MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Spectral Theory of Singular Ordinary Differential Operators

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The conference was held under the leadership of Professor *H.-D. Nießen* (Essen) and Professor *A. Schneider* (Dortmund).

In the 22 lectures given and in the discussion of the 29 participants, not only classical selfadjoint differential operators but also non–selfadjoint ones, indefinite ones and problems with interior singularities were dealt with. The main subjects of this meeting can be described by the following key–words:

integral inequalities with applications to spectral theory,

Titchmarsh–Weyl m–functions,

expansion theorems,

properties of the spectrum.

Abstracts

F.V. Atkinson

Estimation of the Titchmarsh m-coefficient in the matrix case

In the scalar case of the eigenvalue problem $-(py')' + qy = \lambda wy$ on the semi-infinite interval $[a, \infty)$ under the standard conditions there is a family of functions $m(\lambda, x)$ defined for $Im\lambda > 0$, $x \geq a$, which are holomorphic in the open upper half-plane in λ, satisfy in x the Ricatti differential equation $m' = -p^{-1} - (\lambda w - q)m^2$, have positive imaginary part, and are basic to the spectral resolution of the operator concerned. This Riccati property can serve as a foundation of the theory of these Titchmarsh-Weyl-

Krein functions, and yield important information on their values (Argonne National Laboratory Reports, 1987-88). The talk describes extensions to the case of a matrix Sturm-Liouville equation, in which the Riccati equation deals with matrices in the (matrix) unit disc. Modifications for highly singular or highly oscillatory coefficients are also discussed.

H.E. Benzinger

The convergence of spectral families

H. Weyl's method of expanding intervals is considered for Banach spaces other than Hilbert space. The method is applied to the first-order operators which are the infinitesimal generators of the translation groups of $L^p(-a,a)$ for $1 < p < \infty$ and $0 < a \leq \infty$. These operators are well-boundable (see *Benzinger, Berkson* and *Gillespie*, Trans. AMS, 1983) and thus have spectral families. For fixed p, $1 < p < \infty$, the spectral family for $(-a,a)$ converges, uniformly to the spectral family for $(-\infty,\infty)$, as a tends to ∞.

R.C. Brown

Weighted higher order Hardy and interpolation inequalities on R

In this lectures based on recent joint work with *D.B. Hinton*, we develop sufficient conditions on W,P for the Hardy inequality

$$\int_a^b W(t) \mid u(t) \mid^p dt \leq K \int_a^b P(t) \mid u^{(m)}(t) \mid^p dt$$

to hold. We also study the product or "interpolation" inequality, $0 \leq j \leq m-1$,

$$\int_a^b N \mid u^{(j)} \mid^p \leq K \left(\int_a^b W \mid u \mid^p\right)^{1-(j+1)/m} \left(\int_a^b P \mid u^{(m)} \mid^p\right)^{(j+1)/m}$$

and note that if $m = 1$ or if $N = W$ when $m > 1$ and $j = 0$ implies a "Hardy-like" inequality whose domain includes at least the appropriate u having compact support in the interior of I. We give several examples for which $W = M^\alpha \mid M' \mid^\beta \omega$ and $P = M^\Delta \mid M' \mid^\Gamma \omega$ where the functions M,ω and the parameters $\alpha,\beta,\Delta,\Gamma$ satisfy various conditions and show how to extend the one variable theory to R^n for radial weight and shell domains. Recent results of *Stepanov* and *Reyes* and *Sawyer* on Hardy's inequality are also discussed.

