is reduced, graded and finitely generated as an algebra over the field $A_0 = k$. Let $B = \bigoplus_{i \geq 0} B_i$ be the seminormalization of A,

$GW(B) = \{f = 1+b_1t + \ldots \in W(B) \mid b_i \in B_i\}$. There is an injection $\gamma : Pic(A) \to GW(B)/GW(A)$ of $W(k)$-modules. If $A_n = B_n$ for $n >> o$ then γ is an isomorphism. If $char(k) = 0$, composing γ with the ghost map gives an isomorphism of k-modules $Pic(A) \to B/A$.

C. KASSEL: <u>Bivariant Chern character</u>

The Chern character (also called generalized Dennis trace map) $ch : K_*(A) \to HC^-(A)$ from algebraic K-theory to negative cyclic homology can be extended to a <u>bivariant Chern character</u> $ch : K^*(A,B) \to HC^*(A,B)$ from a suitably defined bivariant algebraic K-theory to a bivariant version of cyclic cohomology. Both bivariant theories are covariant in B and contravariant in A. One recovers the usual Chern character when $A = \mathbb{Z}$. As an immediate consequence of the multiplicativity of the bivariant Chern character, two Morita-equivalent algebras have isomorphic (bivariant) cyclic (co)homology groups.
The bivariant K-groups are obtained from the exact category of A-B-bimodules which are finitely generated projective over B.
The bivariant cyclic cohomology groups have the following properties

i) (Product) There exists a graded product

$$HC^*(A_1,B_1 \otimes C) \otimes HC^*(C \otimes A_2,B_2) \to HC^*(A_1 \otimes A_2, B_1 \otimes B_2)$$

ii) (Bivariant Connes exact couple) There exists an exact couple

$$\begin{tikzcd}
HC^*(A,B) \arrow[rr, "S"] & & HC^*(A,B) \arrow[dl, "I"] \\
& HH^*(A,B) \arrow[ul, "B"] &
\end{tikzcd}$$

where $deg(S) = 2$, $deg(I) = 0$, $deg(B) = -1$ and $HH^*(A,B)$ is a bivariant version of Hochschild homology.

iii) For any extension of algebras $0 \to I \to R \to S \to 0$ such that I is H-unital in the sense of Wodzicki, one has the exact triangles

$$\begin{tikzcd}
HC^*(A,I) \arrow[rr] & & HC^*(A,R) \arrow[dl] \\
& HC^*(A,S) \arrow[ul] &
\end{tikzcd}
\qquad
\begin{tikzcd}
HC^*(I,A) \arrow[dr] & & HC^*(R,A) \arrow[ll] \\
& HC^*(S,A) \arrow[ur] &
\end{tikzcd}$$

iv) If SA is the suspension of the algebra A, one has the following isomorphisms:

$$HC^n(A,B) = HC^o(A,S^nB) \qquad \text{and} \qquad HC^{-n}(A,B) = HC^o(S^nA,B) \qquad (n \geq o).$$

J.L. LODAY: Operations in cyclic homology of commutative algebras

The notion of descents for a permutation $\sigma \in S_n$ permits us to define the Eulerian partition of S_n : $S_n = S_{n,1} \cup \ldots \cup S_{n,n}$. The elements $\ell_n^k = (-1)^{k-1} \sum_{\sigma\in S_{n,k}} \mathrm{sgn}(\sigma)\sigma$ of the group algebra $K[S_n]$ have very nice properties. They lead to $\lambda_n^k = \sum_{i=o}^{k-1} (-1)^i \binom{n+i}{i} \ell_n^{k-i}$.

Let S_n act from the left on $A \otimes A^{\otimes n}$ where A is a commutative K-algebra. Denote by b the Hochschild boundary and by B the map defined by Connes.

PROP. $b\,\ell_n^k = (\ell_{n-1}^k + \ell_{n-1}^{k-1})b$ and $\ell_n^k B = B(k\ell_n^k + (n-k+1)\ell_{n-1}^{k-1})$.

COR. $b\,\lambda_n^k = \lambda_{n-1}^k\, b$ and $\lambda_n^k B = B\, k\, \lambda_{n-1}^k$.

Therefore these λ_n^k maps permit us to endow Hochschild homology and cyclic homology with a special λ-ring structure.

In the rational case it implies a natural splitting:
$HH_n = HH_n^{(1)} \oplus \ldots \oplus HH_n^{(n)}$ and $HC_n = HC_n^{(1)} \oplus \ldots \oplus HC_n^{(n)}$, with $HH_n^{(n)} = \Omega^n$, $HC_n^{(n)} = \Omega^n/d\Omega^{n-1}$ and $HH_n^{(1)} = \mathrm{Harr}_n = HC_n^{(1)}$ ($n \geq 3$ for this last equality) where Harr_n is Harrison homology.

