MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Reelle algebraische Geometrie

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The fourth conference on real algebraic geometry at Oberwolfach was organized by professors E. Becker (Dortmund), L. Bröcker (Münster) and M. Knebusch (Regensburg). The participants presented their most recent work in the meeting. The lively scientific atomsphere of the conference resulted in very many stimulating discussions, which maybe will influence future directions and will contribute to further progress in the field.

Vortragsauszüge

On filtrations of sheaves on semi-algebraic sets

Ralph Berr, Dortmund

Let R be a real closed field and $S \subset R^n$ a semialgebraic set. We denote by C_S the sheaf of continuous semialgebraic functions on S. Now let F be a subsheaf of C_S. We will associate with F a direct system $F_n^* \subset F$ of subsheaves with $\varinjlim F_n^* = F$.

It is the aim of this approach to use these subsheaves for a more detailed analysis of the morphisms $(S, F) \to (T, F')$ with S, T semialgebraic and $F \subset C_S$, $F' \subset C_T$. Let $A := R[X_1, \ldots, X_n]$. In order to construct the sheaves F_n^* we use the real spectrum of higher level $\mathbb{P}$-Sper$A$ of A. This space has the following properties:

Theorem 1: a) $\mathbb{P}$-Sper$A$ is a spectral space.

b) Sper$A$ is a dense subspace of $\mathbb{P}$-Sper$A$.

c) There is an open retract $\pi: \mathbb{P}\text{-Sper}A \to \text{Sper}A$.

Hence the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}\text{-Sper } A & \xrightarrow{\ \pi\ } & \text{Sper } A \\ \uparrow & \nearrow_{\text{id}} & \\ \text{Sper } A & & \end{array}$$

Now let $S \subset R^n$ be semialgebraic, $\tilde{S} \subset \text{Sper}A$ the corresponding constructible subset and $F \subset \mathcal{C}_{\tilde{S}}$. For $n \in \mathbb{N}$ we associate with S a subspace $\tilde{S}_n \subset \mathbb{P}\text{-Sper}A$ and a sheaf F_n on $\tilde{S}_n$ such that

a) $\tilde{S}$ is dense in $\tilde{S}_n$.

b) $i: \tilde{S} \hookrightarrow \tilde{S}_n$ extends to a morphism $(\tilde{S}, F) \to (\tilde{S}_n, F_n)$ of ringed spaces.

Moreover the family $(\tilde{S}_n, F_n)$ forms an inverse system and we have the basic result:

Theorem 2: $\varprojlim(\tilde{S}_n, F_n) = (\tilde{S}, F)$.

Now let $F_n^* = \pi_* F_n$, where $\pi: \tilde{S}_n \to \text{Sper}A$ (see Theorem 1). Then F_n^* is a subsheaf of F and $F = \varinjlim F_n^*$. For $U \subset \tilde{S}$ open let $\ell(f) := \min\{n \in \mathbb{N} \mid f \in F_n^*(U)\}$. $\ell(f)$ is called the level of f. The rings $F_n^*(U)$ and the levels $\ell(f)$ can be characterized as follows: For $\alpha \in U$ let $h(\alpha)$ be the henselization of $(k(\text{supp}(\alpha)), \bar{\alpha})$ with respect to the natural valuation w_α of $\bar{\alpha}$. For $f \in F(U)$ we have

$$\begin{array}{ccccc} f: U & \longrightarrow & \coprod_{\alpha \in U} h(\alpha)(f(\alpha)) & \longrightarrow & \coprod_{\alpha \in U} k(\alpha) \\ & & \Big| & e_{\alpha,f} = \text{ramification index} & \\ & & \coprod(h(\alpha), w_\alpha^h)). & & \end{array}$$

Here $e_{\alpha,f}$ denotes the ramification of the unique extension of w_α^h to $h(\alpha)(f(\alpha))$

Theorem 3: a) $F_n^*(U) = \{f \in F(U) \mid e_{\alpha,f} | n \text{ for all } n \in \mathbb{N}\}$.

b) $\ell(f) = \text{l.c.m.}\{e_{\alpha,f} | \alpha \in U\}$.

