MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 24/1991

Diskrete Geometrie

2.6. bis 8.6. 1991

Die Tagung fand unter der Leitung von L. Danzer (Dortmund) und G. C. Shephard (Norwich) statt. Sie hatte 35 Teilnehmer, von denen 33 Vorträge hielten. Die beschränkte Teilnehmerzahl bei gleichzeitig hoher Aktivität der Teilnehmer sorgte für ein diskussionsreiches Tagungsklima, das sich auch in zwei Problemsessions am Montag und am Donnerstag dokumentierte.

Die Vorträge behandelten vielfältige Themen aus dem weiten Spektrum der Diskreten Geometrie, es ergaben sich aber einige eindeutige Themenschwerpunkte. Etwa ein Drittel der Vorträge war Packungs-, Überdeckungs- und Pflasterungsproblemen gewidmet, wobei der aufstrebende Problemkreis der Aperiodizität bei Pflasterungen mit mehreren Vorträgen vertreten war. Das Wechselspiel von Gruppen und Geometrie – Regularitätsfragen im weitesten Sinne – kann als übergreifender Gesichtspunkt zu einer Reihe von Vorträgen genannt werden. Eine weitere Gruppe von Vorträgen war mehr kombinatorischen Fragestellungen gewidmet – Anordnungsproblemen und Anzahlproblemen. Mehrere Vorträge befassten sich mit Maß- und Identifikationsproblemen bei konvexen Körpern. Neben all dieser Schwerpunktbildung zeigten einige Einzelvorträge mit der Behandlung weniger bekannter oder manchmal fast in Vergessenheit geratener Themen, daß auch in solchen Bereichen wie der Dreiecksgeometrie oder der klassischen hyperbolischen Geometrie noch ein großes Potential an Forschungsmöglichkeiten steckt.

Die Tagung wurde nicht zuletzt deshalb von vielen Teilnehmern als besonders fruchtbar empfunden, weil trotz aller Verwandtschaft der Arbeitsgebiete sehr unterschiedliche Arbeitsmethoden und Ideen vorgestellt wurden; dazu hat vor allem auch die Anwesenheit einiger Teilnehmer beigetragen, die der Diskreten Geometrie im engeren Sinne nicht zuzurechnen sind.

Vortragsauszüge

A. BEZDEK:
Cylinder packing and covering problem

The set of points whose distance from a given line does not exceed a given positive number is called an infinite circular cylinder (cylinder in short). A subtle construction of K. Kuperberg (88) produces a positive-densitiy packing with cylinders in which no two of them are parallel. W. Kuperberg and A. Bezdek (88) showed that the ratio of the portion covered by the cylinders of a packing to the whole of the space cannot exceed $\frac{\pi}{\sqrt{12}}$. The later two authors also determined the gapradius (measured by the radius of the largest sphere lying in the gaps). The following new results were discussed during the talk.

1. In connection with a paper of A. Heppes and L. Szabó we prove that the largest radius r_4 such that 4 cylinders of radius r_4 can touch a unit sphere without overlapping each other is $1+\sqrt{2}$.

2. The cupradius of a cylinder covering (with cylinders of unit radius) is at least $1-\frac{\sqrt{3}}{2}$ if not all cylinders are parallel, then it is at least $1-\frac{1}{\sqrt{2}}$. Note that the cupradius of a covering is measured by the largest sphere lying in the double covered regions.

3. More than $R_3(4,5)$ cylinders cannot be arranged so that any two of them touch each other.

4. For any N there is a domain B in E^2 such that one can arrange N disjoint translates of B so that they touch each other at the same point.

G. BLIND:
Packings with circles of two sizes

In the Euclidean plane we consider packings with circles of two sizes, namely with radius 1 and ρ $(0<\rho<1)$. The problem is to find the density $D(\rho)$ of the densest such packing. Clearly $D(\rho) \geq \frac{\pi}{\sqrt{12}}$, the density of the densest packing with congruent circles, and it is well known that $D(\rho) = \frac{\pi}{\sqrt{12}}$ for $0{,}743\ldots \leq \rho < 1$.

Our aim is to enlarge this r-interval to $\rho_0 = 0{,}645\,707\ldots \leq \rho < 1$, the largest possible such interval. We are able to do this under the condition, that the small circles are surrounded by at most 6 circles and the large circles are surrounded by at least 6 circles, which is satisfied by all the known examples with high density.

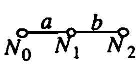
K. BÖRÖCZKY:
Sphere packing in $\mathbb{R}^3$.