All the properties are valid for any functor $\underline{\underline{\mathrm{Fin}}} \to$ (K-modules) where $\underline{\underline{\mathrm{Fin}}}$ is the category of finite sets. In fact, the relations in PROP and COR above may be seen as relations in the universal ring $\mathcal{L} = K[\underline{\underline{\mathrm{Fin}}}]$.

Ref.: J.-L. LODAY, Partition eulérienne et opérations sur homologie cyclique, Cptes Rend. Acad. Sci. Paris (1988).

D. WEBB: G-theory of integral group rings

Let G be a finite group, and consider $G_*(\mathbb{Z}G)$ (or more generally, $G_*(RG)$, for a Noetherian ring R). We first deduce a Lenstra-type decomposition for G nilpotent.

PROP.: Let G be finite nilpotent, and write $\mathbb{Q}G \simeq \prod_\rho \mathbb{Q}(\rho)$, where ρ ranges over irreducible rational representations and $\mathbb{Q}(\rho)$ is simple; let $\mathbb{Z}(\rho)$ be a maximal $\mathbb{Z}$-order in $\mathbb{Q}(\rho)$, $\mathbb{Z}\langle\rho\rangle = \mathbb{Z}(\rho)[\frac{1}{|\rho|}]$, where $|\rho| = [G : \ker \rho]$. Then $G_*(\mathbb{Z}G) \simeq \bigoplus_\rho G_*(\mathbb{Z}\langle\rho\rangle)$.

I. Hambleton, L. Taylor, and B. Williams prove this result independently, and they conjecture a general answer:

CONJECTURE (HTW): Let G be a finite group, and write $\mathbb{Q}G \simeq \prod_\rho M_{n_\rho}(D_\rho)$, $D_\rho = \mathrm{End}_{\mathbb{Q}G}(V_\rho)$ the division algebra associated to the irreducible rational representation $\rho : G \to GL(V_\rho)$. Let $k_\rho = |\ker(G \xrightarrow{\rho} GL(V_\rho))|$, ℓ_ρ the degree of any irreducible constituents of $\mathbb{C} \otimes_{\mathbb{Q}} V_\rho$, $w_\rho = \frac{|G|}{k_\rho \ell_\rho}$, $\mathcal{D}_\rho$ a maximal $\mathbb{Z}$-order in D_ρ. Then $G_*(\mathbb{Z}G) \simeq \bigoplus_\rho G_*(\mathcal{D}_\rho[1/w_\rho])$.

PROP.: The HTW conjecture holds for dihedral extensions of finite abelian groups.

PROP.: The HTW conjecture holds for $|G|$ square-free.

The proofs use Lenstra-type techniques; one defines the Lenstra functor, a self homotopy equivalence of $B\mathbb{Q}\underline{M}(m)$, m a $\mathbb{Z}$-order in $\mathbb{Q}G$ containing $\mathbb{Z}G$; this induces a map of the homotopy fibre sequence $\underline{M}^{tor}(\mathbb{Z}G) \to \underline{M}(\mathbb{Z}G) \to \underline{M}(\mathbb{Q}G)$ to the sequence $\underline{M}^{tor}(a) \to \underline{M}(a) \to \underline{M}(\mathbb{Q}G)$, where a is a ring whose G_* is the desired answer.

C. OGLE: Generalized Trace Map for K-theory of Spaces, and Applications

A conjecture due to T. Goodwillie asserts that

$$\tilde{A}(\Sigma X) \simeq \tilde{D}(|X|) = \prod_{q\geq 1} \tilde{D}_q(|X|),\ \tilde{D}_q(|X|) \overset{\text{def.}}{=} \Omega^\infty\Sigma^\infty(\Sigma(E\,\mathbb{Z}/q_+ \wedge_{\mathbb{Z}/q} |X|^{[q]})),$$

where $A(Z)$ denotes the Waldhausen K-theory of the space = simplicial set Z and $\tilde{A}(Z)$ = cofibre $(A(Z) \to A(*))$. A proof of this conjecture has been announced by G. Carlsson, R. Cohen, T. Goodwillie, and W.-C. Hsiang [CCGH] and independently by myself. Both previous proofs are incorrect. We correct this.