A. Dijksma (jointly with H. Langer and H.S.V. de Snoo)

Hamiltonian systems with λ depending boundary conditions and their spectral properties

We consider the boundary eigenvalue problem

$$\text{BEP}\quad \begin{cases} Jy' - Hy = \ell\Delta y + \Delta f \text{ on } (a,b) \\ \mathcal{A}(\ell)y^1(a) + \mathcal{B}(\ell)y^2(a) = 0, y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}, \end{cases}$$

where the $2n \times 2n$ differential system is symmetric, Hamiltonian, definite, regular at a, limit point at b, and the $n \times n$ matrix functions $\mathcal{A}(\ell)$, $\mathcal{B}(\ell)$ are locally holomorphic on $\mathbf{C} \setminus \mathbf{R}$ in the eigenvalue parameter ℓ of the system. The conditions on $\mathcal{A}$, $\mathcal{B}$ are such that the linearization A of the BEP is a selfadjoint extension in a Π_κ space $\mathcal{K}$ of the minimal symmetric relation in $\mathcal{H} = L^2(\Delta dt, a, b)$ associated with the system. Thus, for $f \in \mathcal{H}$, and almost all $\ell \in \mathbf{C} \setminus \mathbf{R}$, $y \in \mathcal{H}$ is a solution of the *BEP* if and only if y $= R(\ell)f$, where $R(\ell)$ is the compressison of $(A - \ell)^{-1}$ to H. We relate the spectral properties of the *BEP* to those of A via a model connected with the Weyl coefficient $\Omega(\ell)$, as $2n \times 2n$ matrix appearing in the formula of $R(\ell)$.

M.S.P. Eastham

Higher–order differential equations with small oscillatory coefficients

We consider the asymptotic form of the solutions of

$$y^{(2n)} + (s_1 y^{(n-1)})^{(n-1)} + \dots + (s_{n-1} y')' + s_n y = 0$$

as $x \to \infty$, where the coefficients $s_m(x)$ have the form

$$s_m(x) = x^{-m\alpha} p_m(x)$$

with $0 < \alpha \leq 1$ and $p_m(x)$ has period 2π and mean-value zero. It is shown how to transform the $2n - th$ order equation into a first-order system of the type

$$Y'(x) = \{x^{-\alpha} P(x) + x^{-1} C + R(x)\} Y(x),$$

where P is an $2n \times 2n$ periodic matrix, C is constant, and R is $L(x_o, \infty)$. The existing asymptotic theory of such systems can then be used. The results extend known ones for the case $n = 1$. However, certain unexpected features arise when $n > 1$, and the lecture draws attention to these features as well as mentioning some open problems.

W. Eberhard

On the distribution of the eigenvalues of a class of indefinite eigenvalue problems

For eigenvalue problems $l(y) = \lambda r(x)y, U_\nu(y) = 0,\ 1 \leq \nu \leq n$ with a piecewise continuous indefinite weight function r we derive asymptotic formulas for the distribution of the eigenvalues λ_k. Assuming that the boundary conditions are regular (i.e. a generalization of the well known Birkhoff–regularity for the definite case $r(x) \equiv 1$) it will be proved that for $n = 2\mu$ there exist two sequences $\lambda_k^{(j)}$ of eigenvalues with the asymptotic behaviour
$\lambda_k^{(j)} = \pm(-1)^{n/2}\left(\frac{k\pi}{R_\pm}\right)^n\left[1 + O(\frac{1}{k})\right]$ (+ for $j = 1, -$ for $j = 2$) where
$R_\pm = \int_0^1 \sqrt[n]{r_\pm(t)}dt, r_\pm(t) = \max\{\pm r(t), 0\}$.
An analogous formula holds in the case $n = 2\mu - 1$.

W.D. Evans

Boundary conditions for general ordinary differential operators and their adjoints

A characterisation is obtained of all the regularly solvable operators and their adjoints generated by a general differential expression M in $L^2_w(a,b)$. The domains of these operators are described in terms of boundary conditions involving the $L^2_w(a,b)$ solutions of $Mu = \lambda wu$ and the adjoint equation $M^+v = \overline{\lambda}wv (\lambda \in \mathbb{C})$. The results include those of *Sun Jiong* concerning self-adjoint realisations of a formally symmetric M and also those of *Zai-jiu Shang* for the J-self-adjoint realisations in the case when M is formally J-symmetric.

