MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t   21/1992

Quadratische Formen

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This conference was the **sixth** one on quadratic forms in Oberwolfach. It was organized by Knebusch (Regensburg), Pfister (Mainz) and Scharlau (Münster). The most interesting event probably was the historical discovery by J.Minaĉ and L.Hallock who convinced the audience that the algebraic theory of quadratic forms had already been known in the Middle Ages at famous Prince Hamlet's court. This knowledge, however, subsequently got lost by unfortunate circumstances until about thousand years later it was discovered again by Witt and Pfister.

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Vortragsauszüge

E. Bayer
Self-dual normal bases (joint with J.-P. Serre)

Let K be a field of char $\neq 2$, and let G be a finite group. Let L be a G-Galois algebra over K. Let $q_L : L \to K$, $q_L(x) = Tr_{L/K}(x^2)$, be the trace form. This form is invariant under G. The problem considered in this talk is to determine the isomorphism class of this form as a G-form. In particular, L/K is said to have a "self-dual normal basis" if this form is the unit G-form; in other words, if there exists $a \in L^*$ such that $q_L(a, ga) = 1$ if $g = 1$ and 0 otherwise. We solve this problem when the 2- Sylow subgroups of G are elementary abelian or quaternionian of order 8.

E. Becker
The trace formula and some geometric applications

Let B be an A-algebra, f.g. projective as an A-module, (M, φ) a bilinear B-space, possibly degenerate. In this situation one has the following trace formula for every $\alpha \in \mathrm{Sper} A$:

$$\mathrm{sgn}_\alpha \, tr^*_{B/A}(\varphi) = \sum_{\substack{\beta \in \mathrm{Sper}B \\ \beta/\alpha}} \mathrm{sgn}_\beta(\varphi)$$

This formula and the ideas behind allow at least the following applications:

1) Counting real points on 0-dimensional varieties (which can be considered as a multivariate version of the classical Sturm's theorem)

2) a constructive approach to the 0-dimensional case of the Bröcker-Scheiderer theorem about the description of basic-open semialgebraic sets

3) a short proof of Tarski's quantifier elimination result in the theory of real closed fields

4) a short proof of the open mapping theorem of Elman, Lam and Wadsworth.

J.-L. Colliot-Thélène (travail en commun avec A.N. Skorobogatov)
Groupe de Chow des zéro-cycles sur les fibrés en quadriques

Soit k un corps, car $(k) \neq 2$, puis C/k une courbe projective et lisse et $p : X \to C$ un k-morphisme propre, avec X/k une variété projective et lisse. Supposons que la fibre générique $X_\eta/k(\eta)$ est une quadrique lisse sur le corps $k(C)$, corps des fonctions

$\hat{\prod}_{\bar{K}\in\mathcal{K}}\bar{K}^*/(\bar{K}^*)^2$ is surjective, where the "sheaf product" $\hat{\prod}_{\bar{K}\in\mathcal{K}}\bar{K}^*/(\bar{K}^*)^2$ is the subgroup of $\prod_{\bar{K}\in\mathcal{K}}\bar{K}^*/(\bar{K}^*)^2$ consisting of all "locally constant" elements. We prove (with $\hat{\prod}$ denoting the appropriate sheaf products):

Theorem: The following conditions are equivalent:

(a) $W(K) \cong \hat{\prod}_{\bar{K}\in\mathcal{K}} W(\bar{K})$ naturally

(b) $\hat{W}(K) \cong \hat{\prod}_{\bar{K}\in\mathcal{K}} \hat{W}(\bar{K})$ naturally

(c) The maximal pro-2 Galois group $\mathcal{G}(K(2)/K)$ of K is the free pro-2 product of the groups $\mathcal{G}(K(2)/\bar{K}), \bar{K} \in \mathcal{K}$;

(d) K has the SAP with respect to $\mathcal{K}$ and every K-quadratic form which is $\bar{K}$-isotropic for all $\bar{K} \in \mathcal{K}$ is K-isotropic.

This is used to classify the quadratic forms over the field of totally real numbers and, more generally, over fields K with cd $K(\sqrt{-1}) \leq 1$.

A.J. Earnest

Developments in the spinor genus theory for integral quadratic forms

For $n = 2, 3$ and 4, examples of positive definite integral quadratic forms of rank n are known for which the spinor genus and class coincide, but for which the genus and class do not.
Theorem: Let f be a positive definite integral quadratic form of rank exceeding 4. Then the spinor genus and class of f coincide if and only if the genus and class of f coincide.

For forms of rank 3, the theory of spinor genus representations in some cases plays a key role in determining which integers are represented by a given form. Explicit computations of certain local spinor norm groups are given which make possible the determination of all integers which are primitively represented by a genus of ternary forms, but are not primitively represented by every spinor genus in that genus. These results are applied to analyze the primitive representation properties of an interesting list of positive definite ternary quadratic forms first found by Jones and Pall. Some of the results presented here were obtained in joint work with J.S. Hsia and D.C. Hung.

de C. Soit d la dimension de cette quadrique. On suppose $d \geq 1$.
Soit $CH_0(X)$, resp. $CH_0(C)$, le groupe de Chow des 0-cycles de degré zéro modulo l'équivalence rationnelle sur la variété X, resp. C.
On donne une *formule* pour le groupe

$$CH_0(X/C) := \ker p_* : CH_0(X) \to CH_0(C).$$

Cette formule identifie $CH_0(X/C)$ à un sous-quotient du groupe multiplicatif $k(C)^*$.

- Lorsque k est de dimension cohomologique ≤ 1, cette formule identifie $CH_0(X/C) = 0$.

- Supposons $d = 1$. Dans ce cas, une formule essentiellement équivalente est apparue dans des travaux antérieurs de S. Bloch (1981), Sansuc et l'auteur (1981), P. Salberger (1985) (lorsque C est la droite projective), puis de M.Gros (1987) et Ōkōchi (1987) (lorsque C est une courbe quelconque). Si de plus k est un corps p-adique, ou un corps de nombres, cette formule a permis d'établir la finitude du groupe $CH_0(X/C)$ (résultat que l'on sait aussi maintenant obtenir par des méthodes plus générales de K-théorie algébrique).