Therefore the level of $f \in F(U)$ measures the ramification of f on U. Similarly one can define levels of morphisms $\varphi: (\tilde{S}, F) \to (\tilde{T}, F')$. Again this level measures the ramification of φ. For example, let $S_1 = \{0 \leq x, \quad 0 \leq y \leq x^n\} \subseteq \mathbb{R}^2$, $S_2 = \{0 \leq x, \quad 0 \leq y \leq x\}$. Then $f: (S_1, \mathcal{C}_{S_1}) \to (S_2, \mathcal{C}_{S_2})$, $[(x, y) \mapsto (x, \sqrt[n]{y})]$ is an isomorphism and the level of f is n.

A few open problems about geometry and topology of real algebraic varieties

J. Bochnak, Amsterdam

Let X and Y be affine real algebraic varieties and let $\mathcal{R}(X, Y)$ be the space of real regular mappings from X into Y. Let $S^n = \{x \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$ be the unit n-sphere.

1. Is $\mathcal{R}(S^n, S^k)$ dense in $C^\infty(S^n, S^k)$, where the space of C^∞ mappings is equipped with C^∞ topology? Or, at least is every mapping $f \in C^\infty(S^n, S^k)$ homotopic to a $g \in \mathcal{R}(S^n, S^k)$. (It is so if $k = 1, 2$ or 4, or of course if $n < k$).
2. Does there exist a regular mapping $S^2 \times S^2 \to S^4$ of odd topological degree?
3. Assume that X is compact connected nonsingular and oriented of dimension n, and denote
$$\mathrm{Deg}_{\mathcal{R}}(X) = \{k \in \mathbb{Z} | k = \deg f\ ,\ f \in \mathcal{R}(X, S^n)\}$$
where $\deg f$ is the topological degree of f. Is $\mathrm{Deg}_{\mathcal{R}}(X)$ a subgroup of $\mathbb{Z}$?
(It is so if n is odd and in such a case $\mathrm{Deg}_{\mathcal{R}}(X)$ is always $\mathbb{Z}$ or $2\mathbb{Z}$. However no example of odd dimensional X with $\mathrm{Deg}_{\mathcal{R}}(X) = 2\mathbb{Z}$ is known.
If $n = 2$ or 4, then $\mathrm{Deg}_{\mathcal{R}}(X)$ is also a subgroup of $\mathbb{Z}$. In such a case $\mathrm{Deg}_{\mathcal{R}}(X)$ can be even an arbitrary subgroup of $\mathbb{Z}$).
4. Let Z be a complex projective nonsingular irreducible algebraic variety and let $Z_\mathbb{R}$ be its underlying real algebraic variety (which is known to be affine). Let $n = \dim_\mathbb{C} Z$ (and hence $\dim_\mathbb{R} Z_\mathbb{R} = 2n$). Is $\mathcal{R}(Z_\mathbb{R}, S^{2n})$ dense in $C^\infty(Z_\mathbb{R}, S^{2n})$? (It is so if $n = 1$. If $n = 2$ then $\mathrm{Deg}_{\mathcal{R}}(Z_\mathbb{R})$ is always $\mathbb{Z}$ or $2\mathbb{Z}$, but again no example of Z with $\mathrm{Deg}_{\mathcal{R}}(Z_\mathbb{R}) = 2\mathbb{Z}$ is known. The simplest unknown case is $Z = \mathbb{C}P' \times \mathbb{C}P'$; then of course $Z_\mathbb{R} \simeq S^2 \times S^2$; see question 2). It is known that $\mathrm{Deg}_{\mathcal{R}}(Z_\mathbb{R}) \neq \{0\}$ in any dimension.
5. Find all (n, k) such that each polynomial mapping $S^n \to S^k$ is constant. (It is known each couple of the type $(2^p, k)$, $k < 2^p$, has this property. Are there others?). It suffices to solve this problem for couples of the type $(n, n-1)$. The lowest unknown case is $(48, 47)$.
6. Describe all regular automorphisms of $\mathbb{R}P^n$ and S^n. Are they all linear? Unknown even for $n = 2$.
7. Let Z_1 and Z_2 be complex projective algebraic varieties. When their underlying real algebraic structures $Z_{1\mathbb{R}}$ and $Z_{2\mathbb{R}}$ are biregularly isomorphic? (The problem has been solved for complex nonsingular curves and for a large family of complex abelian varieties (Huisman)). It is possible to have an infinite family $(Z_j)_{j\in\mathbb{N}}$ of complex mutually nonisomorphic algebraic varieties with all $Z_{j\mathbb{R}}$ isomorphic?

