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Elementare und Analytische Zahlentheorie

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This conference on *"Elementary and Analytic Number Theory"* was organized by Prof. Dr. H.-E. Richert (Ulm), Prof. Dr. W. Schwarz (Frankfurt) and Prof. Dr. E. Wirsing (Ulm).
It was attended by 43 participants, and in 39 high leveled and well prepared lectures, the most recent results were communicated from research areas as:

> algebraic linear recurrences, arithmetical functions, character sums, continued fractions, Dirichlet series, the divisor function (in arithm. progr.), divisor problems, exponential sums, Goldbach numbers, moments of the Riemann Zeta-function, multiplicative functions, quadratic forms, Waring's problem and others.

Further details can be found in the abstracts of the lectures.
On the background of the uniquely stimulating atmosphere of the Institute we all had a great time of learning and exchanging ideas. The organizers express their thanks to the Land Baden-Württemberg, the Director of the Institute, Prof. Barner and his staff for providing this enjoyable environment.

Abstracts

Pairs of quadratic forms modulo one

This is joint work with J. Brüdern. Let Q_1 and Q_2 be quadratic forms with real coefficients. We consider the problem of finding $\lambda > 0$ for which:
Given $N > C(s,\varepsilon)$ there is $\underline{n} \in \mathbb{Z}^s$, $0 < |\underline{n}| \le N$ satisfying

$$\max(\|Q_1(\underline{n})\|, \|Q_2(\underline{n})\|) < N^{-\lambda+\epsilon}.$$

The equivalent problem with a single quadratic form was first considered by Danicic and subsequently by Schinzel, Schlickewei and Schmidt and then by Baker and Harman. A further sharpening is due to Heath–Brown following new bounds for least solutions of quadratic congruences. This work is helpful for evaluating λ only for $s \ge 6$. We use Schmidt's lattice method to get (e. g.) $\lambda(2) = \frac{1}{5}$. Our results improve all previous work for $s \ge 2$.

R. C. Baker, Egham, Surrey.

Diagonal equations in p–adic fields

Given integer coefficients a_{ij} we consider the following system over $\mathbb{Q}_p$:

$$\begin{aligned} a_{11}x_1^k + \ \dots \ + a_{1s}x_s^k &= 0 \\ \vdots \qquad\qquad & \ \ \vdots \\ a_{r1}x_1^k + \ \dots \ + a_{rs}x_s^k &= 0\ . \end{aligned} \tag{1}$$

Theorem (Atkinson, Brüdern, Cook). Let $s > 2rk$ and $p > k^{2r+2}$. Then (1) has a non–trivial solution in $\mathbb{Q}_p$.
This result is best possible with respect to s, and settles a special case of a recent conjecture by T. Wooley. Some applications to integer solutions of (1) are also mentioned.

J. Brüdern, Göttingen.

Positive integers for which $\Omega(n) = k$

Let $\nu_p(n)$ be the exponent of the prime p in the factorization of n, $S(x,k) = \{n \le x : \Omega(n) = k\}$, $N(x,k)$ the number of elements of $S(x,k)$, $y = x/2^k$ and $Q(\lambda) = \lambda\log\lambda - \lambda + 1$. It is supposed here that $2\lambda\log\log x \le k \le \log\frac{x}{3}/\log 2$, λ fixed $\in(1,\frac{3}{2})$. A theorem stated in the last meeting and proved at the conference in honour of Paul Bateman in 1989 shows that, if $y \to \infty$, then the distribution of the numbers $(\nu_2(n) - (k - 2\log\log y))/\sqrt{2\log\log y}$ (1) on $S(x,k)$ converges to the normal distribution of Gauss. Now we give the following

Theorem.

$$\frac{1}{N(x,k)} \sum_{n \in S(x,k)} \nu_2(n) = k - 2\log\log y + a + O\left((\log\log y)^{1/2} \log^{-2Q(\lambda)} y\right),$$

$$\frac{1}{N(x,k)} \sum_{n \in S(x,k)} \omega(n) = 2\log\log y + b + O\left((\log\log y)^{1/2} \log^{-2Q(\lambda)} y\right),$$

$$a = 2(1 + \log 2 - \gamma - \sum_{p>2} \{\log(1 - \tfrac{1}{p}) + \tfrac{1}{p-2}\}), \qquad b = 1 - a - 4 \sum_{p>2} \frac{1}{p(p-2)}.$$

I can also obtain formulas for the moments of order q of the numbers (1) on $S(x,k)$.

H. Delange, Orsay.

Multiplicative functions on arithmetic progressions (with application to correlations of multiplicative functions and sums of additive functions)

Let the complex-valued multiplicative function g satisfy $|g(n)| \leq 1$ for all $n \in \mathbb{N}$. A survey of a series of papers concerning such functions was given (I–VI). A typical result (from paper VI): Given $\varepsilon > 0$, $x \geq 2$,

$$\sum_{\substack{n \leq x \\ n \equiv r \bmod D}} g(n) - \frac{1}{\varphi(D)} \sum_{\substack{n \leq x \\ (n,D)=1}} g(n) \ll \frac{x}{\varphi(D)} \left(\frac{\log Q}{\log x}\right)^{\frac{1}{8}-\varepsilon},$$

uniformly for $(r, D) = 1$, all $D \leq Q$ save possibly for the multiples of a $D_0 > 1$. If $0 < \beta < 1$, then this result is uniform (i. e. there is the same exceptional D_0, if at all) for $N^\beta \leq x \leq N$. As a consequence, a proof of the estimate

$$\pi(x; D, r) = \frac{x}{\varphi(D)\log x} \left\{ 1 + O\left(\left(\frac{\log Q}{\log x}\right)^{\frac{1}{8}-\varepsilon}\right)\right\}$$

with the same uniformities and $Q \leq N^{\beta-\varepsilon_0}$, holds for primes. The novelty is that the proof does not use estimates for the density or non-existence of zeros of L-series in the half-plane $\mathrm{Re}(s) \leq 1$.

As an application (via the correlation of multiplicative functions):

Let f_j be real additive functions, $\beta_j(x) > 0, \to \infty$, and for $y > 0$, $\beta_j(x^y)/\beta_j(x) \to 1$ as $x \to \infty$. Then

$$\nu_x\left(n; \sum_{j=1}^{2} \frac{f_j(a_j n + b_j) - \alpha_j(x)}{\beta_j(x)} \leq z\right) \implies F(z), \qquad \text{as } x \to \infty,$$

for a suitable $\alpha_j(x)$ where $a_j > 0$, $a_1 b_2 \neq a_2 b_1$, iff there are constants B_j so that independent random variables X_p distributed according to

$$X_p = \begin{cases} (f_1(p) - B_1 \log p)/\beta_1(x) & \text{with probability } 1/p, \\ (f_2(p) - B_2 \log p)/\beta_2(x) & \text{with probability } 1/p, \\ 0 & \text{with probability } 1 - 2/p, \end{cases}$$

satisfy $P\left(\sum_{p\le x} X_p - \gamma(x) \le z\right) \underset{x\to\infty}{\Longrightarrow} F(z)$, for a suitably chosen $\gamma(x)$.