- Supposons $d = 2$. Dans ce cas, on peut associer à X/C une fibration Y/D de dimension relative 1, avec D/C le revêtement discriminant associé à X/C. Notre formule permet d'établir une *injection*

  $$CH_0(X/C) \hookrightarrow CH_0(Y/D).$$

  De ceci résulte, lorsque k est un corps p-adique ou un corps de nombres, la *finitude* du groupe $CH_0(X/C)$.
  (Dans une autre direction, rappelons le théorème de finitude obtenu par Salberger (1985) pour les fibrations en variétés de severi-Brauer au-dessus de la droite projective.)

- Lorsque $d \geq 3$, et k est un corps p-adique ou un corps de nombres, il est très vraisemblable que $CH_0(X/C)$ est toujours un groupe fini, et que dès que la dimension relative d est assez grande, ce groupe est nul si k est p-adique, et contrôlé par les complétions réelles si k est un corps de nombres. Mais notre formule ne permet pour l'instant d'obtenir ce résultat que dans des cas particuliers.

I. Efrat
Local-global principles for quadratic forms and Galois groups

Let $\mathcal{K}$ be a collection of pro-2 extensions of a field K of characteristic $\neq 2$ which is closed in the inverse limit topology. We say that K has the strong approximation property (SAP) with respect to $\mathcal{K}$ if the natural homomomorphism $\bar{K}^* \to$

Proposition: There exists a s.s. $\varphi_0, \dots, \varphi_p$ for φ such that $p = \dim\alpha$.

Corollary : $s_K(\varphi) \leq \frac{\dim\varphi - 1}{2}$

Remark: 1. The corollary tells us that if φ_K is isotropic there exists an isotropic vector (over K) of the form

$$v = v_0 + \sum_{i=0}^{p} v_i s t^{i-1} + \sum_{i=0}^{p} u_i t^i$$

with $p \leq 2\dim\varphi - 2$.

2. There is another way to interpret the s.i. of φ : $s_K(\varphi) = n$ iff φ contains a $2n+1$ dimensional K-minimal form and no K-minimal form of smaller dimension. (For the definition of K-minimal form cf. D. Hoffmann's talk).

O. Heddinga

Residue and transfer maps

Let R be a discrete valuation ring with maximal ideal $\mathfrak{p}$, prime element π, residue field k and quotient field F. Let $L : F$ be a finite extension s.t. the integral closure D of R in L is finitely generated over R. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the prime ideals of D with prime elements Π_i and residue fields $k(\mathfrak{q}_i)$ $(i = 1, \dots, r)$. Denote by $\partial^2_{\mathfrak{p}}$(resp. $\partial^2_{\mathfrak{q}_i}$) the second residue map with respect to π (resp. Π_i). Let $S : L \to F$ be a nontrivial F-linear map and define $\mathcal{D}_S := \{y \in L | \ S(yD) \subseteq R\} = xD$.

Define $S_i : k(\mathfrak{q}_i) \to k$ by $a \longmapsto S\left(\pi\Pi_i^{-1} x pr_i^{-1}(a)\right)$ mod $\mathfrak{p}$, where $pr_i^{-1}(a)$ is any element $\alpha \in L$ s.t. α mod $\mathfrak{q}_i = a$ and $\alpha\Pi_i^{-1} \in \mathfrak{q}_i^{-1}$. Then S_i is a well defined nontrivial k-linear map. Using WT-groups one can prove the commutativity of the following diagram

$$\begin{array}{ccc} W(L) & \xrightarrow{\oplus\partial_{i,x}} & \bigoplus_{i=1}^{r} W(k(\mathfrak{q}_i)) \\ S_* \downarrow & & \downarrow \sum S_{i*} \\ W(F) & \xrightarrow{\partial^2_\pi} & W(k) \end{array}$$

where $\partial_{i,x} = \partial^2_{\Pi_i} \circ m_{<x>}$ and $m_{<x>}$ is the multiplication by $< x >$.

D. Hoffmann

Isotropy of quadratic forms over the function field of a quadric

Let F be a field of char.$\neq 2$ and let K be an extension of F.

Definition: An anisotropic quadratic form φ over F is called K-minimal if $i)$ $\varphi_K = \varphi \otimes_F K$ is isotropic $ii)$ Any proper subform ψ of φ stays anisotropic over K.

R.W. Fitzgerald
Graded Wittrings and Ext-algebras

Let F be a field with Char $F \neq 2$. We prove:

Theorem: If F^*/F^{*2} is finite then Milnor's map $t_* : k_*F \longrightarrow H^*(G, \mathbb{Z}/2)$ is injective.

Corallary: If F^*/F^{*2} is finite and the map $e^* : grWF \rightarrow H^*(G, \mathbb{Z}/2)$ is well-defined then e^* is injective. Here G is the Galois Group of a quadratic closure F_q over F, $H^*(G, \mathbb{Z}/2)$ the Galois cohomology algebra, k_*F Milnor's mod 2 k-groups and $grWF$ the graded Witt ring. The proof involves computations with Ext-Algebras, particulary the subalgebra of $\mathrm{Ext}_{grWF}(\mathbb{Z}/2, \mathbb{Z}/2)$ generated by Ext^1.

J. Van Geel (joint work with D.W. Lewis)

Let F be a field of char $\neq 2$, and K the generic splitting field of a conic $< 1, -a, -b >$ defined over F. Then K is isomorphic to the field of fractions of the domain $R = F[s,t]/_{(s^2-at^2-b)}$. M. Rost proved the following result:

Proposition: Let φ be an anisotropic form over F. If $\varphi_K = \varphi \otimes_F K$ is isotropic then there exists a sequence of forms over F, $\varphi_i, i = 0, \ldots, p$, satisfying : (1) $\varphi = \varphi_0$, (2) $\varphi_i \cong c_i < 1, -a > \perp \psi_i$, (3) $\varphi_{i+1} \cong c_i b < 1, -a > \perp \psi_i$, (4) $((\varphi_p)_K)_{\mathrm{an}} \cong ((\varphi_P)_{\mathrm{an}})_K$.