8. Does there exist an infinite family $\mathcal{F}$ of nonisomorphic real algebraic varieties having the same (isomorphic) complexification? The answer is negative for nonsingular real curves (in which case $\#\mathcal{F} \leq 2(\sqrt{g}+1)$, where g is the genus of the complexified curve).

9. Algebraic embeddings of real algebraic varieties in $\mathbb{R}P^n$ or $\mathbb{R}^n$. Several open problems. Example: Let V be a complex projective nonsingular curve. Does $V_{\mathbb{R}}$ (its real algebraic underlying surface) embed algebraically into $\mathbb{R}^3$ or $\mathbb{R}P^3$?

10. Does there exist uncountably many nonisomorphic complex projective nonsingular curves V with $H_1(V_{\mathbb{R}}, \mathbb{Z}/2) = H_1^{\text{alg}}(V_{\mathbb{R}}, \mathbb{Z}/2)$? Here $H_1^{\text{alg}}(V_{\mathbb{R}}, \mathbb{Z}/2)$ denotes the subgroup of $H_1(V_{\mathbb{R}}, \mathbb{Z}/2)$ of homotopy classes represented by real algebraic curves contained in $V_{\mathbb{R}}$. (It is known that there exist exactly countably many complex elliptic curves E with $H_1^{\text{alg}}(E_{\mathbb{R}}, \mathbb{Z}/2) = H_1(E_{\mathbb{R}}, \mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2$).

Differential geometry of semialgebraic sets (first steps)

Ludwig Bröcker, Münster

Let $S \subset \mathbb{R}^n$ be semialgebraic compact (for simplicity). We provide S with the intrinsic path metric d. Then (S, d) has the structure of a Riemann polyhedron. By this we mean the following:

A Riemann metric on the standard simplex Δ^n is given by a scalar product G_{ij} which is bounded, C^∞ on open subsimplices and continuous when passing from $\Delta^k \to \Delta^{k-1}$. A Riemann polyhedron is patched together from Riemann simplices. Under extraconditions, which hold for semialgebraic sets, one has a Gauß-Bonnet formula in dimension 2.

$$\int_S k\,d\omega + \int_{\text{sing } S} k\,ds + \sum \alpha_i = 2\pi\chi(S)$$

For instance, in the cube the curvature is concentrated in the vertices, being $\frac{\pi}{2}$ at each of them.

Semialgebraic Riemann-polyhedra appear also in the theory of the reduction of semialgebraic sets via real valuations.

Local-global principle for semi-local rings in étale cohomology

Jerôme Burési, Rennes

Let A be a ring, $2 \in A^*$, $H^n(A) := H^n_{\text{et}}(A, \mu_2)$ and $\varphi|_B$ the image of $\varphi \in H^n(A)$ in $H^n(B)$ for a homomorphism $A \to B$. $\varphi \in H^n(A)$ is said to be (-1) torsion free if $(-1)^k \vee \varphi \neq 0$ for all $k \in \mathbb{N}$. We prove the following.

Theorem: Let A be a semilocal ring, $2 \in A^*$. If $\varphi \in H^n(A)$ is (-1) torsion free then there exists a real point α such that $\varphi|_{k(\alpha)} \neq 0$, where $k(\alpha)$ is the real closure of α.

Proof: By Arason Theorem the result is true if A is a field. So we are reduced to prove there exists a real ideal p such that $\varphi|_{A_p^h}$ is (-1)-torsion free where A_p^h is the henselization of A_p. We have

Proposition 1 (Localization): Let A be a ring, $a, u \in A$ such that $2d = 2(u^2 + a) \in A^*$ then kernel and cokernel of $H^n(A) \to H^n(A_a)$ are killed by cup-product with (d)

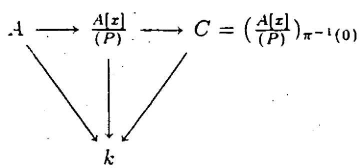
Proposition 2: Let A be a semi-local ring. If φ is (-1)-torsion free, then $P = \{a \in A,$ $\forall u = \sum_{i=1}^n u_i^2$ such that $u + a \in A^* \Rightarrow (u+a) \vee \varphi$ is (-1) torsion $\}$ is a preordering.

The proof is using the property of transversality of quadratic forms over semilocal rings. Mixing the two propositions and picking an ordering α containing P we have that for $p = \text{supp}(\alpha)$ $\varphi|_{A_p}$ is (-1) torsion free.