W.N. Everitt (jointly with W.D. Evans)

Some results in integral inequalities and spectral theory

The lecture discusses two classes of Hardy-Littlewood type integral inequalities on the real line:

(HELP) $$\left(\int_a^b (pf'^2 + qf^2)\right)^2 \leq K \int_a^b wf^2 \int_a^b w(w^{-1}(-(pf')' + qf))^2$$

(KZ) $$\left(\int_a^\infty wf'^2\right)^2 \leq K \int_a^\infty wf^2 \int_a^\infty wf''^2.$$

Both inequalities can be represented as problems in the singular theory of second-order functionals in the calculus of variations. The Euler-Lagrange equations, for both inequalities, are linear quasi-differential equations of the fourth order. The spectral theory of these equations, in the weighted Hilbert function spaces $L^2_w(a,b)$ and $L^2_w(a,\infty)$, gives general results which lead to the characterization of the best possible numbers K and the cases of equality.

G. Freiling

Expansion theorems for non-selfadjoint differential operators

We show that the classical results of *L. Fejer* and *M. Riesz* concerning norm convergence of trigonometric Fourier series and that the results of *L. Carleson* and *R.A. Hunt* concerning pointwise convergence of trigonometric Fourier series can be carried over to eigenfunction expansions arising from regular indefinite boundary eigenvalue problems of the form

$$y^{(n)} + \sum_{\nu=2}^{n} f_\nu(x) y^{(n-\nu)} = \lambda r(x) y, \ x \in [0,1], \qquad U_\nu(y) = 0, \ 1 \leq \nu \leq n;$$

here $r : [0,1] \to \mathbf{R} \setminus \{0\}$ is assumed to be a step-function.

H.S.P. Grässer (jointly with W.N. Everitt)

Integral inequalities, spectral theory and the calculus of variations

The integral inequalities concerned, are of the Hardy-Littlewood type; in particular, the ones with monotonic weight factor investigated by *Kwong* and *Zettl*, and *Everitt* and *Guinaud*. It is indicated that these can be considered as well-posed unconstrained second order problems in the calculus of variations over infinite intervals, and that the Weierstraß–Carathéodory field theoretic approach (through Mayer fields) appears to be applicable. However, difficulties are pointed out regarding the construction of the Mayer fields in the KZ(EG) case. Hence the theory of quasi-differential equations and spectral theory seem to provide the best possibility of progress in these singular problems for which there appears to be no general theory.

B.J. Harris

An eigenvalue problem with interior singularities

In a recent paper (Argonne National Laboratory, 1988), *Atkinson* considers the asymptotic form of the eigenvalues of the linear differential equation

$$(*) \qquad -y'' + qy = \lambda y \quad -\infty < a < x < b < \infty, a < 0 < b.$$

In particular *Atkinson* considers (*) where q is singular at 0. In the case $q(x) = x^{-K}$ his results cover the case $1 \leq K < \frac{4}{3}$. We show how to extend Atkinson's results to cover more singular q, in the power case $1 \leq K < \frac{3}{2}$.

D. Hinton

Titchmarsh-Weyl m-functions

Sturm-Liouville and Dirac operators are considered on $a \leq x < \infty$, and it is assumed that the coefficients are integrable on $[a, \infty)$. For such operators, the Titchmarsh-Weyl m-function is given as the ratio of two series which converge on $Im\lambda \geq 0$ for $|\lambda|$ sufficiently large. For coefficients where Nth derivatives exist and are integrable on

$[a, \infty)$, an asymptotic expansion to order λ^{-N} is derived which is valid on $Im\lambda \geq 0$ as $\lambda \to \infty$. These results contrast with previous work in that the representations are valid for real as well as complex λ.