Rost's proof of the proposition is based on the filtration of $R = F[t] \oplus sF[t]$ defined by : $\deg(P + sQ) = \max\{\deg P, 1 + \deg Q\}$ for $P, Q \in F[t]$.

Definition :

1. Let φ be as in the proposition. A sequence $\varphi_0, \varphi_1, \ldots, \varphi_p$ of smallest length satisfying (1) – (4) is called a splitting sequence for φ (s.s.).
2. Let $\varphi_0, \ldots, \varphi_p$ be a s.s. for φ. Let l be the smallest number such that φ_l is isotropic. Then l is called the splitting index (s.i.) of φ, denoted by $s_K(\varphi)$.
3. Let (V, φ) be a quadratic space over F such that φ_K is isotropic. Then the splitting degree (s.d.) of φ, $\bar{s}_K(\varphi)$, is the smallest number n such that $\exists v \in V \otimes R_n : \varphi(v) = 0$.

Lemma: $s_K(\varphi) = \bar{s}_K(\varphi)$.

Let φ be a form over F as before. By the exellence property there exists a form χ over F such that $(\varphi_K)_{\mathrm{an}} \cong \chi_K$.
Therefore $\varphi \perp -\chi \sim \alpha << a, b >>$ with α anisotropic.

D.G. James
Representations by unimodular quadratic forms

Necessary and sufficient conditions were given for the primitive representation of a lattice by a unimodular quadratic lattice over the p-adic integers, and the number of inequivalent representations determined under the action of the orthogonal group. Global results over $\mathbb{Z}$, for forms corresponding to the Dynkin diagrams, and their orthogonal sums, are then obtained via strong approximation.

P. Jaworski
Quadratic field extensions and residue homomorphisms

It is well-known that with every quadratic field extension $F[z] : F$, $z^2 = f$, there is associated an exact triangle of Witt groups

$$\begin{array}{ccc} & & W(F) \\ & \overset{s_*}{\nearrow} & \\ W(F[z]) & & \downarrow \quad <1,-f> \\ & \overset{i^*}{\nwarrow} & \\ & & W(F) \end{array}$$

The residue homomorphisms induce the morphisms of this diagram to an exact triangle of Witt groups of residue fields.
Using this technique we obtain exact triangles for function fields of algebraic curves and algebroid surfaces, of the following type:

$$\begin{array}{ccc} & & \oplus W(K_p) \\ & \overset{\Delta}{\nearrow} & \\ W(F[z]) & & \downarrow \\ & \nwarrow & \\ & & \oplus_{p\in S} W(K_p) \end{array}$$

where F is a rational functions field $K(x)$ (resp. the field of quotients of the ring of formal power series in two variables $K((x,y))$) over the field K of characteristic different from $2, \Delta$ is a sum of second residue homomorphisms and may be one or two first, K_p are the associated residue fields and S is the set of ramification points (resp. curves). Moreover in special cases Δ may be an isomorphism, for example for quasihomogenous surfaces.

N. Karpenko
Filtrations on the Grothendieck group of a quadric

For a nondegenerate quadratic form φ consider the Grothendieck group of the projective quadric $\varphi = 0$. Although this group is known from Swan's work and looks

Definition:

$$t_{\min}(K/F) := \min\{\dim\varphi \mid \varphi \text{ is } K-\text{minimal}\}$$
$$t_{\max}(K/F) := \sup\{\dim\varphi \mid \varphi \text{ is } K-\text{minimal}\}.$$

If there are no K-minimal forms over F we define $t_{\min}(K/F) = t_{\max}(K/F) = 1$.

Of particular interest is the case $K = F(\psi)$ with ψ an anisotropic form over F. In this case we have

Theorem: *i)* If $\dim\psi = 2$ then $t_{\min}(K/F) = t_{\max}(K/F) = 2$. *ii)* If $\dim\psi \geq 3$ then $t_{\min}(K/F) \geq 3$. *iii)* If $\dim\psi \geq 5$ then $t_{\min}(K/F) \geq 5$. *iv)* If $\dim\psi \geq 9$ then $t_{\min}(K/F) \geq 7$. If $\dim\psi \in \{2,3,5\}$, $\dim\psi = 4$ and $\psi \notin I^2F$, $\dim\psi = 6$ and φ not a Pfister neighbor, then $t_{\min}(K/F) = \dim\psi$.
Definition: An anisotropic form ψ has property (I') (resp.(I)) if $t_{\min}(F(\psi)/F) > \frac{1}{2}\dim\psi$ (resp. $t_{\min}(F(\psi)/F) > 2^{n-1}$ for each $n \geq 0$ s.t. $\dim\psi > 2^{n-1}$). F has property (I') (resp. (I)) if each anistropic form ψ over F has property (I') (resp.(I)). F has property (IP) if each anisotropic Pfister form π over F has property (I).
The above theorem implies for anisotropic ψ :

Proposition: If $\dim\psi \leq 13$ then ψ has property (I'). If $\dim\psi \leq 8$ then ψ has property (I).

Remark: One easily verifies that $(I) \Rightarrow (I') \Rightarrow (IP)$ for any F.

Theorem: If $\tilde{u}(F) \leq 6$ or if F is linked then F has property (I).
($\tilde{u}(F)$ = Hasse number of F).

Theorem : Let π be an anisotropic n-fold Pfister form. TFAE:

i) $F(\pi)/F$ is excellent

ii) φ is $F(\pi)$-minimal over $F \Rightarrow \exists\tau \in WF$ s.t. a) $\varphi \subset \pi\otimes\tau$ b) $\pi\otimes\tau$ is anisotropic c) $\dim\varphi \geq \frac{1}{2}\dim\pi\otimes\tau + 1$. In this situation we have equality in c).

There are examples of F and $\pi \in P_2F$ s.t. $t_{\max}(F(\pi)/F) = 5$ and, in another example, s.t. $t_{\max}(F(\pi)/F) > 7$. Furthermore we improve a result by Elman, Lam, Wadsworth:

Theorem : Let $\pi \in P_nF$ be anisotropic and F linked or $\tilde{u}(F) \leq 6$, and suppose that if F is linked and $\tilde{u}(F) = \dim\pi = 8$ then anisotropic 3-fold Pfister forms are determined by their total signature. Then $F(\pi)/F$ is excellent.