Proposition 3 (projection formula): Consider $A \to \frac{A[x]}{(P)} = B$, where P is monic. Then there exists a map $N \colon H^n(B) \to H^n(A)$ such that $N[(b)] = (N[b])$ with abuse of the notation, and $\forall \varphi \in H^n(A)$ $\forall \psi \in H^m(B)$ $N(\varphi|_B \vee \psi) = \varphi \vee N(\psi)$.

Proposition 4: Let A be a local ring and suppose the maximal ideal m is real. Let P be a monic polynomial such that $\frac{k[x]}{(P)} \cong \frac{k[x]}{(x-x_0)} \times \frac{k[x]}{P_0}$ where $P_0(x_0) \neq 0$ (P has a simple root in k). Let $\pi \colon \frac{A[x]}{(P)} \to k$, $x \mapsto x_0$. If $s \in B$ is such that $\pi(s) < 0$, then there is $u \in B$ such that $s + u^2 \in B^*$ and $\overline{N(u^2+s)} < 0$

Proposition 5: Let A be a local ring, P and π as in proposition 4. If $\varphi \in H^n(A)$ is (-1)-torsion free and the image of P is on the positive part in k, then $\varphi|_C$ is (-1) torsion free, where



This suffices to prove $\varphi|_{A_p^h}$ is (-1) torsion free.

Zero-cycles and cohomology of real algebraic varieties

J.-L. Colliot-Thélène, Orsay

(Joint work with C. Scheiderer, Regensburg)

Let $X/\mathbb{R}$ be an algebraic variety. Assume $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ connected and $X(\mathbb{R}) \neq \emptyset$. Let s be the number of connected components of $X(\mathbb{R})$ and t the number of compact connected components. Let $b_i = \dim H^i(X(\mathbb{R}))$ when $H^i(X(\mathbb{R})) = H^i_{\text{classical}}(X(\mathbb{R}), \mathbb{Z}/2)$. Let $\pi: X_{\mathbb{C}} \to X$ be the projection. Given an abelian group A, and $n > 0$ an integer, let ${}_nA = \{x \in A \mid nx = 0\}$.

(1) Zero-cycles

Let $CH_0(X)$ be the Chow group of zero-cycles modulo rational equivalence. If X is complete, let $A_0(X) \subset CH_0(X)$ be the subgroup of classes of degree 0. Let $D(X) \subset CH_0(X)$ be the maximal divisible subgroup.

Theorem 1.1. a) $CH_0(X)/2 \cong (\mathbb{Z}/2)^t$

b) $D(X) = \pi_* A_0(X_{\mathbb{C}}) = 2A_0(X)$ if X is complete

$D(X) = \pi_* CH_0(X_{\mathbb{C}}) = 2CH_0(X)$ if X is not complete.

In the complete case, this is due to CT/Ischebeck (1981) – the proof there is by reduction to the classical case of curves.

Theorem 1.2. Assume $X/\mathbb{R}$ smooth. Then for any integer $n > 0$, the group ${}_nCH_0(X)$ is finite. If X is affine, or if X is complete and $H^1(X, \mathcal{O}_X) = 0$, then $D(X)$ is uniquely divisible. If X is complete, $q = \dim H^1(X, \mathcal{O}_X)$, $A =$ Albanese variety of X and 2^r is the number of connected components of $A(\mathbb{R})$, then the order of ${}_2CH_0(X)$ is bounded by $2^{2q+s-1+r}$.

The proof uses Roitman's theorem on 0-cycles on $X_{\mathbb{C}}$.

(2) Partial degenerescence of the Bloch-Ogus spectral sequence

Let $H^i(X) := H^i_{\text{et}}(X, \mathbb{Z}/2)$. Let $\mathcal{H}^i$ be the Zariski sheaf on X associated to the presheaf $U \mapsto H^i(U)$. There is a spectral sequence

$$E_2^{p,q} = H^p_{\text{Zar}}(X, \mathcal{H}^q) \Rightarrow H^{p+q}(X) \qquad (*)$$

$E_2^{p,q} = 0$ for $p > d = \dim X$. If X is smooth, $E_2^{p,q} = 0$ for $p > q$ (Bloch-Ogus) and $H^i(X, \mathcal{H}^i) \cong CH^i(X)/2$. It is known (see my talk at Oberwolfach, 1990, see also Scheiderer's Habilitationsschrift) that

$$H^i(X, \mathcal{H}^n) \cong H^i(X(\mathbb{R})) \quad \text{for} \quad n \geq d+1$$