We follow the techniques used by Waldhausen in his proof of the splitting $A(Y) \simeq Wh^{\mathrm{Diff}}(|Y|) \times \Omega^\infty\Sigma^\infty(|Y|_+)$, and the outline of the proof of Goodwillie's conjecture given in [CCGH] in showing

<u>THEOREM 1</u> There exists a trace map $\bar{T}r_X(Y)$ natural in X and Y, (X a connected simplicial set, X and Y base pointed):

$$\bar{T}r_X(Y) : \varinjlim_n \Omega^n \text{ fibre } (\bar{A}(\Sigma(X \vee \Sigma^n Y)) \to \bar{A}(\Sigma X)) \longrightarrow$$
$$\Omega^\infty \Sigma^\infty (\Sigma(\bigvee_{q \geq 1} |X|^{[q-1]} \wedge |Y|)) \simeq \prod_{q \geq 1} \Omega^\infty \Sigma^\infty (\Sigma(|X|^{[q-1]} \wedge |Y|)).$$

The decomposition on the right decomposes $\bar{T}r_X(Y)$ as $\prod_{q\geq 1} \bar{T}r_X(Y)_q$.

There exist maps $\tilde{\rho}_q : \tilde{D}_q(|X|) \to A(\Sigma X)$ as constructed in [CCGH] and [O]. These constructions, as well as the entire proof of the above Theorem, admit and require a precise simplicial formulation. This we do. We then get

<u>THEOREM 2</u> $(\bar{T}r)_X(Y)_q \circ (D_1\tilde{\rho}_p)_X(Y) = \begin{cases} * & \text{if } p \neq q \\ (-1)^{q-1} & \text{if } p = q \end{cases}$

This homotopy is natural in X and Y. Here $(D_1\tilde{\rho}_p)_X(Y)$ denotes the 1^{st} derivative of the map $\tilde{\rho}_p$ at X, evaluated at Y in the sense of Goodwillie. It now follows from the fundamental results of Goodwillie and Waldhausen, who have computed $(D_1\bar{A}\Sigma)_X(Y)$ that

<u>COR. 3</u> $\bar{A}(\Sigma X) \simeq \tilde{D}(|X|)$ by a homotopy equivalence natural in X.

A.O. KUKU: <u>Higher K-theory of orders and integral group-rings</u>

This talk gives an exposition of the speaker's recent results on the higher K-theory of orders and group-rings. First solutions were given to recent questions on finite generation of K_n, G_n of orders as well as finiteness of SK_n and SG_n of orders as follows. More precisely we prove the following results:

(I) Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, P any prime ideal of R, then for all $n \geq 1$

(i) $K_n(\Lambda)$ is a finitely generated abelian group.

(ii) $K_n(\Lambda) \to K_n(\Gamma)$ is an isomorphism mod torsion if Γ is the maximal R-order containing Λ.

(iii) $SK_n(\Lambda)$ is a finite group.

(iv) $SK_n(\hat{\Lambda}_P)$ is finite where $\hat{\Lambda}_P$ is the completion of Λ at P

(II) Let R, Λ, F, Σ be as in (I). Then $\forall\, n \geq 1$

(i) $G_n(\Lambda)$ is a finitely generated abelian group

(ii) $G_{2n-1}(\Lambda_P)$ is a finitely generated abelian group.

(iii) $SG_{2n}(\Lambda_P) = SG_{2n}(\Lambda) = SG_{2n}(\hat{\Lambda}_P) = 0$.

(iv) $SG_{2n-1}(\Lambda)$ is finite; $SG_{2n-1}(\Lambda_P)$, $SG_{2n-1}(\hat{\Lambda}_P)$ are finite groups of order relatively prime to the prime p lying below P.

We also have the following results on Cartan maps: For all $n \geq 1$

(III)(i) If k is a field of characteristic p and π any finite group, then $K_{2n}(k\pi)$ is a finite p-group and $\mathrm{Ker}(K_{2n-1}(k\pi) \twoheadrightarrow G_{2n-1}(k\pi))$ is the Sylow p-subgroup of $K_{2n-1}(k\pi)$.

(ii) $K_n(\Lambda) \to G_n(\Lambda)$ induces a surjection $SK_n(\Lambda) \to SG_n(\Lambda)$.

(iii) $G_{4n+3}(\mathbb{Z}\pi)$, $K_{4n+3}(\mathbb{Z}\pi)$, $G_{4n+3}(\mathbb{Z}_p\pi)$ are finite groups.

Finally we show that reduction theory can be used to reduce the study of K-theory of integral group-rings of finite groups to the study of the K-theory of group-rings over the p-hyperelementary subgroups of π.

S. LICHTENBAUM: Motivic Cohomology

It would be highly desirable to have an algebraic cohomology theory bearing the same relation to algebraic K-theory as ordinary singular cohomology bears to topological K-theory. This theory should also have serious applications to the study of special values of zeta-functions and to arithmetic duality theorems.