P. D. T. A. Elliott, Boulder.

The divisor function over arithmetic progressions

This paper – in collaboration with H. Iwaniec with an appendix by N. Katz – deals with the problem of evaluating the sum

$$D_\tau(x;q,a) = \sum_{n\le x,\, n\equiv a \bmod q} \tau(n) \qquad (a,q)=1, \quad \tau \text{ the divisor function, by}$$

$$D_\tau(x;q) = \frac{1}{\varphi(q)} \sum_{n\le x,\,(n,q)=1} \tau(n).$$ We prove the following

Theorem 1. Let r be squarefree with $(a,r)=1$, $r\le x^{3/8}$ we have

$$\sum_{\substack{s\le r^{-1/2}x^{(1-6\epsilon)/2}\\ (s,ar)=1}} \left| D_\tau(x;rs,a) - \frac{1}{\varphi(rs)} D_\tau(x;rs)\right| \ll_\epsilon r^{-1}x^{1-\epsilon}.$$

This theorem gives an approximation of $D_\tau(x;rs)$, for a special type of averaging, when rs is around the critical value $x^{2/3}$. We require the

Theorem 2. Let $S(m,n;q)$ be the Kloosterman sum $\sum_{uv\equiv 1 \bmod q} e\left(\frac{mu+nv}{q}\right)$. Let r be squarefree with $(a,r)=1$ and $\lambda_l\in\mathbb{C}$. Then we have

$$\sum_{s\le S,\,(s,r)=1} \left| \sum_{l\le L} \lambda_l\, S(a,l;rs)\right|^2 \ll \left(r^{-1/4}+r^{1/4}S^{-1/2}+SL^{-1}\right) LS^2 r^{1+\epsilon}\sum|\lambda_l|^2.$$

The proof of Theorem 2 is based on an upper bound for an exponential sum of dimension five. Such a sum is treated by very deep techniques of algebraic geometry.

E. Fouvry, Orsay.

Some estimates for character sums

We state two theorems from joint work with H. Iwaniec and very briefly sketch the ideas behind the proofs.

Theorem 1. For sequences α_k, β_ℓ, $k\le K$, $\ell\le L$, let

$$S^* = \frac{1}{\varphi(q)-1} \sum_{\substack{\chi \bmod q\\ \chi\ne\chi_0}} \left| \sum_k \alpha_k\,\chi(k) \sum_\ell \beta_\ell\,\chi(\ell) \sum_{m\le M} \chi(m)\right|^2.$$

We have

$$S^* \ll \sum_k |\alpha_k|^2 \sum_\ell |\beta_\ell|^2\, M \left\{1+q^{-3/4}(K+L)^{1/4}(KL)^{5/4}+q^{-1}(KL)^{1/4}\right\} q^\epsilon.$$

The proof depends on nothing more complicated than the Poisson summation formula. By adding on estimates for Kloosterman sums it can be easily sharpened. By choosing $\alpha_k = \overline{\chi}(k)$, $\beta_\ell = \overline{\chi}(\ell)$ we get the following

Corollary. For χ a non-principal character mod q and $M > q^{\frac{5}{11}+\epsilon}$ we have

$$\sum_{m \leq M} \chi(m) \ll M^{1-\delta(\epsilon)}.$$

This improves the Polya–Vinogradov range without using advanced tools.

The second theorem deals with sums $S = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \chi(a+b)$ where χ has prime modulus p and $\mathcal{A} \subset [M, M+A]$ has cardinality $|\mathcal{A}|$, $\mathcal{B} \subset [N, N+B]$ has cardinality $|\mathcal{B}|$. As the theorem is complicated we state two corollaries.

Corollary 1. Let $|\mathcal{A}| = |\mathcal{B}|$, $A \ll p^{1/2}$. Then $|\mathcal{A}| > p^{\frac{11}{24}+\epsilon} \Rightarrow S \ll |\mathcal{A}|\,|\mathcal{B}|\,p^{-\delta}$.

(Note: In case $M = N = 0$, $\frac{11}{24}$ can be replaced by $\frac{9}{20}$.)

Corollary 2. Let $p^\epsilon \leq B \leq A$, $p^{\frac{1}{2}+\epsilon} \leq AB \leq p$. Then $\exists\ \delta(\epsilon) > 0$ such that $\forall \mathcal{A}, \mathcal{B}$ contained in intervals of length A, B and such that $|\mathcal{A}| > Ap^{-\delta}$, $|\mathcal{B}| > Bp^{-\delta}$, we have $S \ll |\mathcal{A}|\,|\mathcal{B}|\,p^{-\delta}$.

J. B. Friedlander, Scarborough, Ontario.

Squarefree values of binary forms of degree six

A paper to appear in the *Quart. J. Math. (Oxford) Ser.* will contain the following result: Suppose $k \geq 2$. Let $f(a,b)$ denote an integral binary form (homogeneous in two variables) with non-zero discriminant, of degree d, and containing non-zero terms in a^d and b^d. Suppose $f(a,b)$ has no non-trivial k-th power divisors, and that its irreducible factors have degrees not exceeding D. Let $N(x)$ denote the number of pairs of integers a, b with $1 \leq a; b \leq x$, for which $f(a,b)$ is k-free. Then, for a certain $C = C_f > 0$,

$$N(x) = C\,x^2 + O(E(x))$$

where

i. $E(x) = x^2 \log^{-1/3} x$ if $k = 2$, $D \leq 6$

ii. $E(x) = x^2 \log^{-1} x$ if $D \leq 2k+1$

We discuss the case $k = 6$ and f irreducible, describing an improvement noticed by K. Ramsay at Harvard which leads to an improved estimate $E(x) = x \log^{-1/2} x$ in i.

G. Greaves, Cardiff.

On consecutive k–th power residues

A. Brauer showed in 1928 that for any given positive integers $k \geq 2$, $\ell \geq 2$ and every sufficiently large prime p there exists a positive integer n such that $n, n+1, \ldots, n+\ell-1$ are k–th power residues modulo p. Let $\Lambda(k,\ell,p)$ denote the least such n and set $\Lambda(k,\ell) = \limsup_{p\to\infty} \Lambda(k,\ell,p)$. In the 1960's a number of authors have investigated $\Lambda(k,\ell)$ for small values of k and ℓ, partly with aid of computers. In particular, it has been shown that $\Lambda(k,2) < \infty$ for $2 \leq k \leq 7$, and it had been conjectured that $\Lambda(k,2)$ is finite for all $k \geq 2$. In *Monaths. Math.* **102** (1986), I proved the conjecture in the case $k \in \mathbb{P}$ and recently extended the argument to all $k \geq 2$.

A. Hildebrand, Urbana.

Remarks on exponential sums

Bombieri and Iwaniec's method: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **13** (1986) for estimating general exponential sums $\sum_n e(f(n))$ rests on a local polynomial approximation $f(m+x) \approx f(m) + \frac{b+\kappa}{q} x + \frac{a}{q} x^2 + \mu x^3$, where a, b, q are integers, and the residual coefficients κ, μ are real and suitably bounded. We must choose the parameters of the method to be sure that the Fourier transforms of these polynomials are usually distinct. We can now calculate how the approximations are related for nearby values of m, and use this information to relax the constraints on the parameters. This leads to slightly sharper bounds for exponential sums. As applications $\zeta(\frac{1}{2}+it) = O\left(t^{89/570+\epsilon}\right)$ as $t \to \infty$, and the error terms in the circle problem and the Dirichlet divisor problem are $O\left(x^{23/73+\epsilon}\right)$. There is some hope of a further improvement, but only in the fifth decimal place.

M. N. Huxley, Cardiff.

Elementary proofs of Halász's theorem

In this talk we present a paper of Daboussi and Indlekofer, which is dedicated to the memory of Prof. Th. Schneider and will appear in the *Math. Z.* The aim of this paper is to give two new and elementary proofs of the following

Theorem of G. Halász: Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative and $|f| \leq 1$. Then there exist constants $c \in \mathbb{C}$, $a_0 \in \mathbb{R}$ and a slowly oscillating function $L(u)$ with $|L(u)| = 1$ so that, as $x \to \infty$

$$x^{-1} \sum_{n \leq x} f(n) = c x^{ia_0} L(\log x) + o(1) . \qquad (1)$$

The common and main idea of both proofs is to show that the means $\frac{1}{x}\int_0^x \alpha(w)\,dw$ (2) tend to a limit α as $x \to \infty$, when $\alpha(w) = e^{-w} \big| \sum_{n \leq e^w} f(n) \big|$. In the first proof we

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deduce from this the relation

$$\int_0^\infty \alpha^2(w)\, e^{-2w(\sigma-1)}\, dw \;=\; \frac{\alpha^2}{2(\sigma-1)} + o\Big(\frac{1}{\sigma-1}\Big) \qquad \text{as } \sigma \to 1^+ \tag{3}$$

which leads to the asymptotic formula (1). Concerning the second proof we observe that $\alpha(w)$ is slowly oscillating and thus fulfills a Tauberian condition for the summability method (2). This allows us to prove the existence of the limit $\lim_{w\to\infty} \alpha(w)$ (which equals α), and again via (3) we deduce the asymptotic formula (1).