(X arbitrary). This is proved by using the isomorphism

$$H^n(X) \cong \bigoplus_{i=0}^{d} H^i(X(\mathbb{R})) \quad \text{for} \quad n \geq 2d+1$$

(Artin-Verdier, Cox, others, ...). Using these facts together with counting arguments, we show:

Theorem 2.1. The differentials in $(*)$ ending in (p,q) with $p+q \geq 2d+1$ vanish. If X is smooth, they vanish if $p+q \geq 2d$. If X is smooth and $H^{2d-1}(X_{\mathbb{C}}) = 0$, they vanish if $p+q \geq 2d-1$.

Theorem 2.2. Let $X/\mathbb{R}$ be smooth. Hence $CH_0(X)/2 = CH^d(X)/2 \cong H^d(X, \mathcal{H}^d)$.

a) $CH^d(X)/2 \cong H^d(X(\mathbb{R}), \mathbb{Z}/2) \cong (\mathbb{Z}/2)^t$ [proof independent of (1)]

b) The cycle map $CH^d(X)/2 \to H^{2d}(X)$ is injective.

c) The group $H^{d-1}(X, \mathcal{H}^d)$ is finite. It admits a filtration whose successive quotients are: $H^{d-1}(X(\mathbb{R}))$; $\mathrm{Ker}[CH^d(X)/2 \to CH^d(X_{\mathbb{C}})/2] \cong (\mathbb{Z}/2)^{t-\varepsilon}$, where $\varepsilon = 1$ if X is complete and $\varepsilon = 0$ if X is not complete; Image $[H^{2d-1}(X_{\mathbb{C}}) \to H^{d-1}(X, \mathcal{H}^d)]$.

d) If $H^{2d-1}(X_{\mathbb{C}}) = 0$, there is a surjection:

$$H^{d-2}(X, \mathcal{H}^d) \twoheadrightarrow H^{d-2}(X(\mathbb{R})).$$

[for $d=2$ this gives a surjective ${}_2\mathrm{Br}X \twoheadrightarrow (\mathbb{Z}/2)^s$]

(3) Connection with $\mathcal{K}$-cohomology

Let $\mathcal{K}_i$ be the Zariski sheaf associated to $U \mapsto K_i(\Gamma(U, \mathcal{O}_X))$. From K-theory (results of Merkur'ev-Suslin) we have an exact sequence for $X/\mathbb{R}$ smooth:

$$0 \to H^{d-1}(X, \mathcal{K}_d)/2 \to H^{d-1}(X, \mathcal{H}^d) \to_2 CH^d(X) \to 0.$$

The results of §2 yield another proof of the finiteness of ${}_nCH^d(X)$ (s. §1). If X is affine of dimension ≥ 2 or X is complete and $H^{2d-1}(X_{\mathbb{C}}) = 0$, we show

$$H^{d-1}(X, \mathcal{K}_d)/2 \cong H^{d-1}(X(\mathbb{R})).$$

A completely normal spetral space that is not a real spectrum

Charles N. Delzell and James J. Madden
Louisiana State University

A spectral space is said to be *completely normal* if whenever points x and y are in the closure of a singleton $\{z\}$, then either x is in the closure of $\{y\}$ or y is in the closure of $\{x\}$.

Let A be a ring, and let $\mathrm{Spec}_r A$ denote the real spectrum of A. In one of the earliest and most influential papers on the real spectrum, Coste and Roy showed that $\mathrm{Spec}_r A$ is a completely normal spectral space, and they asked whether every completely normal spectral space is the real spectrum of some ring. Up to the present, all known topological properties of $\mathrm{Spec}_r A$ have been consequences of complete normality. In this talk, we gave an example showing that the answer to Coste and Roy's question is "no". Our example is one-dimensional. The ideal is to imitate $\mathrm{Spec}_r\mathbb{R}[X]$. It will appear in Journal of Algebra, 1994.

On the homology of the space of non-singular real plane algebraic curves

Th. Fiedler, Toulouse

Let $X \subset \mathbb{C}P^2$ be a smooth real algebraic curve of degree $2k$ and let $\mathfrak{M}_X$ denote the connected component in the space of all real smooth curves of degree $2k$ which contains X.