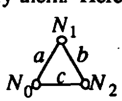
One of the most famous problems of discrete geometry in E^3 is to find the densest packing consisting of congruent balls. The best known bound on the largest possible density δ is due to C. A. Rogers (1958) who proved that the density is $\delta \leq 0{,}7796355\ldots$.
The best published bound is the work of J. H. Lindsey II. He obtained that the density is $\delta \leq 0{,}7784429\ldots$. In the talk we present an improvement, which leads to the inequality $\delta \leq 0{,}7783683\ldots$.

J. BOKOWSKI:
Pseudoplane arrangements of non-representable oriented matroids

Sphere systems (= pseudoline arrangements in the rank 3 case) are among the very useful models for investigating oriented matroids. We consider uniform pseudoplane arrangements with n elements in the rank 4 case, and we denote its number of simplicial cells by $c(n)$. In the representable case, we have Shannon's result $c(n) \geq n$. In the general case of arbitrary uniform oriented matroids, the Cordovil / Las Vergnas conjecture $c(n) \geq 1$ is still open. Classification results of J. Richter - Gebert and the author have shown that there is just a single case with $c(n) < n$ for $n = 8$. Starting with this example, it can be shown that there is an infinite class of examples with $c(n) = n - 1$. The author has a proof for the Cordovil / Las Vergnas conjecture up to $n = 11$.

F. BUEKENHOUT:
A search for the small regular thin geometries of rank 3 , alias "generalised combinatorial polyhedra"

This work is a small part of a general project whose purpose is to gather data on "small geometries", to systemise their study and to classify them. Here we deal with thin geometries Γ over Coxeter diagrams 

and 

whose "vertex-figures" over any element are polygons of a (resp. b , c) vertices with $a, b \geq 3$, $c = 2$ in the first case and

$c \geq 3$ in the second. We assume that Aut Γ is flag transitive, that Γ is residually connected and multiplicity free. Also, Aut Γ is supposed to act faithfully on the elements of each type. If the number of maximal flags is at most 100, there are 14 different Γ and only 7 groups, Alt(5) being represented by 4 geometries. If Aut Γ is Sym(5) (resp. PSL(2,7), PGL(2,7)) there are 10 (resp. 3, 47) possible Γ.

R. CONNELLY:

Globally rigid symmetric tensegrities and group representations

Consider a finite configuration of points in three-space where some pairs are constrained not to get further apart (they have a *cable* between them), while some other pairs are constrained not to get closer together (they have a *strut* between them). This strucutre is called a *tensegrity*, and it is said to be *globally rigid in three-space* if all the configurations of points in three-space that satisfy the distance constraints are congruent.

One can often define an n-by-n symmetric matrix Ω called the *stress matrix*, where n is the number of points in the configuration. The following conditions are sufficient for global rigidity of the tensegrity: i) The lines through the struts and cables, regarded as points in the projective plane at infinity, do not lie on a single conic. ii) Regarding Ω as quadratic form it is positive semi-definite with rank $n-4$. Furthermore, if the configuration has the property that there is a finite point group of rigid symmetries that is transitive on the points and invariant on the cables and struts, then the irreducible representations of this finite group can greatly simplify the calculation of whether Ω is positive semi-definite.

H. S. M. COXETER:

Orthogonal trees (this talk was presented by E. Schulte)

Any tree, with n edges and $n+1$ vertices, may be realized in Euclidean n-space so that its edges, of any chosen lengths, are straight line-segments, mutually perpendicular. Let the edges emanating from a vertex P (of valency q) be PP_ν ($\nu = 1, \ldots, q$), with lengths l_ν. The convex hull of such an *orthogonal tree* is an *orthogonal simplex*. Let a be the altitude of this simplex from the vertex P, that is, the distance from P to the opposite hyperplane of the simplex; and let a_1 be the altitude from an adjacent vertex P_1 of the tree. Then

$$a = \left(\sum_{\nu=1}^{q} l_\nu^{-2}\right)^2 .$$

Let α be the dihedral angle opposite to the edge PP_1 (of length l_1), that is, the acute angle between the hyperplanes opposite to P and P_1. Then

$$l_1^2 \cos\alpha = a a_1 .$$

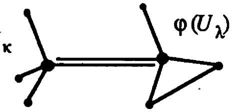
L. DANZER:

Global consequences of local conditions on discrete structures

A naive model for the atomic structure of condensed matter (espec. crystals and quasicrystals) can be described as follows:
The centers of the atoms form a DELONE-set M with some parameters r, R (i.e. $x,y \in M \Rightarrow |x-y| > 2r$; $z \in \mathbb{R}^d \Rightarrow \exists x$: $x \in M \wedge |x-z| \leq R$). The bonds are represented by straight edges. So M becomes the vertex set $V(\mathcal{G})$ of a "DELONE-graph" $\mathcal{G}$. We suppose $\mathcal{G}$ to meet also the condition

> every two vertices with distance less than $3R$ can be connected by a path, consisting of less than N edges.