K.-H. Indlekofer, Paderborn.

On the error term for the fourth moment of the Riemann Zeta–function

This is joint work with Y. Motohashi on

$$E_2(T) := \int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^4 dt \;-\; T\,P_4(\log T),$$

where $P_4(y)$ is a suitable polynomial in y of degree four with leading coefficient $\frac{1}{2\pi^2}$. We use Motohashi's formula for

$$\frac{1}{\Delta\sqrt{\pi}} \int_{-\infty}^{\infty} \left|\zeta(\tfrac{1}{2}+iT+it)\right|^4 e^{-(t/\Delta)^2} dt \qquad (0<\Delta\le T/\log T)$$

and some of his recent results on spectral mean values of Maass wave form L–functions to prove

$$E_2(T) = O(T^{2/3}\log^{c_1} T), \qquad E_2(T) = \Omega(T^{1/2}),$$
$$\int_0^T E_2(t)\,dt = O(T^{3/2}), \qquad \int_0^T E_2^2(t)\,dt = O(T^{9/4}\log^{c_2} T)$$

with effectively computable $c_1, c_2 > 0$. We also slightly improve the well–known result of H. Iwaniec (*Sém. Théorie des Nombres, Bordeaux* 1979/80) on sums of integrals of $|\zeta(\frac{1}{2}+it)|^4$ over short intervals, by replacing T^ϵ by log–powers.

A. Ivić, Beograd.

The Weyl – van der Corput method for exponential sums involving the divisor function

The classical Weyl – van der Corput method in the theory of ordinary exponential sums consists of two procedures: the Weyl shift and the transformation of exponential sums by harmonic analysis and the saddle point method. Let $d(n)$ be the usual divisor function and consider the exponential sum $\sum_{N\le n\le N'} d(n)\, e(f(n))$, where f is a sufficiently smooth function.

The Weyl shift leads to sums $\sum_{N \le n \le N'} d(n)\,d(n+k)\,\mathrm{e}\,(f(n+k)-f(n))$;
so the problem is reduced to sums of the type $S = \sum_{N \le n \le N'} d(n)\,d(n+k)\,\mathrm{e}\,(f(n))$.
This can be transformed in two ways:

i. by using an identity of N. V. Kuznetsov for sums of the type $\sum_{n=1}^{\infty} d(n)\,d(n+k)\,w(\frac{n}{k})$,

ii. via the sum function $D_k(x) = \sum_{n \le x} d(n)\,d(n+k)$.

The latter possibility is considered in some detail. A function analogous to $\zeta_k(s)$, $\zeta_k(s) = \sum_{n=1}^{\infty} d(n)\,d(n+k)\,n^{-s}$ can be constructed in terms of non–holomorphic Eisensteins' series as shown by L. A. Tahtadjan and A. I. Vinogradov in 1984, and this function, say $\zeta_k^*(s)$ plays the role of $\zeta_k(s)$ in Perron's formula for $D_k(x)$. On the other hand, $\zeta_k^*(s)$ can be represented in terms of the hyperbolic Laplacian $\mathcal{L} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, by using its spectral resolution. Combining these facts, one may express S (or a smoothed variant) in terms of the discrete spectrum of $\mathcal{L}$, the contribution of the continuous spectrum being less significant. Applied to Dirichlet polynomials $\sum_{N \le n \le N'} d(n)\,n^{-\frac{1}{2}-it}$, $N \asymp t$, these arguments lead to an analogue of a recent formula due to Y. Motohashi for the mean value

$$\int_{-\infty}^{\infty} \left|\zeta(\tfrac{1}{2} + iT + it)\right|^4 \mathrm{e}^{-(t/\Delta)^2} dt\,.$$

M. Jutila, Turku.

On zeros of certain Dirichlet series

Theorems are proved on the estimates from below of the number of zeros lying on the critical line for certain Dirichlet series, in particular, for the Davenport–Heilbronn function and zeta–function of Epstein. The proof essentially uses the fact that the Dirichlet series under consideration are divisible by some Euler products.

A. A. Karacuba, Moscow.

On arithmetical functions

Let the set of primes $\mathcal{P}$ be subdivided into disjoint classes $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_d$, where $\mathcal{P}_0$ contains only finitely many elements. Let $\mathcal{A} = \{1, \ldots, d\}$ and $\mathcal{A}^*$ be the set of words over $\mathcal{A}$, that is $\mathcal{A}^* = \{\alpha = i_1 \ldots i_k : i_\nu \in \mathcal{A}\}$. Let $\lambda(\alpha) = k$ denote the length of α, and $\mathcal{A}_t = \{\alpha : \alpha \in \mathcal{A}^*, \lambda(\alpha) = t\}$, $\mathcal{A}_0 = \{\Lambda\}$, Λ the empty word. Let $H : \mathcal{P} \to \mathcal{A}_0 \cup \mathcal{A}_1$ defined by $H(p) = \Lambda$ if $p \in \mathcal{P}_0$, $H(p) = i$ if $p \in \mathcal{P}_i$. For $n \in \mathbb{N}$

having the prime decomposition $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ $(p_1 < \dots < p_r)$ we define $H(n) = H(p_1) \dots H(p_r)$, thus $H\colon \mathbb{N} \to \mathcal{A}^*$. For $n \in \mathbb{N}$, $\alpha \in \mathcal{A}^*$ let $U_\alpha(n)$ be the number of occurency of α in $H(n)$, that is $U_\alpha(n) = \#\{\xi \in \mathcal{A}^*\colon H(n) = \xi\alpha\eta\}$. Let $\sigma(n)$ be the largest t such that each $\gamma \in \mathcal{A}_t$ occurs as a subword in $H(n)$, i. e. $H(n) = \xi\alpha\eta$ can be solved, and let $\mathcal{L}_k$ be the k–fold iterates of $\log x$.

Theorem 1. Assume that $\sum_{y \le p \le x,\, p \in \mathcal{P}_i} \frac{\log p}{p} = q(i) \log\left(\frac{x}{y}\right) + O\left(\frac{1}{\log y}\right)$ as $y < x \to \infty$ holds for $1 \le i \le d$. Let $Q(\alpha) = q(i_1) \cdots q(i_k)$ for $\alpha = i_1 \cdots i_k$ and with $c_2(\alpha)$ an explicitly calculable positive constant, $\tau_\alpha(n) = c_2^{-1}(\alpha)\, \mathcal{L}_2^{-1/2}(U_\alpha(n) - Q(\alpha)\mathcal{L}_2)$. Then for every fixed $\alpha_0, \dots, \alpha_h \in \mathcal{A}^*$

$$\lim_{x \to \infty} x^{-1} \#\{n \le x\colon \tau_{\alpha_\ell}(n+\ell) < y_\ell,\ \ell = 0, \dots, h\} = \prod_{\ell=0}^{h} \varphi(y_\ell)\,.$$

Let from now on $\mathcal{P}_j = \{p : p \equiv \ell_j \bmod D\}$, $\mathcal{P}_0 = \{p : p \mid D\}$, where $\ell_1 < \dots < \ell_d$, $d = \varphi(D)$ are the reduced residue classes mod D, and let $p(n)$, $P(n)$ be the smallest, largest prime factors of n resp.