Let p denote the number of ovals of the real part $X_\mathbb{R} = X \cap \mathbb{R}P^2$ which are surrounded by an even number of ovals and let n^- denote the number of ovals which are surrounded by an odd number of ovals and which bound moreover from the outside a surface of negative Euler characteristics in $\mathbb{R}P^2 \backslash X_\mathbb{R}$.

Theorem: If the curve $X_\mathbb{R}$ contains $k-1$ ovals which surround one and the same point in $\mathbb{R}P^2$ and if $\dim H_*(\mathfrak{M}_X; \mathbb{Z}/2\mathbb{Z})$ is odd then

$$p - n^- \leq \frac{3}{2}k(k-1) - \frac{1}{2}(k-3).$$

Remark. For *any* curve of degree $2k$ *Arnold's inequality* holds: $p - n^- \leq \frac{3}{2}k(k-1)$. This inequality is sharp, even for curves which satisfy the first condition of the theorem. Hence the theorem implies that from information about the components of non-singular curves in the space of all curves one can derive an improvement of Arnold's inequality.

The algebra of the center problem

J. P. Francoise, Paris

I study a generalization of Bautin's theorem to polynomial vector fields of $\mathbb{R}^{2m}$ of the following type:

$$X = \sum_{j=1}^{m} \lambda_j (x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}) + \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta \frac{\partial}{\partial x} + \sum c^*_{\alpha\beta} x^\alpha y^\beta \frac{\partial}{\partial y}$$

where λ_j are $\mathbb{Z}$-independent. In a neighbourhood of 0:

Theorem 1: The set of X so that X has m formal first integrals generically independent is an algebraic manifold, denotend by C.

Theorem 2: X has m analytic first integrals iff X has continuum of invariant torii. We denote by Z the set of such vector-fields. Then $Z \underset{\neq}{\subset} C$.

Theorem 3: We can bound uniformly (but the bound cannot be computed) the number of invariant torii for $X \notin C$.

Gabrielov theorem for subpfaffian sets

Zbigniew Hajto, Valladolid

In the talk we give an introduction to the theory of subpfaffian sets. The motivation for developing that kind of transcendental analogy of subanalytic geometry was the following question posed by R. Moussu and M. Shiota (Trento, September 1992): "What can we get adding to subanalytic sets the solutions of Pfaffian equations?"

Briefly, the theory is composed of three parts.

A) Semipfaffian geometry, where basic objects are defined. The building blocks of our geometry are the intersections of leaves of Pfaffian foliations with the strata of normal decompositions of S. Łojasiewicz. This part of the theory was suggested in several papers of R. Moussu, C. Roche and J.-M. Lion.

B) Subpfaffian geometry, which includes the analog of Gabrielov theorem about the complement of subanalytic set. The main obstruction in building a self-contained theory of semipfaffian sets is the lack of the theorem about the closure of a semipfaffian set. However using the theorem about the complement of a subpfaffian set it is possible to prove that the closure of a subpfaffian set is subpfaffian. The proof of the Gabrielov theorem is possible if we know that the boundary of a subpfaffian set E is contained in a closed subpfaffian set of dimension smaller than the dimension of E. This property we call subregularity of E.

C) The most technical part of the theory where the proof that any basic semipfaffian set is subregular is presented.

The theorem about the complement and its consequence theorem about the closure together with subpfaffian version of finiteness theorem are the fundamentals of the basic subpfaffian geometry. They form a good starting point for further study of the solutions of systems of Pfaffian equations, e.g.

$$dx_1 + \frac{\partial f}{\partial x_1} dt = 0$$

$$\vdots$$

$$dx_n + \frac{\partial f}{\partial x_n} dt = 0$$

in $\mathbb{R}^{n+1}$, for f analytic in an open set in $\mathbb{R}^n$. In this context, it seems to be valuable to reprove the triangulation theorem from subanalytic geometry and to establish a kind of equisingularity theory for subpfaffian sets. Finally an interesting connection with the work of Lou van den Dries should be mentioned as well [1].

[1] L. van den Dries, Tarski's problem and Pfaffian functions, Logic Colloquium 84, Willie and Wilmers, ed., North Holland (1986), 59-90.