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In addition it can be proved that for any ideal $\mathfrak{a} \neq \mathfrak{o}$ the group $T(\mathfrak{a})$ is generated by the pure double transvections of order $\subseteq \mathfrak{a}$ and that $SSp(\mathfrak{a})$ is generated by the transvections of order $\subseteq \mathfrak{a}$.

M. Krüskemper
Trace forms of Hilbertian Fields

The purpose of the talk is to introduce a theorem of Mestre and to give some applications of it. The theorem states that if F is a hilbertian field then any trace form of some etale algebra is isometric to the trace form of some field extension. In particular, if F is a number field, any positive quadratic form φ over F with $\dim \varphi \geq 4$ is isometric to the trace form of some field extension.

D. Leep
The u-invariant of a rational function field

Let F be a field, char $F \neq 2$. Define $u_F(2,i)$ to be the min n such that every pair of quadratic forms q_1, q_2 defined over F in more than n variables vanishes on an i-dimensional linear subspace over F.

Theorem: $u(F(t)) = \sup_{i \geq 1}\{u_F(2,i) - 2(i-1)\}$

The proof depends on Amer's theorem which states that q_1, q_2 vanish on an i-dimensional space over F if and only if $q_1 + tq_2$ vanishes on an i-dimensional space over $F(t)$.

D.W. Lewis
Trace forms and splitting fields of central simple algebras

The reduced trace on a central simple algebra over a field gives rise to a quadratic form known as the trace form. A criterion, in terms of the trace form, is given which is necessary and sufficient for the central simple algebra to have a formally real splitting field. Also for a cyclic algebra of degree four a criterion, using the Clifford algebra of the trace form, is given which is necessary and sufficient that the algebra is also a biquaternion algebra (or equivalently that it is of exponent two).

L. Mahé
Pfister's theorems for rings

If R is a real closed field and K a field extension of transcendence degree d over R, then Pfister proved that $\forall f \in K,\quad (f > 0 \text{ in every real closure of } K) \Leftrightarrow \quad f = \boxed{2^d}$

quite easy the topological filtration on it is not so. Some computations show that it depends rather thinly on properties of the form φ. Because factor groups of this filtration contain an information about Chow groups of the quadric the problem to compute the filtration is still more interesting.
Being not able to solve this problem completely one can try to obtain some information computing the other standard filtration namely the j-filtration (which is contained in the topological one).
Some theorems on both filtrations are presented in the talk. In particular, j-filtration is completely discribed for quadrics of dimension ≤ 40.

O.H. Körner

Symplectic groups over 2-adic rings

Let $\mathfrak{o}$ be a local commutative ring with maximal ideal $\mathfrak{p}$ and residue class field $k = \mathfrak{a}/\mathfrak{p}$. I consider a symplectic $\mathfrak{o}$-lattice L of dimension n. Let $Sp(L)$ be the symplectic group of L. For $\sigma \in Sp(L)$ the order $o(\sigma)$ is defined to be the smallest ideal $\mathfrak{a}$ of $\mathfrak{o}$ for which there exists a unit α of $\mathfrak{a}$ such that $\sigma \equiv \alpha id_L \ (\text{mod}\mathfrak{a})$. For a subgroup G of $Sp(L)$ its order $o(G)$ is defined to be the ideal of $\mathfrak{o}$ generated by the $o(\sigma)$ with $\sigma \in G$. For any ideal $\mathfrak{a}$ of $\mathfrak{o}$ the general congruence subgroup modulo $\mathfrak{a}$, defined as $GSp(\mathfrak{a}) := \{\sigma \in Sp(L) \mid o(\sigma) \subseteq \mathfrak{a}\}$, and the special congruence subgroup modulo $\mathfrak{a}$, defined as $SSp(\mathfrak{a}) := \{\sigma \in Sp(L) \mid \sigma \equiv id_L(\text{mod}\ \mathfrak{a})\}$, are known to be normal subgroups of $Sp(L)$ of order $\mathfrak{a}$. I want to deal with the case where L is unimodular, but $\mathfrak{o}$ is 2-adic. By the latter I mean that $\mathfrak{o}$ is a discrete valuation ring statisfying $3 < |k| < \infty$, $\mathfrak{p} = 2\mathfrak{o}$, in particular char $k = 2$. In this case Lacroix's results (1969) on $GL_2(\mathfrak{o})$ imply that for $n = 2$ a subgroup G of $Sp(L)$ is normal iff $G \supseteq SSp(\mathfrak{a})$where $\mathfrak{a} := o(G)$. In a paper from 1974 Chang asserts that Lacroix's criterion remains valid for $n \geq 4$. But I found counterexamples to his assertion which show that for $n \geq 4$, the normal subgroups of $Sp(L)$ cannot be characterized by their orders alone. A second invariant is needed which I call lower order. For $\sigma \in Sp(L)$ the lower order $lo(\sigma)$ is defined to be the ideal of $\mathfrak{o}$ generated by the elements $x(\sigma x)$ with $x \in L$. Then always $lo(\sigma) \subseteq o(\sigma)$. From now on let $n \geq 4$. Then $lo(\sigma) \neq o(\sigma)$ may occur for some σ. For any ideal $\mathfrak{a} \neq \mathfrak{o}$ of $\mathfrak{o}$ the set $\tilde{G}Sp(\mathfrak{a}) := \{\sigma \in GSp(\mathfrak{a}) \mid lo(\sigma) \subseteq 2\mathfrak{a}\}$ is a normal subgroup of $Sp(L)$ of order $\mathfrak{a}$ and of lower order $2\mathfrak{a}$. Therefore, if $\mathfrak{a} \neq \mathfrak{o}$ and $\mathfrak{a} \neq (0)$, then $\tilde{G}Sp(\mathfrak{a}) \not\supseteq SSp(\mathfrak{a})$ because of $lo(SSp(\mathfrak{a})) = \mathfrak{a}$. Thus $\tilde{G}Sp(\mathfrak{a})$ is a counterexample to Chang's assertion in this case. Let $T(\mathfrak{a}) := [Sp(L), \tilde{G}Sp(\mathfrak{a})]$.