Theorem 2. Assume that $y = y(x) \to \infty$ such that $y \ll \mathcal{L}_3$. For every $n \in \mathbb{N}$ we write $n = An_1$, where $P(A) \le y$ and $p(n_1) > y$ or $n_1 = 1$. Then there exists $\delta = \delta(x) \to 0$ such that uniformly as $1 \le A \le \mathcal{L}_2$, $|(\lambda(\alpha) - \mathcal{L}_2)\mathcal{L}_2^{-1/2}| < 1/\delta$ with $K(x) = [(\mathcal{L}_3 - \mathcal{L}_4) \log^{-1} d]$, for every $\alpha \in \mathcal{A}^*$ we have

$$\frac{\#\{n = An_1 \le x : H(n_1) = \alpha\}}{\#\{n = An_1 \le x : \omega(n_1) = \lambda(\alpha)\}} \longrightarrow \frac{1}{d^{\lambda(\alpha)}} \quad \text{as } x \to \infty, \quad \text{and}$$

$$x^{-1} \#\{n \le x\colon \sigma(n) \ne K(x) \text{ or } K(x) + 1\} \longrightarrow 0\,.$$

I. Kátai, Budapest.

Estimates in the general divisor problem

Let $d(a;n)$ denote the number of representations of $n \in \mathbb{N}$ in the form $n_1^{a_1} n_2^{a_2} \cdots n_p^{a_p} = n$, $a = (a_1, \dots, a_p)$, $a_1, \dots, a_p \in \mathbb{N}$ with $a_1 \le \dots \le a_p$ $(p \ge 2)$. It is well–known that $D(a;x) = \sum_{n \le x} d(a;n) = H(a;x) + \Delta(a;x)$, where the main term is given by $H(a;x) = \sum_{i=1}^{p} \prod_{\substack{j=1 \\ j \ne i}}^{p} \zeta\left(\frac{a_j}{a_i}\right) x^{1/a_i}$ $(a_1 < \dots < a_p)$. We are interested in estimations of the remainder $\Delta(a;x)$ without or with weak conditions on $a_1, \dots, a_p$. A trivial estimation is $\Delta(a;x) \ll x^{(p-1)/A_p}$, $A_p = a_1 + a_2 + \dots + a_p$. By means of the theory of estimating multiple exponential sums we proved the following theorems:

Theorem 1. $\Delta(a;x) \ll x^{\frac{p-1}{a_1+A_p}} \log^p x.$

This is the only non–trivial result without any conditions on $a_1, \ldots, a_p$. The next theorem is an improvement of Theorem 1 with weak conditions.

Theorem 2. Let $p \geq 3$ and let (k,l) be any exponent pair. Then

$$\Delta(a;x) \ll x^{\frac{k+l+p-2}{(k+l)a_1+A_p}} \log^p x.$$

This estimation holds under the conditions (1) $(2k+1)(a_2+a_3) \geq 2(k+l+1)a_1$, (2.1) if $la_1 \leq ka_2$ then $(k+l+1)a_1 \geq k(a_2+a_3)$ or (2.2) if $la_1 \geq ka_2$ then $(l-k)(2k+1)a_3 \leq (2l-2k-1)(k+l+1)a_1 + (2k(k-l+1)+1)a_2$.

E. Krätzel, Jena.

The radius of convergence of power product expansions

Let $f(z)$ be a complex function analytic in some neighbourhood of the origin with $f(0) = 1$. It is known that $f(z)$ admits a unique 'power product' expansion of the form $f(z) = \prod_{n=1}^{\infty}(1 + b_n z^n)$ convergent in a suitably small neighbourhood of the origin. In joint work with A. Knopfmacher the following result was obtained in 1990: The radius of convergence of the above power product expansion is at least $(\sup\{ \sqrt[n]{|a_n|} : n \in \mathbb{N} \})^{-1}$, where $\log f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} z^n$. This bound is in general best possible.

L. Lucht, Clausthal.

Abstract harmonic analysis and arithmetical functions

In "Generalized multiplicative functions" *Acta Arith.* **32** (1977), §5.3 I. Rusza considers group valued additive arithmetical functions and sets the following problem: "Can one formulate and prove the analogues of the well–known global limit theorems for additive arithmetical functions with values in a group G which is not only $\mathbb{R}$, Π or a discrete group?" A complete solution to this question has been provided when the group G is a locally compact abelian metrizable group. (So, this result contains the cases of $G = \mathbb{R}$, Π or a discrete group.) The method leads also to the following

Theorem. If G is a locally compact abelian group, and f is an additive arithmetical function with values in G, satisfying $\lim_{n\to\infty}\{f(n+1) - f(n)\} = 0$, then there exists a continuous homomorphism φ: $\mathbb{R} \to G$ such that $f(n) = \varphi(\log n)$ for

$n \in \mathbb{N}$. The case $G = \mathbb{R}$ is due to P. Erdős, $G = \Pi$ to E. Wirsing, and the extension to a metrical compactly generated locally compact group was considered by Daróczy and Kátai.

J.-L. Mauclaire, Paris.

Some explicit continued fractions in the field of formal power series

This is joint work with J. P. Allouche, A. J. van der Poorten.

Let K be an arbitrary field and let $K((X^{-1}))$ be the field of formal power series over K. Then $f = f(X) \in K((X^{-1}))$ can be expanded in a continued fraction $f = [c_0(X), c_1(X), c_2(X), \dots]$ where $c_j(X)$ are polynomials and $j \geq 1 \Rightarrow \deg c_j(X) \geq 1$.

Theorem. Suppose K has characteristic 0. Let $f = \prod_{n=0}^{\infty}\left(1 + X^{-k^n}\right)$ where $k \geq 2$ is a given integer.

i. If $k = 2$, $f = (1 - X^{-1})^{-1}$ (Euler).

ii. If $k = 3$, $\deg c_j(X) = 1$ for all $j \geq 1$.

iii. If $k \geq 5$ is odd, $\liminf_j \deg c_j(X) = 1$, $\limsup_j \deg c_j(X) = +\infty$.

iv. If $k \geq 4$ is even, then $c_j(X)$ can be explicitly computed:

$$c_{2n+2}(X) = X^{\frac{k^n-1}{k-1}} \prod_{i=0}^{n-1} (X^{k^i} + 1),$$

$$c_{2n+1}(X) = X^{\frac{k^n - 2k^{n-1} + 1}{k-1}} (X^{k^n - 2k^{n-1}} - 1) \prod_{i=0}^{n-1} (X^{k^i} + 1)^{-1}$$

so that $\lim_{n\to\infty} \deg c_n(X) = +\infty$.

M. Mendès France, Talence.

Distribution of small powers of a primitive root

Let g be a primitive root modulo the prime p. Put $\mathcal{N} = \{g^i : 1 \leq i \leq N\}$. Suppose that $0 < n_1 < n_2 < \dots < n_N < p$ are the elements of $\mathcal{N}$ in increasing order. We consider two questions:

1. For which N is it true that the discrepancy of $\mathcal{N}$ is $o(N)$ (i. e. $\mathcal{N}$ is approximately uniformly distributed).
2. For which N is it true that $n_{i+1} - n_i = O(\frac{p}{N})$ for $\gg N$ values of i.

We conjecture that $\mathcal{N}$ has both properties if $N > p^{\epsilon}$. We prove that $\mathcal{N}$ has the first property if $\frac{N}{p^{1/2}\log p}$ is large, and that $\mathcal{N}$ has the second property if $N \geq p^{3/4}$. It seems that this latter exponent may be reduced, perhaps to $\frac{5}{7}$, but probably not as far as $\frac{2}{3}$, using known techniques. Question 2 was raised by M. Tompa (Computer Sci., U. of Washington) in connection with the running time of the quicksort algorithm.

H. L. Montgomery, Ann Arbor.

Kuznetsov's paper on the eighth power moment of the Riemann Zeta–function

This is a largely expanded version of my letter to Prof. N. V. Kuznetsov which I mailed on Jan. 9th, 1991. There I asked him to clarify some dubious points which I had found in his important paper (*T. I. F. R. Studies in Math.* **12** (1989), 57–117) [1]. I have not received his response yet, perhaps because of the chaotic situation allegedly prevailing his country.