The neighbourhood $U(x,\mathcal{G})$ of a vertex x of $\mathcal{G}$ is the subgraph spanned by x and its neighbours (i. e. the vertices adjacent to x).
Given a finite family $\mathcal{F} := \{U_1, U_2, \ldots, U_\kappa\}$ of protoneighbourhoods, a finite set L of matching types and r, R, N , we consider the species $S := S(\mathcal{F},L,r,R,N)$ of all DELONE-graphs with parameters r, R, N, such that every neighbourhood is congruent to some U_κ and every union of two matching neighbourhoods is congruent to some member of L. For such species among others the following statements are proved:



Theorem 2: Suppose S contains a DELONE-graph $\mathcal{G}$, which is weakly repetitive but not multiregular (not permitting d linearly independent translations). Then there are

$$2^{\aleph_0} \text{ congruence classes in } S.$$

Theorem 4: Let $S := S(\mathcal{F},L,r,R,N)$ and suppose:

- S is not empty
- $\mathcal{F}$ and L are minimal under the first condition
- there are only finitely many congruence-classes in S,

then every member of S admits d linearly independent translations (is multiregular).

N. DOLBILIN:

Periodic tilings

Some finiteness theorems on space tilings are considered. The results are formulated in terms of coronas and are based on the concept of the stable rank for a tiling.

Theorem 1. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is periodic if $\mathcal{T}$ possesses a finite stable rank. Moreover, the index m of regularity of $\mathcal{T}$ is equal to the number m_k $(m = m_k)$ of different coronas of a rank k.

Theorem 2 (a generalization of the Delone theorem). Let $\mathcal{T}$ be an m-regular face-to-face tiling $\mathcal{T}$ with convex tiles. For the number f of facets of an arbitrary tile of $\mathcal{T}$ holds

$$f \leq 2 \cdot (2^d - 1) + (hm - 1) \cdot 2^d,$$

where $h \leq h(d)$ is a value depending only on the dimension d.

From theorems 1 and 2 follows

Theorem 3. For $m \in \mathbb{N}$ there exists a value $c(m,d)$ such that any given tiling $\mathcal{T}$ of $\mathbb{R}^d$ is m-regular if and only if $\mathcal{T}$ has a finite stable rank k and we have $k < c(m,d)$.

Theorem 4. There exists such a number $t(d,m)$ such that the number of all combinatorial classes of m-regular face-to-face tilings of $\mathbb{R}^d$ with convex (finite) tiles is less than $t(d,m)$.

Some words were said about a special method to construct 3-dimensional tilings. The method is describing the combinatorial structure by the continuous fraction representation of some parameter for tilings of the 3-sphere. For any finite sequence $a_1, \dots, a_n \in \mathbb{N}$ one can construct a periodic tiling of $\mathbb{R}^3$ with convex polytopes of the same structure well-defined by the sequence $a_1, \dots, a_n$. This implies that the structure of such tiles can be as complicate as desired.

The method may also be used for computing the Dirichlet-Voronoi (D-V-)tiling for an orbit of screw group Γ of motions in $\mathbb{R}^3$. Let $\Gamma = \langle \gamma \rangle$ (γ is a screw motion given by the rotation angle ϕ around a chosen screw axe and translational shift along it) and $\Gamma \cdot x$ an orbit. Then the strucutre of theD-V-tiling for $\Gamma \cdot x$ is well-defined by the representation $\frac{2\pi}{\phi} = [a_1, a_2, \dots]$.

From this it follows that the strucuture of D-V-tilings must change continuously in dependence of ϕ. Thus, there are *uncountably* many different structures of D-V-tilings for screw orbits corresponding to a value of the angle ϕ.