Theorem: Let L be unimodular, $n \geq 6$, $\mathfrak{o}$ be 2-adic, G a subgroup of $Sp(L)$ of order $\mathfrak{a}$ and of lower order $\mathfrak{b}$. If $\mathfrak{a} = \mathfrak{o}$ and G is normal, then $G = Sp(L)$. If $\mathfrak{a} \neq \mathfrak{o}$ and $\mathfrak{a} = \mathfrak{b}$, then G is normal iff $G \supseteq SSp(\mathfrak{a})$. If $\mathfrak{a} \neq \mathfrak{o}$ and $\mathfrak{a} \neq \mathfrak{b}$, then G is normal iff $G \supseteq T(\mathfrak{a})$.

2. If $\dim X_T = d < \infty$, then each basic closed $S \subseteq X_T$ is expressible with m inequalities $f_1 \geq 0, \ldots, f_m \geq 0$, $m \leq s_0 + \ldots + s_d$: Here $\dim X_T := \sup \dim \frac{A}{\mathfrak{p}} \mid X_{T(\mathfrak{p})} \neq \emptyset\}$ and $s_i = \sup\{s_{T(\mathfrak{p})} \mid \dim(\frac{A}{\mathfrak{p}}) \leq i,\ X_{T(\mathfrak{p})} \neq \emptyset\}$, $i = 0, \ldots, d$ and $s_{T(\mathfrak{p})}$ is the stability index of the induced preordering $T(\mathfrak{p})$ on the residue field at $\mathfrak{p}$.

Using a generalization of a result of L. Bröcker on the behaviour of the stability index under field extension, we obtain the following application to real algebraic geometry:

Theorem 2: Let (F, P) be an ordered field and let $h_1, \ldots, h_k \in F[X_1, \ldots, X_n]$. Let V be the variety in R^n defined by $h_1 = 0, \ldots, h_k = 0$, R any real closed extension of (F, P). Then

1. Any basic open s.a. set in V defined over F is describable by m inequalities $f_1 > 0, \ldots, f_m > 0$, $f_1 \ldots, f_m \in F[\underline{X}]$ with $m \leq d + \bar{s}_p + \delta$.
2. Any basic closed s.a. set in V defined over F is describable by m inequalities $f_1 \geq 0, \ldots, f_m \geq 0$, $f_1 \ldots, f_m \in F[\underline{X}]$, with $m \leq \frac{d(d+1)}{2} + (d+1)(\bar{s}_P + \delta)$, $\bar{s}_p, \delta$ are computable: $\delta \in \{0, i\}$. $\bar{s}_p = 0$ if P is archimedian, $\bar{s}_p = \delta = 0$ if (F, P) is real closed or hereditarily euclidean.

J. Mináč(joint work with L. Hallock)
Hamlet and Pfister forms. A Tragedy in four acts

Let F be a field, $1 + 1 \neq 0$, $K = F(X_1, X_2, \ldots, X_{2^n})$, φ is an anisotropic Pfister form $< 1, a_1 > \otimes \ldots \otimes < 1, a_n >$ over F, φ_K extended form from F to K. A classical result of Pfister shows that $\varphi_K \cong \varphi(X_1, \ldots, X_{2^n})\varphi_K$. Pfister found a very nice matrix proof and Witt later produced a very short and elegant proof. However for $n \geq 5$ there wasn't known a proof which uses some underlying algebra together with its multiplication and "norm like map" as can be done in the cases $n = 1, 2, 3, 4$ using quadratic field extensions, quaternion algebras and Cayley-Dickson's algebras. In this lecture it was observed that one can set $L = K\left(\sqrt{-a_1}; \sqrt{-a_2\frac{\hat{\psi}_1}{\psi_1}}; \ldots; \sqrt{-a_n\frac{\hat{\psi}_{n-1}}{\psi_{n-1}}}\right)$; where $\psi_1 = x_1^2 + a_1x_2^2$; $\hat{\psi}_1 = x_3^2 + a_1x_4^2$; $\psi_2 = x_1^2 + a_1x_2^2 + a_2x_3^2 + a_1a_2x_4^2$; $\hat{\psi}_2 = x_5^2 + a_1x_6^2 + a_2x_7^2 + a_1a_2x_8^2$; $\ldots$; $\Theta = (x_1 + x_2\sqrt{-a_1})\left(1 + \sqrt{-a_2\frac{\hat{\psi}_1}{\psi_1}}\right)\ldots\left(1 + \sqrt{-a_n\frac{\hat{\psi}_{n-1}}{\psi_{n-1}}}\right) \in L$, $\omega = \otimes\omega_i$, where ω_i is the norm form L_i to K. Then one can identify ω with φ_K and show that the multiplication by the element Θ is the required isometry between forms φ_K and $\psi(x_1, \ldots, x_{2^n})\varphi_K$. The proof is quite transparent.
The lecturers claimed that the main theorem was proved by Rosenkrantz and Guildenstern as well as many other surprising historical revelations concerning King's Claudius's family attempts to solve the mystery of Pfister forms. Both authors died during the lecture thereby bringing it to an abrupt end. These revelations were further discussed on the subsequent wine party.

(sum of 2^d squares). Here we try to push as far as possible this theorem in rings. We obtain

Theorem 1: If $R \to A$ is an R-algebra of tr.d. d and $f \in A$ is totally positive, then

1. if $f \in A^*$ and $d \leq 4$ $\quad f\boxed{2^d} = 1 + \boxed{2^d - 1}$
2. if $f \in A^*$ and $d \geq 4$ $\quad f\boxed{2^d} = 1 + \boxed{2^d + d - 4}$
3. if $f \notin A^*$ and $d \leq 3$ $\quad f\boxed{2^{d+1}} = 1 + \boxed{2^{d+1} - 1}$
4. if $f \notin A^*$ and $d \geq 3$ $\quad f\boxed{2^d + 7} = 1 + \boxed{2^{d+1} + d - 4}$

Corollary: If $R \to A$ has tr.d. d and no real point then

$$-1 = \boxed{2^{d+1} - 1} \quad if \quad d \leq 4$$
$$-1 = \boxed{2^{d+1} + d - 4} \quad if \quad d \geq 4$$