It should be stated explicitly that I have the opinion that [1] has not yielded anything definitive on the resolution of the eighth power moment problem. I doubt the validity of the formula (5.1) of [1] which is the core of Kuznetsov's argument. Nevertheless his ideas are very interesting and deserved of much discussions.

Y. Motohashi, Tokyo.

An upper bound for the first eigenvalue of the noneuclidean Laplacian for $\Gamma(q)$ and $\Gamma_0(q)$

Theorem. Let $\Gamma = \Gamma(q)$ or $\Gamma_0(q)$, q squarefree. There exists an absolute positive constant A such that for $q \geq A$: $\lambda_1 \leq \dfrac{1}{4} + \dfrac{475}{\log^2 q}$.

The result follows from the Selberg trace formula with test function for each q

$$h(t) = \frac{\cos^2(ktL)\ \sin^6(ctL)}{\left[1 - t^2\left(\frac{2kL}{\pi}\right)^2\right] L^4 t^4}$$

where $\dfrac{c}{k} = 0.822$ and $L = \dfrac{\log q}{(2+\varepsilon)(2k+6c)B}$ for $B > 1$.

C. J. Mozzochi, Princeton.

Polynomial mappings

A survey was given on new and old results concerning polynomial mappings with particular emphasis on the existence and length of possible cycles. If $f = x^n + \cdots$

is a polynomial with integral coefficients in an algebraic number field K of degree N, then it has been shown that the cycles of f lying in K cannot have their length larger than $B(N)$, where $B(N)$ is an explicitly given constant. In particular we have $B(1) = 2$ and $B(2) = 6$.

W. Narkiewicz, Wrocław.

On the number of $n \leq x$ which have more than $\log x$ divisors

The results of this talk are contained in two papers: "Grandes déviations pour certaines fonctions arithmétiques" (Balazard, Pomerance, Tenenbaum, Nicolas) which will appear in 1991 in *J. Number Theory* and "Sur les entiers $n \leq x$ dont le nombre de diviseurs est supérieur à $\log x$" (Deléglise, Nicolas) which is submitted to the *Sém. de Théorie des Nombres de Bordeaux.*

Let $\tau(n)$ be the number of divisors of n. Let us define

$$S_\lambda(x) = \#\{n \leq x : \tau(n) \geq (\log x)^{\lambda \log 2}\} \quad \text{if} \quad \lambda > 1 \quad \text{and}$$
$$S_\lambda(x) = \#\{n \leq x : \tau(n) \leq (\log x)^{\lambda \log 2}\} \quad \text{if} \quad \lambda < 1.$$

In the first paper it is proved that

$$S_\lambda(x) = \frac{1}{|\lambda-1|} \sqrt{\frac{\lambda}{2\pi}}\, H(\lambda)\, K(\{\lambda \log\log x\})\, M(x,\lambda) \Big(1 + O\big((\log\log x)^{-1}\big)\Big)$$

where $M(x,\lambda) = x(\log x)^{-\lambda\log\lambda+\lambda-1}(\log\log x)^{-1/2}$, $H(\lambda)$ is the entire function $H(\lambda) = \frac{1}{\Gamma(\lambda+1)} \prod_p (1-\frac{1}{p})^\lambda (1+\frac{\lambda}{p})$, $\{t\}$ is the fractional part of t; K is a 1-periodic function: $K(\theta) = \lambda^\theta \sum_{d=1}^{\infty} \chi(d) h(d,\lambda) d^{-1} \lambda^{[1-\theta+\log\tau(d)/\log 2]}$, $h(d,\lambda) = \prod_{p|d} (1+\frac{\lambda}{p})^{-1}$, χ is the multiplicative function $\chi(p) = 1$ and $\chi(p^k) = 0$ for $k \geq 2$ and $[t]$ is the integral part of t.

In the second paper, it is proved that for $1 < \lambda \leq 2$, we have

$$\inf_\theta K(\theta) = K(0_+) \qquad \text{and} \qquad \sup_\theta K(\theta) = K(1_-).$$

Finally, for $\lambda = \lambda_0 = 1/\log 2$, we find that

$$\limsup_{x\to\infty} \frac{S_{\lambda_0}(x)}{M(x,\lambda_0)} = 1.148\ldots \qquad \text{and} \qquad \liminf_{x\to\infty} \frac{S_{\lambda_0}(x)}{M(x,\lambda_0)} = 0.938\ldots.$$

J.-L. Nicolas, Villeurbanne.

Goldbach numbers in short intervals

Let $r(n) = \sum_{h+k=n} \Lambda(h)\Lambda(k)$, $\sigma(n) = \prod_{p \nmid n}(1 - \frac{1}{(p-1)^2}) \prod_{p|n}(1 + \frac{1}{p-1})$ and $L = \log N$.

It is well known that Hardy–Littlewood's conjecture $r(2n) \sim 2n\,\sigma(2n)$ (1)

holds for almost all even integers up to N. We deal here with similar problems for short intervals.

Theorem 1 (joint with J. Pintz). Let $H \geq N^{\frac{1}{3}+\epsilon}$. Then for any $A > 0$

$$\sum_{N \leq 2n \leq N+H} \left| r(2n) - 2n\,\sigma(2n) \right|^2 \ll_{A,\epsilon} H N^2 L^{-A}.$$

Theorem 2 (joint with J. Pintz). Let $H \geq N^{\frac{7}{36}+\epsilon}$. Then for all but $O(H L^{-A})$ even integers $2n \in [N, N+H]$ we have $r(2n) > 0$.

Theorem 3 (joint with J. Kaczorowski and J. Pintz). Assume GRH. For all but $O(H^{1/2}L^{3+\epsilon})$ even integers $2n \in [N, N+H]$ we have that (1) holds.
We remark that Theorem 3 is non–trivial as soon as $H > L^{6+\epsilon}$.

A. Perelli, Genova.

Continued fractions of formal power series

The uncountably many transcendental binary 'decimals' $2\sum \pm 2^{-2^h}$ all have continued fraction expansions requiring the partial quotients 1 or 2 only. One sees this by noting that the formal power series $X\sum X^{-2^h}$ has the 'folded' continued fraction

$$[1, X, -\dot{X}, -X, -\dot{X}, X, X, -X, -\dot{X}, X, -X, -X, X, X, X, -X, -\dot{X}, \ldots]$$

with the signs reflecting the creases in a sheet of paper repeatedly folded in half and the marked entries reflecting the signs (thus requiring change if the sign is negative). However, the point is that all folded sequences have alternating signs at the entries in the odd–numbered places. Hence specialising one always has $2\sum \pm 2^{-2^h} =$ $[1, 2, a, -2, b, 2, c, -2, d, \ldots]$, $a, b, c \ldots = \pm 2$. On eliminating the improper -2 entries one obtains $[1, 2, a{-}1, 2, b{-}1, 2, c{-}1, 2, \ldots]$ and noting that $[\ldots, 2, -3, 2, \ldots]$ $= [\ldots, 1, 1, 1, 1, \ldots]$ and $[\ldots, 1, 1, -3, 2, \ldots] = [\ldots 2, 1, 1, 1, \ldots]$ the result follows. This is joint work with J. Shallit (Waterloo).