Such a theory should be the hypercohomology (in the ètale and Zariski sites) of a complex of sheaves $\mathbb{Z}(r)$ $(r = 0,1,2\ldots)$ on a noetherian regular scheme X satisfying (at least) the following properties:

(0) $\mathbb{Z}(0) = \mathbb{Z}$ $\quad \mathbb{Z}(1) = G_m[-1]$

(1) For $r \geq 1$, $\mathbb{Z}(r)$ is acyclic of $[1,r]$

(2) There is a product pairing $\mathbb{Z}(r) \overset{L}{\otimes} \mathbb{Z}(s) \to \mathbb{Z}(r+s)$

(3) (a) If n is invertible on X, there is a distinguished triangle in the ètale site

$$\mathbb{Z}(r) \xrightarrow{n} \mathbb{Z}(r) \longrightarrow \mathbb{Z}/n\mathbb{Z}(r) \longrightarrow \mathbb{Z}(r)[1]$$

(b) If X has characteristic p, there is a distinguished triangle in the ètale site

$$\mathbb{Z}(r) \xrightarrow{p^m} \mathbb{Z}(r) \longrightarrow \mu_m(r)[-r] \longrightarrow \mathbb{Z}(r)[1]$$

(4) If α maps the ètale site to the Zariski site,

$$\alpha^*\mathbb{Z}(r)_{zar} = \mathbb{Z}(r)_{\text{ét}}, \qquad t_{\leq r}R\alpha_*\mathbb{Z}(r)_{\text{ét}} = \quad t_{\leq r+1}R\alpha_*\mathbb{Z}(r)_{\text{ét}} = \mathbb{Z}(r)_{zar}$$

In particular, $R^{r+1}\alpha_*\mathbb{Z}(r) = 0$ (Hilbert Theorem 90).

(5) $R^r\alpha_*\mathbb{Z}(r) = \underline{\underline{K}}^M_{r,zar}$.

(6) The homology sheaves $H^i(\mathbb{Z}(r))$ should be isomorphic to the sheaves $gr^{\gamma}_{r}\underline{\underline{K}}_{2r-i}(\mathcal{O}_X)$, up to p-torsion for primes $p < r$.

For $r = 2$, we have constructed a cohomology theory satisfying all of these properties, with the exception that we do not know for property (6) that $gr^{\gamma}_{2}\underline{\underline{K}}_{4-i}(\mathcal{O}_X) = 0$ for $i \leq o$.

A possible candidate for a motivic cohomology complex in the case of a field F is the following:

Let the i-th term of the complex $\mathbb{Z}(r)$ $(o \leq i \leq r)$, be

$$\underset{V,S}{\underrightarrow{\mathrm{Lim}}}\; K_i^{'M} \quad (V-S,\ I_1,I_2,\ldots I_n)$$

where V runs over all reduced i-dimensional subschemes of $\mathbb{A}^r_F$ whose intersection with all faces of the hypercube $X_i(X_{i-1}) = 0$, $i=1,\ldots,r$ is proper. S runs over all finite subsets of V whose intersection with the (r-i)-skeleton of the hypercube is empty, and I_j is the ideal defined by X_j (X_j-1). $K_i^{'M}$ here denotes multirelative Milnor K'-theory.

M. HARADA: Grothendieck-Riemann-Roch for general schemes

Let S be a base scheme, Noetherian and of finite Krull dimension, separated. Let ℓ be a prime number, fixed once for all so that 1) $\ell^{-1} \in \mathcal{O}_S$, 2) all residue fields of S have uniformally bounded,

ℓ-étale cohomological dimension, e.g. $\mathbb{Q}$ if $\ell \neq 2$, $\mathbb{C}, \bar{k}_{alg}, \mathbb{Z}[\frac{1}{\ell}], \ldots$. Schemes we consider are essentially of finite type over S.

<u>THEOREM</u>. There exists a topological G-theory spectrum $G^t(X)$ so that

1) Atiyah-Hirzebruch s.s. $H^*(X_{ét}; i^! \mathbb{Z}_{\hat{\ell}}(*)) \Rightarrow G^t_*(X)$, $i : X \to S$, the str.-morph.

2) Grothendieck-Riemann-Roch: When $f : X \to Y$ is proper morph.,

$$\begin{tikzcd} G^{alg}(X) \arrow[r, "\tau_X"] \arrow[d, "f_*"'] & G^t(X) \arrow[d, "f^t_*"] \\ G^{alg}(Y) \arrow[r, "\tau_Y"'] & G^t(Y) \end{tikzcd}$$

where $G^{alg}(X)$ is the spectrum associated to coherent sheaves on X, f_* is induced by an alternative sum of higher direct image sheaves.