Theorem 2: If $R \to A$ is a semilocal R-alg. of tr.d. d and $f \in A^*$ then f totally positive $\Leftrightarrow$ $\quad$ f $= \boxed{2^d}$

Theorem 3: If $R \to A$ is a Regular Function Ring $(1 + \sum x^2 \subseteq A^*)$ and $f \in A$ then

$$f \text{ totally positive} \Leftrightarrow f = \boxed{2^d}$$

M. Marshall
Minimal generation of constructible sets and semi-algebraic sets

The talk was a summary of results from two papers which will appear in the proceedings of the Ragsquad special year at Berkeley 1989 - 90, and the proceedings of the conference in La Turballe, Brittany in 1991. Part of this work is joint with L. Walter. Several results are presented generalizing work of L. Bröcker and C. Scheiderer. In particular:

Theorem 1: For any commutative ring A with 1 and any preordering $T \subseteq A$

1. each basic open set $S \subseteq X_T$ is describable with m inequalities $f_1 > 0, \ldots, f_m > 0$, $m < \sup\{s_{T(\mathfrak{p})}, 1 : X_{T(\mathfrak{p})} \neq \phi\}$.

following

Theorem: Given $f, g \in R[X]$, $\deg g = \deg f$, $f, g > 0$. Then $\frac{f}{g} = \sum_{i=1}^{n} (\frac{f_i}{g_i})^2$ for some n and $f_i, g_i \in R[X]$ with $\deg f_i = \deg g_i$, f_i, $g_i > 0$ if and only if $\frac{f}{g} \in \sum R(X)^4$. In that case we can even restrict to $n = 2$.
If R is non-archimedean and $\omega > n$ for all n, $\frac{f}{g} = \frac{x^2+\frac{1}{\omega}}{x^2+1}$ does not admit such a representation. As a consequence we get that for all $n \in \mathbb{N}: \frac{x^2+\frac{1}{n}}{x^2+1} = (\frac{f_1^{(n)}}{g_1^{(n)}})^2 + (\frac{f_2^{(n)}}{g_2^{(n)}})^2$ with $f_i^{(n)}, g_i^{(n)} > 0$ and $\deg f_i^{(n)} = \deg g_i^{(n)}$, but $\deg f_i^{(n)} \to \infty$ as $n \to \infty$.

V. Powers
Valuations and higher level orders in commutative rings

Over fields there is a close relationship between valuations and the reduced theory of quadratic forms. As shown by the work of Becker and Rosenberg, this relationship extends to the higher level theory. Our general question is whether we can use the theory of valuation in commutative rings to extend results on higher level orders and higher level reduced Witt rings from fields to commutative rings.
Let R be a commutative ring, we fix a preorder (of level n) and set $O_T = \{\text{orders} P | T \subseteq P\}$. The crucial step is to replace R by $(1+T)^{-1}R$, i.e. we assume $1+t$ is a unit in R for all $t \in T$. Note that this does not change O_T. With this assumption we obtain the following results: For any $P \in O_T, (A(P), I(P))$ is a Manis valuation in R, where $A(P) = \{r \in R | q \pm r \in P$ for some $q \in \mathbb{Q}^+\}$, $I(P) = \{r \in R | q \pm r \in P$ for all $q \in \mathbb{Q}^+\}$. We can define notions of compatibility between valuations and orders (and preorders), and the equivalence relation of dependency on O_T. Finally, using a standard construction originally due to Bröcker, we show that if the set $\{A(P) | P \in O_T\}$ is finite then (R, T) gives rise to a space of signatures (the higher level analogue to a space of orderings). In fact we have that (R, T) is equivalent on the level of T-forms and the reduced Witt ring to a preordered field.

M. Rost
Galois cohomology of algebraic groups and principles of quadratic form theory

Let G be an algebraic group over a field F and let K_i be finite extensions of F of coprime degree. J.P. Serre asked whether for connected G the restriction map

$$H^1(F, G) \longrightarrow \prod_i H^1(K_i, G)$$

is injective. Although there seems to be no a priori reason for a positive answer, the question is a good guide for investigating a specific group G. One gives a survey about classical groups G where one uses Hilbert 90, Witt cancellation and norm

R. Parimala
Non-trivial G_2 - bundles on the affine plane

Let k be a field of characteristic not 2. Suppose k admits a Cayley division algebra O_0. We construct non-trivial Cayley algebras over $k[X,Y]$ with O_0 as the algebra on the fibre, with norm form q, q being an indecomposable rank 7 quadratic form over $k[X,Y]$. This gives examples of principal G_2-bundles on $\mathbb{A}_k^2$ which admit no reduction of the structure group to any proper, connected reductive subgroup, a case left out by Raghunathan in his construction of non-trivial G-bundles on $\mathbb{A}_k^2$, for any connected reductive group. The main ingredient is to first construct a family of Cayley-algebra bundles with norms $< 1, \lambda > \perp q_0$, where q_0 are rank 6 quadratic spaces with discriminant $-\lambda, q_0$ indecomposable. These give G_2-bundles on $\mathbb{A}_k^2$ with reduction of the structure group to SU_3. Then, a patching technique is adopted to produce Cayley algebra bundles of the required type.

W. Plesken
Finite unimodular groups and their invariant forms

For finite rational matrix groups G of degree n two key structures have to be investigated, namely the $\mathbb{Z}G$-lattices in the natural $\mathbb{Q}G$-module $\mathbb{Q}^n$ and the G-invariant bilinear forms on $\mathbb{Q}^n$. Some open problems are mentioned and two topics are discussed in detail. Firstly the positive semidefinite G-invariant integral forms on a $\mathbb{Z}G$-lattice L form a semigroup under addition. The additively indecomposable forms ϕ are the ones which do not allow diagonal embeddings of L in orthogonal sums of $\mathbb{Z}G$-lattices with positive semidefinite G-invariant forms on them. Under some restrictions one can show that $Aut_{\mathbb{Z}G}(L)$ acts on the set of these forms with only finitely many orbits. Secondly the interplay between the forms and the lattices leads to the main tools for classifying maximal finite subgroups of $Gl(n,\mathbb{Q})$. This has been carried out up to degree $n \leq 23$ in collaboration with Gabriele Nebe. Many new lattices with big automorphism groups arise this way.