A. J. van der Poorten, North Ryde.

On Rényi's formula on arithmetical semigroups

A free commutative semigroup G with identity generated by a countable set is called *arithmetical* if in addition there exists a real-valued norm mapping $|\cdot|$ on G such that i.) $|ab| = |a|\cdot|b| \quad \forall\, a, b \in G$ ii.) the total number $N_G(x)$ of elements $n \in G$ of norm $|n| \leq x$ is finite for each real x. The arithmetical semigroup is said to satisfy *Axiom A* if there exist positive constants A, δ, η $(0 \leq \eta < \delta)$ such that

$$N_G(x) = A\,x^{\delta} + O(x^{\eta}) \quad \text{as} \quad x \to \infty\,.$$

Theorem 1. Let G be an arithmetical semigroup satisfying Axiom A, and let $k \in \mathbb{N}$ be such that $k\eta < \delta$. Then for the set $Q_{k,G}$ of k-free integers in G we have

$$Q_{k,G}(x) = \sum_{\substack{n \in Q_{k,G} \\ |n| \leq x}} 1 = A\,x^{\delta}\,\zeta_G^{-1}(k\delta) + R(x),$$

where the estimate $M_G(x) = \sum_{\substack{n \in G \\ |n| \leq x}} \mu_G(n)) = o(x^{\delta})$ yields

$$R(x) = o(x^{\delta/k}) \tag{1}$$

and the estimate $M_G(x) = O(x^{\delta}\log^{-a} x) \quad \forall\, a > 0$ implies

$$R(x) = O(x^{\delta/k}\log^{-a} x) \quad \forall\, a > 0. \tag{2}$$

Theorem 2. Let G be an arithmetical semigroup satisfying Axiom A for which $2\eta < \delta$. Then for every $q \geq 1$ the estimate (1) implies

$$A_{q,G}(x) = \sum_{\substack{n \in G, |n| \leq x \\ \Omega(n)-\omega(n)=q}} 1 = d_{q,G}\,x^{\delta} + o(x^{\delta/2}(\log\log x)^{q})$$

whereas (2) implies

$$A_{q,G}(x) = d_{q,G}\,x^{\delta} + O(x^{\delta/2}(\log\log x)^{q-1}).$$

Š. Porubský, Bratislava.

Character sums in algebraic number fields

Let K be an algebraic number field of degree $[K:\mathbb{Q}] = n = r_1 + 2r_2$ (in the standard notation), d its discriminant, $r = r_1 + r_2 - 1$, $q \subseteq K$ an integral ideal and χ a character for numbers $\bmod\, q$. Moreover, let $x_1, \ldots, x_n > 0$ and $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{r_1} \times \mathbb{C}^{2r_2}$ be such that $x_{p+r_2} = x_p$, $\alpha_{p+r_2} = \overline{\alpha}_p$ for $p = r_1 + 1, \ldots, r+1$. Then the following result of Pólya–Vinogradov type holds:

$$\sum_{|\nu^{(p)}+\alpha_p| \leq x_p} \chi(\nu) = \delta(\chi)\,\frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}}\,\frac{\varphi(q)}{N(q)}\,X + O\left(N(q)^{\frac{1}{r_2+2}}\,X^{1-\frac{2}{r_2+2}}\log^{r+1} X\right).$$

Here, the sum is over all integers $\nu \in K$ with conjugates subject to $|\nu^{(p)} + \alpha_p| \leq x_p$, $p = 1, \dots, n$, $\delta(\chi) = 1$ or 0 according as χ is principal or not, and $X = x_1 \cdots x_n \geq 2$. The O-constant depends only on the field.
This improves on previous results of Lee (1979), Hinz (1983) and Söhne (1990).

U. Rausch, Clausthal.

On the distribution in the arithmetic progressions of reducible quadratic polynomials

We are concerned with the problem of the distribution in arithmetic progressions of the sequence $A = \{ n(n+2) \mid 1 \leq n \leq x \}$ for moduli beyond the trivial level of distribution $x^{1-\epsilon}$. We are able to treat the case of moduli which are the product of two primes. We show that

$$\sum_{\substack{q \sim Q, r \sim R \\ q, r \in \mathbb{P} \\ (q,r)=1}} \gamma_q \, \delta_r \sum_{\substack{1 \leq n \leq x \\ n(n+2) \equiv 0(qr)}} 1 \;=\; \sum_{\substack{q \sim Q, r \sim R \\ q, r \in \mathbb{P} \\ (q,r)=1}} \gamma_q \, \delta_r \left\{ \frac{2x}{qr} + 2 \left[\frac{x}{qr} \right] \right\} + O(x^{1-\epsilon})$$

provided $Q^{3/2} R^{1/2} \ll x^{1-2\epsilon}$, $QR \ll x^{\frac{3}{2}-2\epsilon}$, $R \ll x^{1-2\epsilon}$.
This means that QR can reach the level $x^{\frac{4}{3}-\epsilon}$. We remark that, whilst for irreducible quadratic polynomials we have on average $\sum\limits_{n<x, n^2+1 \equiv 0(qr)} 1 \sim x \dfrac{\varrho(qr)}{qr}$ also for $qr > 2$ (Iwaniec, 1978), in the reducible case we have on average $\sum\limits_{n<x, n(n+2) \equiv 0(qr)} 1$ $\sim x \dfrac{\varrho(qr)}{qr}$ for $qr < x^{1-\epsilon}$ and $\sim x \dfrac{\varrho(qr)}{2qr}$ for $qr > x$ $(q, r \in \mathbb{P})$.
The estimate of the error term is performed using Poisson formula and Weil estimate for Kloosterman sums.

S. Salerno, Salerno.

Prime divisors of binomial coefficients

We prove the following results which answer several questions raised by P. Erdös:

Theorem 1. Let $0 < \varepsilon < 1$, $a \in \mathbb{N}$. Then for sufficiently large m and $0 \leq k \leq m$ with $|m - 2k| < m^{1-\epsilon}$, there is a prime $p > \frac{1}{2} m^{\frac{1}{a+1}}$ satisfying $p^a \mid \binom{m}{k}$.

Theorem 2. Let $0 < \varepsilon \leq 1$, $a \geq 2$. If $|d| \ll n^{1/a} \log^{-(1+\epsilon)} n$, then

$$\log S_a(n,d) = C_a \, n^{1/a} + O\left(n^{1/a} \log^{-2} n\right),$$

where $S_a^a(n,d)$ is the largest a-th power dividing $\binom{2n \pm d}{n}$ and

$$C_a = 2^{1/a}\left(\tfrac{1}{2}\right)^{a-1}\sum_{k=1}^{\infty}\left\{\left(\tfrac{1}{2k-1}\right)^{1/a}-\left(\tfrac{1}{2k}\right)^{1/a}\right\}.$$

This generalizes a result of Sárközy from 1985; $a=2$, $d=0$.

Theorem 3.

$$\sum_{p\le n,\, p\nmid\binom{2n}{n}} \frac{\log p}{p} \sim (1-\log 2)\log n .$$

Further results concerning generalized coefficients (e. g. q–multinomial coefficients) may be obtained by our method, which is based upon a new exponential sum estimate.

J. Sander, Hannover.

Differences between values of quadratic forms

An elementary proof was given for the following

Theorem. Let f be a primitive quadratic form in any number of variables. Then every integer $n \not\equiv 2 \bmod 4$ is a difference of two values of f.

A. Schinzel, Warszawa.

Multiplicities of algebraic linear recurrences

We studied equations

$$\sum_{l=1}^{k} P_l(\underline{\underline{x}})\,\underline{\underline{\alpha}}_l^{\underline{\underline{x}}} = 0 \tag{1}$$

in variables $\underline{\underline{x}} = (x_1,\dots,x_n)\in\mathbb{Z}^n$. Here the P_l are polynomials with coefficients in a number field K and $\underline{\underline{\alpha}}_l^{\underline{\underline{x}}} = \alpha_{l_1}^{x_1}\cdots\alpha_{l_n}^{x_n}$, with $\alpha_{l_i}\in K^*$. We obtain uniform upper bounds for the number of solutions of (1). This is achieved by applying my p–adic generalization of W. M. Schmidt's quantitative Subspace Theorem. E. g. in the case $n=1$ we get the following

Theorem. Suppose that $\alpha_1,\dots,\alpha_k\in K^*$ are such that for $i\neq j$ α_i/α_j is not a root of unity. Then the equation

$$\sum_{l=1}^{k} P_l(m)\,\alpha_l^m = 0 \tag{2}$$

has not more than $\left(4(\omega+d)\,d!\right)^{2^{40(k\delta)!d!}}(\omega+d)^6$ solutions $m\in\mathbb{Z}$.