And it induces the Hirzebruch-Riemann-Roch formula, the main theorem of Baum-Fulton-MacPherson and its generalization to higher K-theory, the theorem of Gillet.

The proof and the construction is based on the facts that 1) f_* can be localized with respect to the étale topology on Y; 2) $K^{alg}(\ )^{\wedge}_{\ell}$ is locally constant on the étale topology.

The projection formula $f_*(x \cap f^*y) = (f_*x) \cap y$ is formulated as the commutative diagram of spectra

$$\begin{tikzcd} G^{alg}(X) \otimes K^{alg}(Y) \arrow[r, "1\otimes f^*"] \arrow[d, "f_*\otimes 1"'] & G^{alg}(X) \otimes K^{alg}(X) \arrow[r, "\cap"] & G^{alg}(X) \arrow[d] \\ G^{alg}(Y) \otimes K^{alg}(Y) \arrow[rr, "\cap"'] & & G^{alg}(Y) \ . \end{tikzcd}$$

The fact gives us

$$\begin{tikzcd} \mathbb{H}(X_{ét}; \underset{\sim}{G}^{alg}/\ell^{\nu}) \overset{L}{\underset{K^t(S)^{\wedge}_{\ell}}{\otimes}} K^t(Y)^{\wedge}_{\ell} \arrow[r, "1\otimes f^*"] \arrow[d, "f_*\otimes 1"'] & \mathbb{H}(X_{ét}; \underset{\sim}{G}^{alg}/\ell^{\nu}) \overset{L}{\underset{K^t(S)^{\wedge}_{\ell}}{\otimes}} K^t(X)^{\wedge}_{\ell} \arrow[r, "\cap"] & \mathbb{H}(X_{ét}; \underset{\sim}{G}^{alg}/\ell^{\nu}) \arrow[d, "f_*"] \\ \mathbb{H}(Y_{ét}; \underset{\sim}{G}^{alg}/\ell^{\nu}) \overset{L}{\underset{K^t(S)^{\wedge}_{\ell}}{\otimes}} K^t(Y)^{\wedge}_{\ell} \arrow[rr, "\cap"] & & \mathbb{H}(Y_{ét}; \underset{\sim}{G}^{alg}/\ell^{\nu}) \end{tikzcd}$$

When X and Y are proper over S , compose the Gysin mapping to S

$$\begin{tikzcd} \mathbb{H}(X_{\acute{e}t};\underset{\sim}{G}^{alg}/\ell^{\nu}) \arrow[d, "f_*"] \arrow[dr] & \\ \mathbb{H}(Y_{\acute{e}t};\underset{\sim}{G}^{alg}/\ell^{\nu}) \arrow[r] & G^{alg}/\ell^{\nu}(S) \end{tikzcd}$$

and taking the adjunction as $K^t(S)^{\wedge}_{\ell}$-module, we get the theorem 2).

To prove the theorem 1), we look at the Postnikov filtration on them.

J. BERRICK: Acyclic groups

Acyclic groups are those groups whose homology (trivial $\mathbb{Z}$ coefficients) is that of the trivial group. This survey attempts to indicate the importance of acyclic groups and examine their group-theoretic structure.

Examples

Acyclic groups are to be found in work of G. Higman (1951), McLain (1954), Baumslag & Gruenberg (1967), Epstein (1968), J. Mather (1971), Wagoner (1972), Kan & Thurston (1976), Baumslag, Dyer & Heller (1980), de la Harpe & McDuff (1983), and elsewhere. Many examples have few normal subgroups.

Ubiquity results

For a group extension $N \hookrightarrow G \twoheadrightarrow Q$ with Q acting trivially on H_*N,

(i) N acyclic $\Leftrightarrow H_*G \xrightarrow{\cong} H_*Q$

(ii) Q acyclic $\quad H_*N \xrightarrow{\cong} H_*G$

[K&T]1976 : $\forall$ group G, $G \trianglelefteq D \trianglelefteq$ acyclic.

This prompts the study of normal-in-acyclic groups, e.g. abelian groups [BD & H 1980, B 1983], $GL(R)$ (R ring) [W 1972].

Group structure $\Rightarrow$ acyclicity

Techniques used to prove acyclicity include Mayer-Vietoris sequences, preservation of dirlim by homology, and binate structure: $G = \cup G_n$ where $G_1 \leq G_2 \leq \ldots$ and $\forall n\ \exists\ \varphi_n : G_n \to G_{n+1}$ and $a_{n+1} \in G_{n+1}$ s.t. $\forall g \in G_n$ $g = [\varphi_n(g), a_{n+1}]$. Binate groups are acyclic [B, to appear in Proc. Singapore Group Theory Conf., de Gruyter] .