A. Prestel
On a variation of Hilbert's 17th problem

At the 1987 conference on real algebraic geometry Schülting was stating the following *problem*: Let $f,g \in \mathbb{R}[X], \deg f = \deg g$, and $f,g > 0$. Is it possible to find $f_i, g_i \in \mathbb{R}[x]$ s.t. $\frac{f}{g} = \sum(\frac{f_i}{g_i})^2$, $\deg f_i = \deg g_i$ and $f_i, g_i > 0$? Recently Joachim Schmid solved this problem positively in the following generalisation: Let $H(K)$ be the real holomorphy ring of a formally real field K. Denote by $U^+(K)$ the group of totally positive units of $H(K)$. Then to every $e \in U^+(K)$ there exists some $n \leq P_2(K)+1$ and $e_i \in U^+(K), 1 \leq i \leq n$ s.t. $e = \sum_{i=1}^{n} e_i^2$. The case $K = \mathbb{R}(x)$ solves Schülting's problem. Replacing $\mathbb{R}$ by a real closed field R in Schülting's problem , we prove the

We end with an open question:
Suppose A is an algebra and K/F is a field extension of odd degree.
If σ is an anisotropic involution on A then must $\sigma \otimes K$ be anisotropic? Springer's theorem says that this is true when A is split. Bayer-Lenstra (1990) proved that $\sigma \otimes K$ cannot be hyperbolic.

R. Scharlau
Unimodular lattices over real quadratic fields

We investigate integral even unimodular lattices L in a vector space with a totally positive definite quadratic form, defined over a real quadratic field F. We give explicit constructions of a number of such lattices in dimension 4, for indeterminate field discriminant d (only depending on d mod 24, i.e. on the ramification of 2 and 3 in F). The lattices we construct here have large automorphism groups. In most cases, the full orthogonal group is known and essentially independent of the field. This is true in particular for the so-called reflective lattices which have a "root system" of maximal rank. As an application we obtain the full classification (essentially independent of the use of computers) of all even unimodular lattices in dimension 4 over the first 11 real quadratic fields with discriminants $d = 5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37$.

J.-P. Tignol
An elementary proof of the existence of fields with arbitrary even u-invariant

The existence of fields of characteristic different from 2 with arbitrary even u-invariant has been proved in 1989 by Aleksandr S. Merkurjev. The aim of this talk was to outline Merkurjev's construction and to give an elementary proof of the following key result:

Theorem (Merkurjev): Let D be a central division algebra over a field F of characteristic not 2 and let ψ be an anisotropic quadratic form over F of dimension at least 2. The algebra $D \otimes_F F(\psi)$ is not a division algebra if and only if D contains a homomorphic image of the even Clifford algebra $C_0(\psi)$.
It follows from this Theorem that tensor products of $(2n-1)$ quaternion algebras which are division algebras over F remain division algebras over the function field of every quadratic form of dimension $2n+1$. This observation provides a way of constructing an anisotropic form of dimension $2n$ which remains anisotropic when all the forms of dimension $2n+1$ are made isotropic in a generic way.

Berichterstatter: O. Heddinga

principles due to Diedonné, Knebusch and Scharlau to give a positive answer in many cases. Moreover the groups $G = G_2, F_4$ were discussed.

D.B. Shapiro (Joint with J.-P. Tignol)
Hyperbolic involutions

$F =$ field of char. $\neq 2$. Every algebra here is central simple F-algebra; every involution is of first kind. Let (A, σ) be an algebra with involution , define $V =$ irreducible left A- module and $D = \mathrm{End}_A V$. Then V is a right D-vector space and $A \cong \mathrm{End}_D V$. Albert proved that D admits an involution. Any nonsingular λ-hermitian form $h : V \times V \to D$ (where $\lambda = \pm 1$) induces an adjoint involution I_h on $\mathrm{End}_D V$, generalizing the transpose. Conversely, any involution σ on A equals I_h for some λ-hermitian form h, which is uniquely determined up to a scalar factor from F^*.

Lemma: Equivalence:

1. (V, h) is anisotropic (i.e. $h(x, x) = 0$ implies $x = 0$)
2. if $a \in A$ then : $\sigma(a) \times a = 0$ implies $a = 0$.
3. if $J \subseteq A$ is a right ideal then $J = fA$ for some idempotent f with $\sigma(f) = f$.

Proposition: Equivalence:

1. (V, h) is hyperbolic.
2. There exists a right ideal $J \subseteq A$ with $\sigma(a)a = 0$ for $a \in J$ and $\dim J = \frac{1}{2} \dim A$.
3. There exists an idempotent $e \in A$ with $\sigma(e) = 1 - e$.
4. There is a σ- invariant subalgebra $M \subseteq A$ such that $M \cong \mathbb{M}_2(F)$ and $\sigma|_M \cong I_{\mathbb{H}}$ (Note that $\mathbb{H} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so that $I_{\mathbb{H}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

We use this "internal" characterization of hyperbolic spaces to investigate the behavior of forms under quadratic extensions.

Theorem: Suppose (A, σ) is anisotropic and $K = F(\sqrt{d})$ is a quadratic field extension of F. Then $(A \otimes K, \sigma \otimes k)$ is hyperbolic $\Leftrightarrow$ there exists $r \in A$ with $r^2 = d$ and $\sigma(r) = -r$.
In the split case, $A = \mathrm{End}_F V$ and $\sigma = I_q$, it is well-known that such r exists if and only if $q \cong < 1, d > \otimes q'$ for some q'.
Suppose $A \cong \mathbb{M}_n(D)$ where D is a quaternion division algebra. This theorem has an application to the question of when an involution σ on A admits an invariant quaternion subalgebra.