J. ECKHOFF:

Graphs, boxes, f-vectors

We address the problem of finding an intrinsic characterization of the set of all f-vectors of (i) finite simple graphs, (ii) finite families of boxes in $\mathbb{R}^d$ with edges parallel to the coordinate axes. The f-vector of a graph G is the sequence $f(G) = (f_0, f_1, f_2, \dots)$, where $f_k = f_k(G)$ denotes the number of complete subgraphs of G having $k+1$ vertices. The f-vector of a family $\mathcal{F}$ of boxes is the f-vector of the intersection graph $G(\mathcal{F})$ of $\mathcal{F}$. Since every graph G is of the form $G(\mathcal{F})$ for some family $\mathcal{F}$ in some dimension d, the two problems are intimately connected. A solution would have important consequences in combinatorial geometry and extremal graph theory. We introduce a new kind of "pseudopowers" similar to the ones used in the Kruskal-Katona theorem but based on elementary symmetric functions rather than binomial coefficients. We believe that these pseudopowers will play a decisive role in the characterization. As a first positive result, we establish the best possible lower bound on f_{k-2} in terms of f_{k-1}, subject to the condition $f_k = 0$, in both the graph-theoretical and the geometrical setting.

G. FEJES TOTH:
Covering with convex bodies

For a convex body K in E^d let $\vartheta(K)$ be the density of the thinnest covering with congruent copies of K and let $\vartheta_L(K)$ be the density of the thinnest lattice covering with K. The main result in the talk, which is a report on joint work with W. Kuperberg, is that if $d > 2$ then for any strictly convex body K in E^d there is an affinity A such that $\vartheta(AK) < \vartheta_L(AK)$.

J. E. GOODMAN:
Finite and infinite families of lines and curves

We discuss three problems involving arrangements of lines in the plane, arrangements of pseudolines, and spreads (i.e., continuously varying arrangements) of both lines and pseudolines, in all of which the idea of extending families, or properties of families, to infinite families, plays a central role.

The first is the conjecture of B. Grünbaum that any arrangement of pseudolines in the plane may be extended to a spread, which we have recently proved in joint work with R. Pollack, R. Wenger, and T. Zamfirescu.

The second is the conjecture of R. Aharoni, P. Duchet, and B. Wajnryb that a family of lines in the plane which is such that any two of its members meet in the unit disk has the property that any sequence of projections onto its members must remain bounded. This we have proved jointly with I. Bárány and R. Pollack.

The last is the problem: given a spread of chords on the unit disk, is it the set of bisectors of some mass distribution on the disk? The corresponding fact is easily seen to hold for a finite arrangement of chords, and the difficulty of the problem in the infinite case seems to depend on function-theoretic considerations. This is work in progress with I. Bárány, J. Pach, and R. Pollack.

P. GRITZMANN:

Successive determination and verification of polytopes by their X-rays

The talk which is based on joint work with Richard Gardner deals with the problem from computational convexity of reconstructing polytopes for which only X-ray images are known.

We consider the general case of k-dimensional X-rays of polytopes P in d-dimensional Euclidean space E^d, functions which return the k-dimensional volumes of parallel k-dimensional sections. For $k = 1$ we have the ordinary X-rays, which for our purposes can be identified with the Steiner symmetral of P.

We will be dealing with the three concepts of *determination* of a class of objects, where the directions of the X-rays are to be specified in advance, of *verification*, where the directions may be chosen depending on the subject (in order to check a given shape, for example) and of *successive determination*, where one does not know the body in advance, but is allowed to consult previous X-ray pictures in selecting the direction for the next X-ray.

We will give various results outline some applications and state some open problems.

H.-C. IM HOF:

The generalized pentagramma mirificum

The pentagramma mirificum, viewed as a cycle of orthoschemes, has a counterpart in hyperbolic geometry of any dimension. The truncated orthoschemes, that necessarily appear, are also known as the fundamental polytopes of certain hyperbolic Coxeter groups.

A. IVIC WEISS:

Polytopes constructed from projective linear groups

In recent years the term "chiral" has been used for geometric and combinatorial figures which are symmetrical by rotation but not by reflection. The correspondence of groups and polytopes is used to construct infinite series of chiral and regular polytopes whose facets or vertex-figures are chiral or regular toroidal maps. In particular, the groups $PSL_2(\mathbb{Z}_m)$ are used to construct chiral polytopes, while $PSL_2(\mathbb{Z}_m[i])$ and $PSL_2(\mathbb{Z}_m[\omega])$ are used to construct regular polytopes.

G. KABATYANSKI:
Packing and covering, coding theory

We provide a new construction for packing and covering in Hamming space, and prove that the density of optimal packing and covering by unit spheres approaches 1 as the dimension of the space tends to infinity. We also discuss the general case when the radius of the spheres is larger than 1. Finally, we consider a similar problem of packing the Hamming space by certain special sets.