Acyclicity ⇒ group structure

T: Any f.d. complex representation of an acyclic group restricts trivially to all finite subgroups.

C1: Finite normal-in-acyclic groups are abelian.

C2: A (non-central) normal subgroup N of a torsion-generated acyclic group has N/N'' f.g. iff N is infinite perfect - by - f.g. abelian. (Possible example $GL(R) \triangleleft GL(CR)$ therefore $GL(R)$ is ER-by-K_1R.)

C3: If perfect $N \triangleleft$ torsion-gen'd acyclic A and $\mathrm{Aut}N$ has a series with factors residually finite and/or hypoabelian and/or torsion-free, then $A \simeq N \times A/N$, so N also torsion-gen'd acyclic.

J. BOCHNAK: Algebraic vector bundles over real algebraic varieties and applications

Let X be an affine nonsingular, compact connected real algebraic variety and let $R(X)$ be the ring of regular functions from X into $\mathbb{R}$. The groups $\mathrm{Pic}\ (R(X))$, $\mathrm{Pic}\ (R(X)\otimes_{\mathbb{R}}\mathbb{C})$, $K_0(R(X))$, $K_0(R(X)\otimes_{\mathbb{R}}\mathbb{C})$ contain precious information about the geometry and topology of X. Each of these groups is a subgroup (in a natural way) of the corresponding group of the ring $C(X)$ of continuous functions from X into $\mathbb{R}$ (embedding is induced by the inclusion map $R(X) \longrightarrow C(X)$).

$\mathrm{Pic}(R(X))$ is naturally isomorphic to a subgroup $H^1_{alg}(X,\mathbb{Z}/2)$ of $H^1(X,\mathbb{Z}/2)$ where $H^1_{alg}(X,\mathbb{Z}/2)$ is the image of

$$H^{alg}_{n-1}(X,\mathbb{Z}/2) = \{\text{homology classes in } H_{n-1}(X) \text{ represented by algebraic hypersurfaces of } X\}$$

by the Poincaré duality isomorphism $H_{n-1} \to H^1$; $n = \dim X$.

Theorem. Let M be a compact connected C^∞ manifold of dimension $\geqslant 3$, and let G be a subgroup of $\mathrm{Pic}\ (C(M))$ containing the first Stiefel-Whitney class of M. Then there is an algebraic model X of M and a diffeomorphism $\varphi : X \longrightarrow M$ such that $\varphi^*(G) = \mathrm{Pic}\ (R(X))$.

(here $\varphi^* : \mathrm{Pic}\ (C(M)) \longrightarrow \mathrm{Pic}\ (C(X))$ is the isomorphism induced by φ).

Remark. A slightly weaker version of this theorem is valid also for surfaces.

Corollary. For each compact connected C^∞ manifold M, orientable of $\dim \geqslant 2$, there exist an algebraic model X of M with $R(X)$ factorial.

$K_o(R(X))$ of real affine surfaces and 3-folds :

Define the following invariants of a nonsingular real algebraic surface X .

$$\beta(X) = \dim_{\mathbb{Z}/2} H^1_{alg}(X,\mathbb{Z}/2)$$

$$\sigma(X) = \dim_{\mathbb{Z}/2} \left\{ v \in H^1_{alg}(X, \mathbb{Z}/2) \mid v \cup v = 0 \right\} .$$

<u>Theorem.</u>

(i) Let X be a compact connected affine real algebraic surface. Then

$$K_o(R(X)) = \mathbb{Z} \oplus (\mathbb{Z}/4)^{\beta(X)-\sigma(X)} \oplus (\mathbb{Z}/2)^{\beta(X)+1-2(\beta(X)-\sigma(X))}$$

(ii) As X runs through all algebraic models of a compact connected smooth surface M of genus g , the groups $K_o(R(X))$ take (up to isomorphism) precisely $q(M)$ values, where

$$q(M) = \begin{cases} 2g+1 & \text{if } M \text{ orientable} \\ g & \text{if } M \text{ nonorientable, } g \text{ odd} \\ 2g-2 & \text{if } M \text{ nonorientable, } g \text{ even} \end{cases}$$

(Remark. Similar results holds true for algebraic 3-folds).

<u>Theorem.</u> Let $M \subset \mathbb{RP}^k$ be a C^∞ compact hypersurface. Then there exists a diffeomorphism $h : \mathbb{RP}^k \to \mathbb{RP}^k$ (which can be chosen arbitrary close to the identity), such that:

(i) $X = h(M)$ is an algebraic nonsingular subset of $\mathbb{RP}^k$

(ii) $\tilde{K}_o(R(X))$ and $\tilde{K}_o(R(X) \otimes_{\mathbb{R}} \mathbb{C})$ are finite groups.