R.M. Kauffman

Continuous spectrum eigenfunction expansions for differential operators

Eigenfunction expansions for the equations $y'' + py = \lambda y$ in $L_2(\mathbf{R}^1)$, with p bounded, continuous and even, are discussed. A number of recent results about the odd and even eigenfunctions f_λ and g_λ which actually appear in the expansion are given; for example, for any $\varepsilon > 0$ $\{\lambda \mid (x^2+1)^{-(\frac{1}{2}+\varepsilon)} f_\lambda \notin L_1\}$ has even spectral measure zero, with a corresponding assertion for the odd eigenfunctions. Attention was called to the remarkable recent result of *Naboko*, which indicates the difficulties which exist in classifying spectral measure.

I. Knowles

Sturm–Liouville equations and the Riemann hypothesis

There exists a differential equation

$$(1) \qquad \frac{d}{dr}\left(e^{(\mu-s)r} b(r) \frac{du}{dr}\right) + (\mu - s)^2 c(r)\, e^{(\mu-s)r} u = 0 \quad 0 \leq r < \infty$$

with the property that the Riemann hypothesis holds if and only if (1) has a solution $u(r,s)$ that is asymptotic to $e^{-\int_{\frac{3}{2}}^{r} [p^{-s}(\log p)^{-1}]dp}$ for all real s, $\frac{1}{2} < s < 1$, and a certain analyticity condition holds. The connection with the *Faddeev–Pavlov* (1972) results for the automorphic wave equation, will also be discussed.

A.M. Krall

The decomposition of $M(\lambda)$ surfaces using Nießen's limit circles

A connection is made between the $M(\lambda)$ matrix of *Hinton–Shaw* and the 2-dimensional subspaces of *H.D. Nießen*. It is shown that the surface on which the $M(\lambda)$ matrices lie can be represented as a generalization of a sum of Nießen circles, namely ellipsoids, some of whose directions may have collapsed.

M.K. Kwong

Uniqueness of some Emden–Fowler boundary value problems

An open problem that I am currently working on is:

for any $b \in (\frac{\pi}{2}, \pi)$, there is exactly one solution to the boundary value problem
$u'' + \frac{2}{7}u' + u^5 + u = 0,\ u > 0$ in $(0, b), u'(0) = 0$ and $u(b) = 0$.

This was first raised by *Brezis* and *Nirenberg* (1983) who established existence but not uniqueness of the solution. A survey on two methods of proving uniqueness for Emden–Fowler equations of more general forms will be given. Neither is powerful enough to resolve the conjecture completely. It is only known that for b larger than some number uniqueness holds.

H. Langer

Some direct and inverse spectral problems for strings and canonical systems

Starting from fundamental results of *M.G. Krein* about spectral functions of a string (that is the equation $df' + \lambda f dM = 0$ on $[0, \ell)$, $f'(0) = 0$) and of *L. de Branges* about 2×2 canonical systems ($Jy' = zHy$ with a trace-normed Hamiltonian H on $[0, L)$), eigenvalue problems of the form $\lambda^2 f dD + \lambda f dM + df' = 0$ on $[0, 1)$ with suitable boundary conditions are considered (D – nondecreasing, $M = M_+ + M_-, M_\pm$ – nondecreasing).The corresponding Weyl coefficient is introduced and in special cases where $supp(D + M_-)$) consists of finitely many points some related inverse problems are solved.

D. Race

Some problems concerning scalar, linear ordinary quasi–differential expressions (in collaboration with A. Zettl)

We ask: when do two ordinary, linear quasi–differential expressions (of the type introduced by *Shin* and *Zettl*) commute? For classical differential expressions, answers to this question are well known. The set of all expressions which commute with a given such expression form a commutative ring. For quasi–differential expressions such an algebraic structure can no longer be exploited. Instead we use the equivalence of matrices which determine the same expression as each other, based upon work by *Frentzen* as well as *Everitt* and *Race*. We thereby obtain a complete classification of all real symmetric expressions of both second order and fourth order which commute with any given real symmetric expression of second order. We can also classify pairs of two–term real symmetric expressions of order $2n$ which commute. In each case, there is a corresponding result for commuting expressions having complex coefficients but which are J–symmetric.