P. KLEINSCHMIDT:
Flag vectors of polytopes

For general convex polytopes there are combinatorial invariants which can be expressed as linear combinations of flag-numbers and which are nonnegative. Additional linear combinations are obtained using a convolution operation. Using all this information we proved several results about face numbers of polytopes and their quotients. The results were obtained with a computer and some of them checked by hand.
Two of the results are:

- Every rational d-polytope, $d \geq q$, has a "small" 3-face with at most 500 vertices.
- Every d-polytope, $d \geq q$, has a 3-dimensional quotient which is a simplex.

(Joint work with G. Kalai and G. Meisinger.)

D. G. LARMAN:
Maximal k-simplices in the n-cube (joint work with V. L. Klee)

It is well known that the vertices of the unit cube $C^{n+1} = [-1,1]^{n+1}$ admit an orthogonal set of vectors of cardinality $n+1$ if and only if the vertices of C^n admit the vertices of a regular n-simplex. Such a regular n-simplex must necessarily be the n-simplex of maximal volume in C^n. It is well known that if the above situation occurs then 4 divides $n+1$ and it is a famous problem of Hadamard to decide if the converse is true. It is reasonable to conjecture that a maximal n-simplex in C^n is regular only if 4 divides $n+1$. This would be the case if the maximal n-simplex necessarily has its vertices as vertices of C^n. Unfortunately this is not true for $n = 2, 6, 10$ and perhaps whenever $n \equiv 2 \bmod 4$. Generalising to maximal k-simplices in C^n, it must be that at least two of the vertices of that k-simplex are also vertices of C^n and perhaps it is true that if that k-simplex is also regular then all of its vertices are vertices of

C^n. For triangles and tetrahedra, the maximal simplex is regular if and only if 3 divides n. However, the maximal 4-simplex is never regular in C^n. Perhaps it is true that for $k \geq 3$, a maximal k-simplex in C^n is regular if and only if $k = 4m-1$ and $n = p(4m-1)$, m, p positive integers.

V. I. LEVENSTEIN:

Optimal packings on the euclidean sphere and in $\mathbb{R}^n$

Let $M_n(\varphi)$ be the maximal cardinality of a finite set of points on the Euclidean sphere S^{n-1} the angular distance between any two of which does not exceed φ. An upper bound on $M_n(\varphi)$ is given which is attained in many cases, for example it follows that $M_8(\frac{\pi}{3}) = 240$ and $M_{24}(\frac{\pi}{3}) = 196\,560$, which gives contact numbers for $n = 8$ and $n = 24$. The following bound on the highest density δ_n for packings of equal spheres in $\mathbb{R}^n$ is also obtained:

$$\delta_n \leq \frac{(j(\frac{n}{2}))^n}{(\Gamma(\frac{n}{2}+1))^2\, 4^n}, \quad n = 2, 3, \ldots$$

Here $j(\nu)$ is the smallest positive zero of the Bessel function $J_\nu(z)$. This bound is better than the known bound of Rogers for sufficiently large n ($n \geq 97$).

H. MARTINI:

Extremal and exposed symmetrizations of convex bodies

A Steiner symmetrization of a convex body $K \subset E^d$ at the hyperplane H is said to be *extremal* if it has at most two extreme points outside H or if it is a cylinder normal to H. A Schwarz symmetrization of K at the line L has this property if all its extreme points outside L form a $(d-2)$-sphere or if it is a cylinder. Additionally, such symmetrizations are called *exposed* if they are given in directions of extremal cross-sectional measures (as thickness or diameter) of K. Discrete methods are sufficient to investigate how large the direction sets of extremal and exposed symmetrizations for convex bodies can be and which classes of convex bodies can be characterized in that way.

P. MCMULLEN:

Higher toroidal polytopes

An abstract regular polytope whose underlying point-set can be identified with a torus of some dimension is called a toroid. One whose facet and vertex-figure are both spheres or toroids (with at least one of the latter) is called toroidal. In this talk, recent work (with Egon Schulte) on classifying the regular toroidal polytopes of rank at least 5 is described. In most cases, the questions of existence and finiteness can be settled. Among the techniques employed are twisting arguments, as well as reduction of the problem to those concerning toroids of lower rank, and direct approaches. Some related polytopes are also discussed.