(iii) If $H^{even}(M, \mathbb{Z})$ is torsion free, then $\tilde{K}_o(R(X) \otimes \mathbb{C}) = 0$

(iii) If M is orientable, and $\dim M = k-1$ is <u>even</u>, then each regular mapping $X \to S^{k-1}$ is homotopic to a constant.

There are many applications of these and similar results to the study of the structure of the set $R(X,S^k)$ of regular mappings from affine real algebraic varieties X into S^k (= the standard sphere).

A sample of results:

Theorem. Given a compact connected C^∞ surface M, the following conditions are equivalent:

(i) For each algebraic model X of M, the set $R(X,S^2)$ is dense in $C^\infty(X,S^2)$ (=set of C^∞ mappings from X into S^2 equipped with the C^∞ topology).

(ii) M is nonorientable of odd genus.

Remark. In particular one gets an algebraic model X of the Klein bottle with $R(X,S^2)$ not dense in $C^\infty(X,S^2)$ by constructing a model with $\tilde{K}_o(R(X)\otimes\mathbb{C}) = 0$.

Theorem. Let Σ_k^2 be a Fermat sphere i.e.

$$\Sigma_k^2 = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^{2k}+y^{2k}+z^{2k} = 1\right\}$$

Then

$$R\left(\Sigma_k^2, S^2\right) \text{ is dense in } C^\infty\left(\Sigma_k^2, S^2\right).$$

Remark. The Fermat spheres are quite exceptional, since for "most" algebraic surfaces X in $\mathbb{R}^3$, the set $R(X,S^2)$ contains only mappings homotopic to a constant!

Theorem. Given a compact connected orientable C^∞ manifold M, $\dim M = 4$, the following conditions are equivalent:

(i) There exists an algebraic model X of M such that each regular map $X \to S^k$ is homotopic to a constant.

(ii) The signature of M is 0.

Theorem. Let C be a nonsingular complex projective curve, and let $C_{\mathbb{R}}$ be the underlying real algebraic variety. Then $R(C_{\mathbb{R}}, S^2)$ is dense in $C^\infty(C_{\mathbb{R}}, S^2)$.

R. Mc CARTHY: Cyclic and Hochschild Homologies of an Exact Category

Let k be a commutative ring and for $\mathfrak{C}$ a small k-linear category; we define the cyclic nerve of $\mathfrak{C}$, $CN(\mathfrak{C})$ to be the cyclic k-module:

$$CN_n(\mathfrak{C}) = \bigoplus_{C_o,\dots,C_n} \mathrm{Hom}(C_1,C_o) \otimes_k \dots \otimes_k \mathrm{Hom}(C_o,C_n)$$

$$\text{i.e. } \begin{tikzcd} C_o & C_1 \arrow[l, "a_o"'] & C_2 \arrow[l, "a_1"'] & \dots \arrow[l] & C_n \arrow[l] \arrow[llll, bend right, "a_n"'] \end{tikzcd} \quad (a_o \otimes \dots \otimes a_n).$$

Face and degeneracy operators are like those of Hochschild homology.

Theorem If A is a unital k-algebra, and P_A = cat. of f.g. projective modules, then

$$CN(A) \simeq CN(P_A) \quad \text{[by def. retract]} .$$

For $\mathfrak{M}$ an exact category, which is also k-linear, we can form CN. S. $\mathfrak{M}$, where S.$\mathfrak{M}$ is Waldhausen simplicial category for a cofibered category.

Def: $HH_*(\mathfrak{M}) = HH_{*+1}(\,CN.\ S.\mathfrak{M}\,)$
$HC_*\ \mathfrak{M} = HC_{*+1}(\,CN.\ S.\mathfrak{M}\,)$

Theorem $CN.\ S_m\, P_A \simeq CN.\ P_A^m$

Cor. The map $CN.\ P_A \longrightarrow \Omega\ CN.\ S.\ P_A$ is a homotopy equivalence.

Cor. We have trace map (by Goodwillie earlier)

$$\begin{tikzcd} \Omega NiS.\ P_A \arrow[r] \arrow[d, phantom, "\wr"] & \Omega CNS.\ P_A & CN.\ P_A \arrow[l, "\sim"'] \arrow[d, phantom, "\wr"] \\ \Omega N.QP_A & & CN.A \end{tikzcd}$$

S. LANDSBURG: Relative Chow Groups

Let $Y \subset X$ be a closed inclusion of regular schemes of finite type over a field. (Regularity can be relaxed in much of what follows.) We want to define a relative Chow Theory $Ch^p(X,Y)$.