B. Schultze

On singular differential operators with positive coefficients

For real formally selfadjoint differential expressions on $I = [a, \infty)$ $a \in \mathbf{R}$, $n \in \mathbf{N}$

$$My = \sum_{i=0}^{n} (-1)^i \left(p_i y^{(i)}\right)^{(i)}, p_i \in C^i(I, \mathbf{R}), p_n > 0$$

with positive p_i $(i = 0, .., n-1)$ it is shown that for the deficiency index all values of the set $\{n + 2j \mid 0 \le j < \frac{n}{2}, j \in \mathbf{N}_o\}$ actually occur. This generalizes results of *Paris* and *Wood* on the conjecture of *McLeod* that all integers k with $n \le k \le 2n - 1$ occur as deficiency index for expressions with non–negative coefficients.

J.K. Shaw

Half–bound states of the perturbed Hill's equation

We consider Hill's equation $-y'' + q(x)y = \lambda y$, $0 \leq x < \infty$, with $q(x)$ is periodic, $q(x+1) = q(x)$, and the corresponding perturbed equation $-y'' + (q(x) + p(x))y = \lambda y$, where $\int_0^\infty (1+x) \mid p(x) \mid dx < \infty$. Both free and perturbed equations have the same essential spectrum, namely the bands associated with $q(x)$, but either may have finitely many eigenvalues for the problem $y(0) = 0$ in the gaps. The ends of gaps are eigenvalues, but may be half–bound states of the perturbed equation; i.e., the solutions satisfying $y(0) = 0$ are bounded but not in $L^2(0,\infty)$ We characterize the half–bound states by studying the behavior of the Titchmarsh–Weyl coefficient at the ends of the gaps.

H.S.V. de Snoo

Generalized resolvents, Weyl coefficients and their kernels

Let S be a closed symmetric relation in a Hilbert space $\mathcal{H}$. Let A be a selfadjoint Krein space extension of S with generalized resolvent $R(\ell) = P_{\mathcal{H}}(A-\ell)^{-1}\mid_{\mathcal{H}}$. Then $R(\ell)$ can be expressed in terms of one fixed (Krein space) generalized resolvent $\mathring{R}(\ell)$ and the characteristic functions describing S, $\mathring{R}(\ell)$ and $R(\ell)$. If we suppose that S has equal defect numbers and we choose a holomorphic basis $s(\ell)$ in the null space $\nu(S^* - \ell)$, then the above description takes the form $R(\ell) = R_o(\ell) + s(\ell)\Omega(\ell)[\cdot, s(\bar{\ell})]$, where $\Omega(\ell) = -(U(\ell)\mathcal{S}(\ell))^{-1}U(\ell)\mathcal{G}(\ell)$ where $U(\ell)$ are boundary conditions for $R(\ell)$, $\mathcal{S}(\ell)$ are boundary values of $s(\ell)$ and $\mathcal{G}(\ell)$ is expressed in terms of $\mathring{R}(\ell)$ and $s(\ell)$. The kernels for $U(\ell)$ and $\Omega(\ell)$ are related and describe the extension space. Our results are applied to canonical systems and Hamiltonian systems in the limit point case, with boundary conditions involving Stieltjes integrals and depending on the eigenvalue parameter. This is joint work with *A. Dijksma* and *H. Langer.*

A. Zettl

Computing eigenvalues of Sturm–Liouville problems

In ACM–TOMS 4 (1978) *Bailey*, *Gordon* and *Shampine* describe a software package called *SLEIGN*.It computes the eigenvalues and eigenfunctions of regular and singular Sturm–Liouville problems. In the regular case for general separated boundary conditions; in the singular case the code automatically selects a special, particular boundary condition. Here we describe a new code, called *SLEIGN 2* to compute the eigenvalues for general, separated, singular self–adjoint ("limit–circle") boundary conditions. This code was developed by *Bailey*, *Everitt* and *Zettl* and is based on a new algorithm.

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