B. MONSON (with A. Ivic Weiss):
Regular maps constructed from linear groups

For $m \geq 2$, the group $L(m)$ of 2×2 matrices over $\mathbb{Z}_m$ with determinants ± 1 is generated by

$$r_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad r_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

Now choose any subgroup H with $\{\pm I\} \subseteq H \subsetneq \mathrm{Centre}(L(m))$. Then $L(m)/_H$ with the generators $\overline{r_j}$ is a C-group yielding, for $m > 2$ a regular map of type $\{3,m\}$, and for $m = 2$ the triangle $\{3\}$. (This generalizes constructions in McMullen's 'Regular Polyhedra Related to Projective Linear Groups', Discrete Mathematics, in press.)

We have determined ways in which one such map can cover another, and the way in which a map decomposes as a blend of simpler maps. In all this, the prime factorization of m plays the key role. In some cases our structural results enable a simple description of the corresponding automorphism groups.

We also describe a 'realization' for the maps based on the 'plane' $\mathbb{Z}_m^2$.

J. PACH:
Remarks on a paper of Danzer and Grünbaum

Some 30 years ago Turán discovered a beautiful connection between packing problems and questions about the distribution of distances determined by a finite set of points in space. Given a compact domain $D \subseteq \mathbb{R}^d$, let c_k be the maximum number such that there are $p_1, \ldots, p_k \in D$ with $|p_i - p_j| \geq c_k$ $(\forall\, i \neq j)$. Then

$$c_2 = c_3 = \ldots = c_{k_1} > c_{k_1+1} = \ldots = c_{k_2} > c_{k_2+1} = \ldots \to 0\,.$$

He proved that, for any set of n points in D, the minimum number of pairs determining a distance $\geq c_{k_i+1}$ is $\sim \frac{1}{k_i}\binom{n}{2}$. This result has various applications in potential theory, for embeddability of graphs etc.

Let $g(n)$ (and $g^*(n)$) denote the minimum number of distinct distances (resp. distinct vectors) determined by n points in the plane in general position (no 3 on a line, no 4 on a circle). It follows from a 30 years old construction of Danzer and Grünbaum that $g(n) \le g^*(n) \le n^{\log 3/\log 2}$. We improve this result to $g(n) \le g^*(n) \le n e^{c\sqrt{\log n}}$, and we show that it follows from some deep results in additive number theory that $\lim_{n\to\infty} \frac{g^*(n)}{n} = \infty$.

We say that a set of n points in $\mathbb{R}^d$ is generic, if all the $\binom{n}{2}$ distances determined by them are distinct. A theorem of Avis, Erdös and Pach states that, for any fixed k, almost all k-element subsets of *any* n-element set in the plane are generic (as $n \to \infty$). This result remains true for slowly increasing values of k (more precisely, if $k = o(n^{1/7})$). However, the situation radically changes in higher dimensions. Let $g_d(k) = g$ denote the largest number with the property that almost all k-element subsets of any n-element set in $\mathbb{R}^d$ are generic (as $n \to \infty$). We can determine exactly all values of $g_d(k)$ $(< \binom{k}{2})$ for $d \ge 3$.

R. POLLACK:

Arrangements of lines in space

We discuss some problems relating to arrangements of lines in $\mathbb{R}^3$. They involve "weaving patterns" of lines, i.e., the over/under patterns of projections of lines onto suitable planes.

We first survey some results that are joint with J. Pach and E. Welzl where we discuss the realizability of certain bipartite perfect weaving patterns and the non-realizability of perfect weaving patterns of more than 4 lines, as well as the nonrealizability of a 4×4 perfect bipartite weaving pattern.

This last nonrealizability result is the basis for showing that the maximum number, $f(n)$, of elementary cycles in a bipartite weaving of n lines satisfies $c_1 n^{4/3} < f(n) < c_2 n^{3/2}$. Though we have been unable to obtain any non-trivial bounds on the number of cycles in an arbitrary weaving pattern, we are able to show that in a family of n lines there are at most $c\, n^{7/4}$ points that are incident to at least three noncollinear lines of the family. These results on counting cycles are joint with B. Chazelle, H. Edelsbrunner, L. J. Guibas, R. Seidel, M. Sharir and J. Snoeyink.

J. F. RIGBY:
Two cubic curves connected with a triangle

The study of 'triangle centres' is largely concerned with particular points: the orthocentre, incentre, Gergonne and Nagel points, Brocard points, and other points of a more exotic nature. One of the few general results is Kiepert's theorem.