Oliver Heddinga
Fachbereich Mathematik
Universität Mainz
Saarstr. 21
Postfach 3980

W-6500 Mainz 1
GERMANY

Dr. Detlev Hoffmann
Aindorferstr. 84

W-8000 München 21
GERMANY

Prof.Dr. Donald G. James
Department of Mathematics
Pennsylvania State University
218 McAllister Building

University Park , PA 16802
USA

Prof.Dr. Piotr Jaworski
Institute of Mathematics
University of Warsaw
ul. Banacha 2

Warszawa 02-097
POLAND

Prof.Dr. Nikita Karpenko
Math. Institut der Universität
Münster
Einsteinstr. 26

W-4400 Münster
GERMANY

Prof.Dr. Manfred Knebusch
Fakultät für Mathematik
Universität Regensburg
Postfach 397
Universitätsstr. 31

W-8400 Regensburg
GERMANY

Prof.Dr. Martin Kneser
Mathematisches Institut
Universität Göttingen
Bunsenstr. 3-5

W-3400 Göttingen
GERMANY

Prof.Dr. Max Albert Knus
Mathematik-Department
ETH Zürich
ETH-Zentrum
Rämistr. 101

CH-8092 Zürich

Prof.Dr. Otto Hermann Körner
Abteilung für Mathematik IV
Universität Ulm
Helmholtzstr. 18

W-7900 Ulm
GERMANY

Prof.Dr. Martin Krüskemper
Mathematisches Institut
Universität Münster
Einsteinstr. 62

W-4400 Münster
GERMANY

Prof.Dr. David B. Leep
Dept. of Mathematics
University of Kentucky
715 POT

Lexington , KY 40506-0027
USA

Tagungsteilnehmer

Prof.Dr. Eva Bayer-Fluckiger
Laboratoire de Mathématiques
Université de Franche-Comte
16, Route de Gray

F-25030 Besançon Cedex

Ido Efrat
Institute for Advanced Studies
The Hebrew University of Jerusalem
Givat Ram

91904 Jerusalem
ISRAEL

Prof.Dr. Eberhard Becker
Fachbereich Mathematik
Universität Dortmund
Postfach 50 05 00

W-4600 Dortmund 50
GERMANY

Frank Esselmann
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31

W-4800 Bielefeld 1
GERMANY

Prof.Dr. Ludwig Bröcker
Mathematisches Institut
Universität Münster
Einsteinstr. 62

W-4400 Münster
GERMANY

Prof.Dr. Robert W. Fitzgerald
Dept. of Mathematics
Southern Illinois University

Carbondale , IL 62901-4408
USA

Prof.Dr. Jean Louis Colliot-Thélène
Mathématiques
Université de Paris Sud (Paris XI)
Centre d'Orsay, Bâtiment 425

F-91405 Orsay Cedex

Prof.Dr. Jan van Geel
Seminarie voor Algebra en
Funct. Analyse
Rijksuniversiteit Gent
Galglaan 2

B-9000 Gent

Prof.Dr. Andrew G. Earnest
Dept. of Mathematics
Southern Illinois University

Carbondale , IL 62901-4408
USA

Prof.Dr. Danielle Gondard
Institut de Mathématiques Pures et
Appliquées, UFR 920
Université de Paris VI
4, Place Jussieu

F-75252 Paris Cedex 05

Prof.Dr. Heinz-Georg Quebbemann
Fachbereich Mathematik/Informatik
Universität Oldenburg
Carl-von-Ossietzky-Str.
Postfach 2503

W-2900 Oldenburg
GERMANY

Dr. Claus Scheiderer
Fakultät für Mathematik
Universität Regensburg
Postfach 397
Universitätsstr. 31

W-8400 Regensburg
GERMANY

Prof.Dr. Carl R. Riehm
Department of Mathematics and
Statistics
Mc Master University

Hamilton Ontario L8S 4K1
CANADA

Dr. Rainer Schulze-Pillot
Mathematisches Institut
Universität zu Köln
Weyertal 86-90

W-5000 Köln 41
GERMANY

Dr. Markus Rost
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26

W-5300 Bonn 3
GERMANY

Prof.Dr. Daniel B. Shapiro
Mathematisches Institut
Universität Dortmund
Postfach 500500

W-4600 Dortmund 50
GERMANY

Dr. Rudolf Scharlau
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31

W-4800 Bielefeld 1
GERMANY

Dr. Ramadorai Sujatha
Fakultät für Mathematik
Universität Regensburg

W-8400 Regensburg
GERMANY

Prof.Dr. Winfried Scharlau
Mathematisches Institut
Universität Münster
Einsteinstr. 62

W-4400 Münster
GERMANY

Prof.Dr. Jean-Pierre Tignol
Institut de Mathématiques
Université Catholique de Louvain
Chemin du Cyclotron,2

B-1348 Louvain-la-Neuve

Prof.Dr. David W. Lewis
Dept. of Mathematics
University College
Belfield

Dublin 4
IRELAND

Prof.Dr. Raman Parimala
School of Mathematics
Tata Institute of Fundamental Research
Homi-Bhabha Road

Bombay 400 005
INDIA

Prof.Dr. Louis Mahe
U. E. R. Mathématiques
I. R. M. A. R.
Université de Rennes I
Campus de Beaulieu

F-35042 Rennes Cedex

Prof.Dr. Albrecht Pfister
Fachbereich Mathematik
Universität Mainz
Saarstr. 21
Postfach 3980

W-6500 Mainz 1
GERMANY

Prof.Dr. Murray Marshall
Dept. of Mathematics
University of Saskatchewan

Saskatoon Sask. S7N 0W0
CANADA

Prof.Dr. Wilhelm Plesken
Lehrstuhl B für Mathematik
RWTH Aachen
Templergraben 64

W-5100 Aachen
GERMANY

Michael Meurer
Fachbereich Mathematik
Universität Mainz
Saarstr. 21
Postfach 3980

W-6500 Mainz 1
GERMANY

Prof.Dr. Victoria Powers
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
Universitätsstr. 31, PF 397

W-8400 Regensburg
GERMANY

Prof.Dr. Jan Minac
Department of Mathematics
Middlesex College
University of Western Ontario

London, Ontario N6A 5B7
CANADA

Prof.Dr. Alexander Prestel
Fakultät für Mathematik
Universität Konstanz
Postfach 5560

W-7750 Konstanz 1
GERMANY