Here $d=[K:\mathbb{Q}]$, $\delta=\max_l \deg P_l$, and ω is the number of prime ideal divisors in the decomposition of the fractional ideals (α_l) in K.

The general case $n>1$ was treated in recent joint work with W. M. Schmidt.

H. P. Schlickewei, Ulm.

Anzahl direkter Faktoren abelscher Gruppen

Sei $t^*(G)$ die Anzahl der Zerlegungen der endlichen abstrakten abelschen Gruppe G in zwei 'teilerfremde' direkte Faktoren, $H(x)$ die Residuensumme von $s^{-1}x^s \cdot \cdot \prod_{k=1}^{\infty} \zeta^2((2k-1)s)\zeta(2ks)$ in der Halbebene $\mathrm{Re}(s) > \frac{1}{3}$ und $\Delta(x)$ das Restglied in $\sum_{\#G \leq x} t^*(G) = H(x) + \Delta(x)$. Ferner sei $\Delta_0(x)$ das Restglied des korrespondierenden Teilerproblems $\sum_{n_1 n_2 n_3^2 \leq x} 1 = H_0(x) + \Delta_0(x)$, worin $H_0(x)$ die Residuensumme von $s^{-1}x^s\zeta^2(s)\zeta(s)$ in der rechten Halbebene bezeichnet. Es genügt, $\Delta_0(x)$ abzuschätzen, da bekanntlich $\Delta(x) \ll x^\theta \log^\vartheta x$ aus $\Delta_0(x) \ll x^\theta \log^\vartheta x$ folgt, sofern $\theta > \frac{1}{3}$ ist.
Bisherige Resultate: Cohen (1960) $\theta = \frac{1}{2}$, Krätzel (1988) $\theta = \frac{11}{29} = 0.379\ldots$, Menzer (1991) $\theta = \frac{31}{82} = 0.378\ldots$.
In meinem Vortrag zeige ich $\theta = \frac{3}{8} = 0.375$. Der Beweis läuft – nach geeigneten Vorbereitungen – auf die Abschätzung einer Exponentialsumme hinaus. Dazu benutze ich lediglich den einfachsten Satz der van der Corput– Methode, um die Wirksamkeit dieser Vorbereitungen deutlich zu machen.
Mit mäßigem Mehraufwand wird man $\theta = \frac{5}{14} = 0.357\ldots$ erreichen können. Die Grenze der Methode dürfte bei $\theta = \frac{17}{48} = 0.354\ldots$ liegen.

P. G. Schmidt, Marburg.

On integers coprime to half the primes

This talk is on recent progress in some joint work with B. J. Birch. We have obtained the following results:
Theorem 1. Given $R \in \mathbb{N}$ and real constants a_r $(r = 0, 1, \ldots, R)$, there exists a sequence $(\varepsilon(n))$ of signs such that, for all sufficiently large N

$$\left| \sum_{n=1}^{N} \varepsilon(n)\, p_n^{-1} \log^r p_n - a_r \right| < \exp\left(-c_0 \log^{1/2} N\right) \qquad \text{for} \quad r = 0, 1, \ldots, R,$$

where $c_0 > 0$ is a constant and p_n denotes the n–th prime.

Theorem 2. Given $k, R \in \mathbb{N}$ $(k \leq R)$ and $c \in \mathbb{R}$, there exists a set of primes $\wp$ such that, for all sufficiently large x,

$$\#\{m \leq x :\, p|m \Rightarrow p \in \wp\} - \#\{m \leq x :\, p|m \Rightarrow p \notin \wp\} = $$
$$= cx(\log x)^{-k-1/2} + O\left(x(\log x)^{-R-\delta-1/2}\right) \qquad (1)$$

for some $\delta > 0$. In particular, this holds when $c = 0$.

Theorem 1 answers a question raised during the Symposium for K. F. Roth held at Imperial College, London, in 1985. The proof is elementary but very delicate. Theorem 2 provides an answer to a question of P. Erdős, who asked how small one could make the left side of (1) if one was free to choose the set $\wp$ of primes. The proof of Theorem 2 uses a modification of Theorem 1 and analytic methods.

E. J. Scourfield, Egham, Surrey.

On the Japanese remainder theorem

Let Q_i, $P_i \in \mathbb{Z}^+$, $i=1,2$, $Q = \gcd(Q_1, Q_2)$, $P = \gcd(P_1, P_2)$, $Q_i' = Q_i/Q$. With J. Pitman and J. Simpson we give a new proof for the following

Theorem of Morikawa. There exist β_1 and β_2 such that the two generalized arithmetic progressions $\left\{ \left[\frac{P_i}{Q_i} n + \beta_i \right], n = 1, 2, \dots \right\}$ $i = 1, 2$ are disjoint, iff there exist positive integers x, y such that

$$Q_1'(x + Q_2 - Q_2') + Q_2'(y + Q_1 - Q_1') = P \qquad (1)$$

We get x, y explicitely and this puts some more light on condition (1). The proof is based on the three distance theorem for $(\{n\alpha\})$ sequences.

With E. Fried we generalize this theorem for groups. We prove the following

Theorem. Let G be abelian, $\leq$ an order on G, $\preceq$ a preorder on G. Let $I_\preceq$ denote an interval in $\preceq$. We call an $u' = L(u; I)$ a lower neighbour of u in $I_\preceq$, if i.) $u, u' \in I$, ii.) $u' < u$, iii.) $x < u$, $x \in I_\preceq$ implies $x \leq u'$. We call G discrete, if $u' = L(u; I)$ exists for every $I_\preceq$ and $u \in I_\preceq$.

Let $G = (G, \leq, \preceq)$ be a discrete abelian group. Then for any $I_\preceq$ $x - x'$ for $x \in I$ has at most *three* different values. Moreover, if it takes exactly three, then $\{x - x' : x \in I\}$ $= \{a, b, a+b\}$ with some a, b. We also have some characterisations for $\{x - x'\}$, and some structural theorems for archimedian resp. nonarchimedian orders.

V. T. Sós, Budapest.

Effective estimates for mean values of complex multiplicative functions

The talk was devoted to present the following theorem, to appear in the *Math. Proc. Cambridge Philos. Soc.*, in a joint work with R. R. Hall.

For given $0 \leq \delta < 1$, $0 \leq \varphi < \pi$, let $\mathcal{E}(\delta, \varphi)$ denote the class of multiplicative functions g such that $|g(n)| \leq 1$ for all n and $\mathrm{Im}\left(e^{-i\varphi} g(p)\right)^2 \leq \delta^2 \left(1 - \mathrm{Re}(e^{-i\varphi} g(p))^2\right)$ for all prime p. Furthermore, for $z \in \mathbb{C}$, put $W(z) := e^{i\varphi}\left(\mathrm{Re}(e^{-i\varphi} z) + i\delta \, \mathrm{Im}(e^{-i\varphi} z)\right)$.

Theorem. The equation

$$\frac{1}{2\pi}\int_0^{2\pi}\left|W(e^{i\theta}-K)\right|\,d\theta \;=\; 1-K$$

has a unique solution $K=K(\delta,\varphi)>0$. Uniformly for $g\in\mathcal{E}(\delta,\varphi)$ we have

$$\sum_{n\le x} g(n) \;\ll\; x\exp\Big(-K\sum_{p\le x}\frac{1-\mathrm{Re}(g(p))}{p}\Big).$$

Moreover, this is sharp: given $(\delta,\varphi)\in[0,1)\times[0,\pi]$ and $x\ge 3$ there exists a function $g\in\mathcal{E}(\delta,\varphi)$ such that

i. $\sum_{p\le x}\frac{1-\mathrm{Re}(g(p))}{p} \;\to\; +\infty$

ii. $\Big|\sum_{n\le x} g(n)\Big| \;\ge\; cx\exp\Big(-K\sum_{p\le x}\frac{1-\mathrm{Re}(g(p))}{p}\Big)$

where c is a positive absolute constant. Two particularly interesting values of K are $K(0,0) = 0,32867\ldots$ (which corresponds to real valued functions g) and $\max K(\delta,\varphi) = K(0,\pi/2) = 1-2/\pi$. The proof rests upon Montgomery's effective approach to Halász mean value theorem.