Here we present further general theorems. Let P and Q be points in the plane of a triangle ABC. The feet of the perpendiculars from P to the sides of the triangle form the pedal triangle of P, and the lines QA, QB, QC meet BC, CA, AB in three points forming the cevian triangle of Q. if the pedal triangle of P is also a cevian triangle, P is a *pedal point*. If the cevian triangle of Q is also a pedal triangle, Q is a *cevian point*. Notable pedal points are the circumcentre, orthocentre and incentre; notable cevian points are the centroid, orthocentre, Gergonne point and Nagel point. The pedal points and the cevian points lie on two cubic curves through A, B, C and have many interesting collinearity properties.

J. RUSH:
Dense lattice packings of certain convex centrally symmetric shapes

By applying construction A of Leech and Sloane to nonbinary error-correcting codes, one obtains surprisingly dense lattice packings. Sometimes the density exceeds $2^{-n+o(n)}$ as assured by the Minkowski-Hlawka bound.

P. SCHMITT:
An aperiodic prototile in space

A set of prototiles in $\mathbb{R}^d$ is called aperiodic if it does not admit a periodic tiling. (A tiling is called periodic if among its symmetries there are translations in d linearly independent directions.) An example of such an aperiodic set of prototiles is the well-known pair (in the plane) found by Penrose. It is not known if there is an aperiodic prototile, i.e. a single prototile that forms an aperiodic set.

The purpose of this talk is to present a construction which shows the following: In space (and therefore in higher dimensions, too) aperiodic prototiles do exist. More precisely, there are prototiles which are spacefillers and for which every tiling of space consists of congruent plane layers that are rotated with respect to each other by an angle irrational to π, and which therefore cannot be periodic. The prototiles can be chosen to be polytopes.

J. J. SEIDEL:

Distance matrices and Lorentz space

1. A distance matrix G is a symmetric matrix with zero diagonal and positive entries elsewhere. Suggested by questions on metric embeddability, we assume G to have signature $+^1\ 0^{n-d}\ -^d$. Then G is the Gram matrix of vectors $g_0, g_1, \ldots, g_n$ on the light cone K of Lorentz space $\mathbb{R}^{1,d}$. Write $x = \sum_{i=0}^{n} \xi_i g_i$, and D for the positive diagonal matrices. Then DGD has the following members:

2. NEGATIVE TYPE G has $g_0, g_1, \ldots, g_n \in F \cap K^+$ for $F // E, F \neq E$, E a Euclidean hyperplane. Then $\sum_{i=0}^{n} \xi_i = 0$ for $x \in E$, and $= 1$ for $x \in F$. The sphere $F \cap K$ has centre $m \perp E, m \in F$.

3. REGULAR G has constant row sums and centre $m = \sum_{j=0}^{n} g_j / (n+1)$. Existence from Sinkhorn's theorem.

4. CAYLEY-MENGER G has shape $\begin{bmatrix} 0 & 1 \\ 1 & B \end{bmatrix}$, where the entries of B are the squared distances of n vectors in Euclidean $(d-1)$-space.

5. INTEGRAL G generates a discrete lattice $L = \langle g_0, \ldots, g_n \rangle_{\mathbb{Z}}$ if G is *hypermetric* (that is, if $F \cap K$ is a hole in $L \cap F$). Then $L \cap E = \langle g_i - g_j \rangle_{\mathbb{Z}}$ is an even Euclidean lattice, and $g_i - m$ are minimal vectors in a coset of $L \cap E$ in its dual.

6. $L \cap E$ is a ROOT LATTICE if $(g_i, g_j) = 1$ is a connected relation. The Euclidean root lattices, and the minimal vectors in their cosets are well-known. This produces the integral connected hypermetric distance matrices, cf. Terwilliger-Deza, Graphs and Combinatorics 3 (1987), 193-198.

M. SENECHAL:

The quasicrystal problem

This talk is a survey of the present status of the development of a geometric model for quasicrystal structures. Quasicrystals are crystals whose diffraction diagrams show bright spots but noncrystallographic symmetry. A suitable model would be a discrete, relatively dense point set in E^n which can be reconstructed from its local neighbourhoods and whose Fourier transform has a discrete component. The vertex sets of projected tilings (such as the Penrose tilings) are the most popular model today. They satisfy the first and third criteria but (apparently) not the second; in any case this class of point sets may be insufficiently general.

G. C. SHEPHARD:
Pick's theorem

The traditional form of Pick's theorem (G. Pick 1899) concerns simple lattice polygons P, that is, polygons whose vertices are points of the unit lattice L, and whose edges form a simple circuit, that is, have no intersections except at their vertices. Then the area $A(P) = i + \frac{1}{2}b - 1$ where i is the number of lattice points in P and b is the number of lattice points on the boundary of P.