To see what this theory should look like, consider the usual absolute Chow theory $Ch^p(X)$. We have

$$gr^p K_0(X) \underset{\substack{\uparrow \\ \text{iso up to torsion.}}}{\longleftarrow} Ch^p(X) = Z^p(X)/\sim \; = H^p(X, \underline{\underline{K}}_p) = E_2^{p,-p}(X)$$

where Z^p is cycles, $\sim$ is rational equivalence, $\underline{\underline{K}}_p$ is sheafified K-theory, and $E_2^{p,-p}(X)$ is from the Quillen spectral sequence.

Here are the relative analogues of some of these objects:

(1) Let $\tilde{Z}^p(X)$ be free abelian on cycles meeting Y properly. Then $Z^p(X,Y)$ is defined by $0 \to Z^p(X,Y) \to \tilde{Z}^p(X) \to Z^p(Y)$.

(2) $\mathcal{K}_p(X)$ is the complex $\underline{\underline{K}}_p(X) \to i_* \underline{\underline{K}}_p(Y)$.

(3) We get a spectral sequence for relative K-theory by taking fibers vertically in the diagram

$$\begin{array}{ccccc}
F^{m+1} & \longrightarrow & F^m & \longrightarrow & F^{m/m+1} \\
\downarrow & & \downarrow & & \downarrow \\
K\mathcal{M}^{m+1}(X) & \longrightarrow & K\mathcal{M}^m(X) & \longrightarrow & K\mathcal{M}^{m/m+1}(X) \\
\downarrow & & \downarrow & & \downarrow \\
K\mathcal{M}^{m+1}(Y) & \longrightarrow & K\mathcal{M}^m(Y) & \longrightarrow & K\mathcal{M}^{m/m+1}(Y)
\end{array}$$

Here $\mathcal{M}^m(X)$ is the category of X-modules of cod $\geq m$. The spectral sequence is $E_1^{pq} = \Pi_{-p-q}\left(F^{p/p+1}\right) \Rightarrow K_{-p-q}(X)$.

The construction of the spectral sequence leads immediately to maps

$$\begin{array}{ccc}
E_2^{p,-p} & \Rightarrow & \mathbb{H}^p(X, \mathcal{K}_p) \\
\downarrow & & \\
Z^p(X,Y)/\sim & &
\end{array} \qquad \text{for appropriate } \sim .$$

We also get a cycle map $Z^p(X,Y)/\sim \;\longrightarrow \mathbb{H}^p(X,\mathcal{K}_p)$ directly by noticing that for $Z \in Z^p(X,Y)$, $\mathbb{H}^p_Z(X,\mathcal{K}_p)$ is free abelian on the components of Z.

Before defining $Ch^m(X,Y)$, we generalize some of this to higher Chow groups. There is a map from Bloch's higher Chow complex to the Gersten-Quillen complex induced by

$$Z^m(X,n) \to \coprod_{x \in X^{m-n}} K_n k(x) \quad \text{via} \quad Z \mapsto \left(p_* Z, \left\{ \frac{Nt_1}{\Sigma Nt_i - 1}, \dots, \frac{Nt_n}{\Sigma Nt_i - 1} \right\}\right)$$

where $N = \mathrm{Norm}_{Z/p_*Z}$ and { } is the Steinberg symbol. This gives

$$Ch^m(X,n) \to H^{m-n}(X, \underline{\underline{K}}_m) \text{ ; this is iso for } n \leq 1.$$

Now define $Ch^m(X,Y,n) = \Pi_n (\mathrm{Cone}\,(Z^m(X,\cdot) \to Z^m(Y,\cdot))[-1])$.
(To define the map, first replace $Z^m(X,\cdot)$ by quasi-isomorphic complex consisting of things that restrict properly to Y.)
Define

$$Ch^m(X,Y) = Ch^m(X,Y,0).$$

Then we get a Bloch Formula

$$Ch^m(X,Y) \xrightarrow{\approx} \mathbb{H}^m(X, \mathcal{K}_m).$$

Finally, to get a cycle map, note that an element of $Ch^m(X,Y)$ is represented by a cycle Z on X with a choice of trivialization of $Z|_Y$. This gives data consisting of compatible cycles on two copies of X and one of $Y \times \mathbb{A}^1$ (namely Z^+, Z^- and the trivialization). Under favorable circumstances, these can be "patched" to give a class in

$$K_0\left(X \amalg_{y \times 0} \left(Y \times \mathbb{A}^1\right) \amalg_{y \times 1} X\right) \approx K_0(X,Y) \oplus K_0(X).$$

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