G. Tenenbaum, Nancy.

Waring's problem

An account was given of the consequences of the new ideas of Wooley as applied to Waring's problem and the Vinogradov mean value theorem. With regard to Waring's problem, let, as usual, $G(k)$ denote the smallest s such that every sufficiently large natural number is the sum of at most s k-th powers. Then Wooley has recently shown that $\limsup G(k)/(k\log k)\le 1$, and for small k Vaughan and Wooley have shown that $G(5)\le 17$, $G(6)\le 24$, $G(7)\le 33$, $G(8)\le 42$, $G(9)\le 51$, $G(10)\le 59$, $G(11)\le 67$, $G(12)\le 75$, $G(13)\le 83$, $G(14)\le 91$, $G(15)\le 99$. Let $J_{s,k}(P)$ denote the number of solutions of the system

$$\begin{aligned} x_1 + \dots + x_s &= y_1 + \dots + y_s\\ &\vdots\\ x_1^k + \dots + x_s^k &= y_1^k + \dots + y_s^k \end{aligned}$$

with $1\le x_i,\,y_i\le P$. Then Wooley has shown that $\exists\ k_0$ such that $\forall k\ge k_0$ we have

$$J_{r,k}(P) \;\ll_{r,k}\; P^{2rk-k(k+1)/2+\eta(r,k)} \qquad \text{where}$$

$$\eta(r,k)\;\begin{cases} < k^2\log k\left(1-\frac{2}{k}+\frac{2}{k\log k}\right)^r & \text{for } 1\le r\le s_0\\ \le 8\log^3 k\left(1-\frac{1}{k}\right)^{r-s_0} & \text{for } r>s_0\end{cases}$$

and $s_0 = [1 + k \log k - k \log\log k]$.

R. C. Vaughan, Princeton.

On the divisor problem

I propose an approximation to the divisor problem by Riesz' logarithmic means; that is, let κ be a non-negative real number and form the Riesz mean

$$\frac{1}{\log^\kappa x} \sum_{n<x} d(n) \log^\kappa\left(\tfrac{x}{n}\right) = H_\kappa(x) + \Delta_\kappa(x) \qquad \text{where}$$
$$\Delta_\kappa(x) = \frac{\kappa}{\log^\kappa x} \int_1^x \Delta(t) \log^{\kappa-1}\left(\tfrac{x}{t}\right) t^{-1}\, dt\,,$$

thus generalizing the divisor problem, which corresponds to the case $\kappa = 0$.
I have proved that $\Delta_\kappa(x) \ll x^{\frac{1}{4}+\epsilon}$ holds true for all $\kappa \geq \frac{13}{110} = 0.118\overline{18}$ and this can even be improved.
The proof is based on the evaluation of the integral $\frac{1}{2\pi i} \int_C \zeta^2(s)\, x^s s^{-\kappa-1}\, ds$ upon a path C – which does not encircle the origin. This leads to the formula

$$\sum_{n<x} d(n) \log^\kappa\left(\tfrac{x}{n}\right) = x \log x + (2\gamma - \kappa - 1)x + O\left(x^{\frac{1}{2}-\frac{\kappa}{2}+\epsilon} N^{-\frac{1}{2}-\frac{\kappa}{2}+\epsilon}\right)$$
$$+ \frac{\sqrt{2}}{(2\pi)^{\kappa+1}} x^{\frac{1}{4}-\frac{\kappa}{2}} \sum_{n\leq N} \frac{d(n)}{n^{\frac{3}{4}+\frac{\kappa}{2}}} \cos\left(4\pi\sqrt{n x} - \tfrac{\pi}{2}(\kappa + \tfrac{1}{2})\right).$$

The main error term arises as a trigonometric sum to which we apply van der Corput's method for exponential sums. Moreover, using two dimensional exponential sums leads to the even better result $\kappa \geq 0.1129\ldots$. Generalizing the above technique, one can obtain f. e. for the divisor problem in arithmetical progressions, the circle problem the conjecture on $\Delta_0(x)$, that is $O\left(x^{\frac{1}{4}+\epsilon}\right)$, for $\kappa \geq 0.1129\ldots$ and for Ramanujan's τ-function the conjecture on $\Delta_0(x)$, that is $O\left(x^{\frac{23}{4}+\epsilon}\right)$, for $\kappa \geq 0.118\overline{18}$, where $\Delta_0(x) = \sum_{n\leq x} f(n) - H(x)$.

U. Vorhauer, Ulm.

On a special class of rational functions

Let K be a field of characteristic zero. A rational function $f \in K(x)$ shall be called 'special' if all residues of f and $\frac{1}{f}$ vanish. The trivial examples are $c(x-x_0)^n$, $c \neq 0$, $n \in \mathbf{Z}$, $n \neq \pm 1$. No other polynomials are special. G. Szekeres gave the example $f = (x^3+1)^2 x^{-2} = x^4 + 2x + x^{-2}$, where $f^{-1} = -\frac{1}{3}\left(\frac{1}{x^3+1}\right)'$.

Theorem. Let $k, \ell \in \mathbb{N}_0$. A pair of coprime and squarefree polynomials $p, q \in K[x]$ with $\deg p = k$, $\deg q = \ell$ such that $\left(\frac{p}{q}\right)^2$ is special exists iff $\{k, \ell\} = \{\frac{i(i+1)}{2}, \frac{i(i-1)}{2}\}$ with $i \in \mathbb{N}_0$.

In fact there are polynomials $p_i \in K[x]$ with $\deg p_i = \frac{i(i+1)}{2}$, $i = -1, 0, 1, \ldots$, such that all p_i are squarefree, $(p_{i-1}, p_i) = 1$ and $(\frac{p_{i-1}}{p_i})^2$ special. These p_i can be constructed recursively by $p_{i+1} := p_{i-1} \int p_i^2 \, p_{i-1}^{-2}$, where the constant of integration must avoid certain exceptional values, that would produce common zeros for p_{i+1} and p_i.

E. Wirsing, Ulm.

Goldbach–Vinogradov's theorem in short intervals

Goldbach–Vinogradov's theorem states that every large odd integer is a sum of three primes. Since 1950's, several mathematicians have studied the problem of representing any large odd integer N in the following form

$$N = p_1 + p_2 + p_3, \quad |p_i - N/3| \leq U, \quad 2 \nmid N, \quad p_i \in \mathbb{P}, \quad 1 \leq i \leq 3, \tag{1}$$

with U as small as possible. Assuming GRH about L-functions, it can be shown, that (1) is solvable for every large odd integer N if $U = N^{\frac{1}{2}+\epsilon}$. In 1989, Pan C.–d. and Pan C.–b. proved, by purely analytic means, that $U = N^{2/3} \log^c N$, $c > 0$ an absolute constant, is permissible. Following the idea of Pan, I obtained a further improvement on their result, namely

Theorem. If $U = N^{5/8} \log^c N$, $c > 0$ a suitable positive constant, then (1) is solvable for every large odd integer N. Moreover, the number of solutions of (1) is

$$T(N,U) = 3\,\sigma(N)\,U^2 \log^{-3} N + O(U^2 \log^{-4} N)$$

where $$\sigma(N) = \prod_{p|N} \left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{p \nmid N} \left(1 - \frac{1}{(p-1)^3}\right) > \frac{1}{2}, \quad \text{if } 2 \nmid N.$$

Recently, Jia C.–h. further showed by sieve methods, that (1) is solvable if $U = N^{\frac{3}{5}+\epsilon}$. Since sieve methods are applied, no asymptotic formula for the number of solutions is given.

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