In this lecture a generalisation of the theorem will be given, that applies to the most general lattice polygons, in which edges may intersect at relatively interior points, or even overlap, and vertices may coincide. The basic idea is that such a polygon P and any point $x \in L$ we can define an index $i(P,x)$. (The definition is too long to include here.) Then the area of P is given by $A(P) = \sum_x i(P,x) - r(P)$ where the summation is over all the points of L and $r(P)$ is the rotation number of P. For the definition of area and also for the proof of the theorem it is essential that the polygon be oriented. Details will appear shortly in Amer. Math. Monthly. This work was done pointly with B. Grünbaum.

H. TVERBERG:
When do three sides or diagonals (prolonged) of a regular *n*-gon have a common point?

The talk will deal with the problem of deciding all the non-trivial occurrences of the phenomenon of the title. "Non-trivial" means that the common point is not a vertex. Let 1 and 2, 3 and 4, 5 and 6 denote the n' th roots of unity, which determine the three lines in question. Then the concurrence is expressed by the equation (12 terms)

$$123 + 124 + \ldots + 561 + 562 - (125 + 126 + \ldots + 564) = 0.$$

We shall give an idea of the general theory of such equations (vanishing of a sum of roots of unity) as developed by H. B. Mann (Mathematika 1965). We shall then sketch some situations arising when we go back from the algebraic solutions to see whether they correspond to geometric ones.

P.S. After the talk, J. F. Rigby informed us that the problem had been solved by him (see Geom. Ded. **9** (1980) 207-238).

G. WEGNER:
Maximal packings

A *maximal packing* in the usual sense is a packing $\mathcal{G}$ of congruent copies of a convex body C such that for each member of $\mathcal{G}$ the number of neighbours is as large as possible, i.e. the number is given by the Newton number $N(C)$ of C. I generalize this notion in two ways by weakening the assumption of congruence: I call a packing $\mathcal{G}$ *locally maximal* iff it is not possible to rearrange the neighbourhood of any member C of $\mathcal{G}$ in such a way that a further convex body D congruent to one of the neighbours of C may be attached to C. And $\mathcal{G}$ is called *globally maximal* iff the same is true for D being congruent to any member of $\mathcal{G}$.

I present here some results concerning maximality of packings and tilings in the plane, especially I investigate tilings consisting of regular polygons (including the archimedian tilings) for maximality.

J. M. WILLS:
Intrinsic volumes and minimal determinants

I) In euclidean d-space $E^d, d \geq 2$, let K be a convex body and L a lattice with $\det L > 0$. Further let V_i, $i = 0, 1, \ldots, d$ denote the intrinsic volumes, $G(K,L) = \text{card}(K \cap L)$ and $D_i(L) = \min\{ |\det L_i| \mid L_i \ i\text{-dim. sublattice of } L \}, i = 1, \ldots, d$, $D_0(L) = 1$. Then

$$\frac{V_d(K)}{D_d(L)} - 2\left(\frac{2}{\sqrt{3}}\right)^{d-1} \frac{V_{d-1}(K)}{D_{d-1}(L)} \leq G(K,L) \leq \sum_{i=0}^{d} i! \frac{V_i(K)}{D_i(L)}$$

For small d or special L or special K there are improvements by various authors (e.g. Pick, Blichfeldt, Davenport, Hadwiger et al.).

II) For 0-symmetric K and Minkowski's successive minima λ_i holds

(1) $\frac{2^i}{i!} \leq \lambda_1(K,L) \ldots \lambda_i(K,L) \frac{V_i(K)}{D_i(L)}$ $\qquad i = 1, \ldots, d$

(2) $\lambda_{i+1}(K,L) \ldots \lambda_d(K,L) \frac{V(K)}{D(L)} \leq i! \, 2^{d-i} \frac{V_i(K)}{D_i(L)}$ $\qquad i = 0, 1, \ldots, d-1$

(3) $\lambda_{i+1}(K,\mathbb{Z}^d) \ldots \lambda_d(K,\mathbb{Z}^d) V(K) \leq 2^{d-i} V_i(K)$ $\qquad i = 0, 1, \ldots, d-1$ (Henk)

(1) and (3) are tight, (2) only for $i = 0$ and 1. (Conjecture: 1 instead of $i!$)
A consequence of (3): $V(\frac{1}{2}\lambda_{i+1} K) \leq V_i(\frac{1}{2}\lambda_{i+1} K)$ $\qquad i = 0, 1, \ldots, d-1$

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