Realizability of cycles on real abelian varieties

Johan Huisman, Utrecht

If X is a real abelian variety (that is, X is a complete, geometrically integral, separated group scheme of finite type over $\mathbb{R}$) of dimension n then it is well known that the set of real points $X(\mathbb{R})$ of X is topologically the disjoint union of 2^i copies of $(S^1)^n$, where $0 \leq i \leq n$. In particular,

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_{n-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 2^i \cdot n$$

In this talk it will be shown that one can compute the subgroup $H^{\mathrm{alg}}_{n-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ of $H_{n-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$ of the codimension-1 cycles that are realizable by a real algebraic subvariety. As a consequence we will prove the following.

Theorem: If X is a real abelian variety of dimension $n > 1$, then the set of real points $X(\mathbb{R})$ is connected whenever $H^{\mathrm{alg}}_{n-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = 0$, or $H^{\mathrm{alg}}_{n-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = H_{n-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$.

Projective modules over fractional real polynomial rings

F. Ischebeck, Münster (joined work with M. Ojanguren, Lausanne)

Let $A := \mathbb{R}[X, Y, Z]$, $\Sigma := \{f \in A | f(x) \neq 0 \, \forall x \in \mathbb{R}^3\}$.
Previous results: Every projective $\Sigma^{-1}A$-module is free. (O. - I.). For $f \in \Sigma$ every projective A_f-module is free. (O. - Parimala). For $f = X^2 + Y^2 + Z^2$ there is a nonfree projective A_f-module. (Murthy or Eisenbud). If the real zero-set $Z_{\mathbb{R}}(f)$ has a compact connected component, there is a nonfree projective A_f-module. (O. - Röver). We prove

Theorem: If $Z_{\mathbb{R}}(f)$ is smooth and has no compact connected component, then every projective $\Sigma^{-1}A_f$-module is free. (We would like to replace $\Sigma^{-1}A_f$ by A_f!).

Using a global version of Gabber's theorem on "projective modules over localized local rings" we are reduced to show, that every projective $\Sigma^{-1}A/(f)$-module splits into rank-1-ones. To this aim we show the following result in differential topology:

Proposition: Let F be a closed C^∞-surface, $P \in F, \tilde{F}$ the blow up of F in P, $Q \in$ exceptional fibre of this blow up, ξ a vectorbundle on F. Then there is a C^∞-section in the pull back of ξ over $\tilde{F}$, which is transversal to the zero section and has its only (if any) zero in Q.

Counter-examples to the Ragsdale conjecture

Ilia Itenberg, DMI, Ecole Normale Supérieure, Paris, France

In 1906 V. Ragsdale proposed the following conjecture: there are two inequalities for a non-singular real algebraic plane projective curve of degree $2k$:

$$p \leq \frac{3k^2-3k+2}{2}$$
$$n \leq \frac{3k^2-3k}{2},$$

where p (n, respectively) is the number of ovals of real point set of given curve lying inside of even (odd, respectively) number of other ovals (such ovals are called even (odd, respectively)).

In 1980 O. Viro constructed curves of degree $2k$ (where k is even and $k \geq 4$) with

$$n = \frac{3k^2-3k}{2} + 1$$

and suggested to change the second inequality in Ragsdale conjecture to inequality

$$n \leq \frac{3k^2-3k+2}{2}$$

(in this form the statement of conjecture was formulated in 1938 by I. Petrovsky).

The following theorem gives counter-examples to the "corrected" Ragsdale conjecture.

Theorem For each integer positive $k > 1$

a) there exists a non-singular real algebraic plane projective curve of degree $2k$ with

$$p = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right]$$

b) there exists a non-singular real algebraic plane projective curve of degree $2k$ with

$$n = \frac{3k^2-3k+2}{2} + \left[\frac{(k-3)^2+4}{8}\right] - 1$$

([a] denotes the maximal integer number no greater than a).

These counter-examples are obtained using Viro's method of construction of real algebraic varieties with prescribed topology.

Hilbert 17^{th} problem and Riemannian vector bundles

Piotr Jaworski - Warszawa

The aim of my talk was to show that the problem of representing the positive definite global real analytic function f may be reduced to the local problem in the neighbourhood of the zero set of f. Namely:

Assume that there exists a positive integer p and a family of components $V_1, V_2, \dots$ of the zero set of f, such that:

a) $f^{-1}(0) = \bigcup V_i$, $f^{-1}(0) \neq \bigcup_{j \neq i} V_j$;

b) to each component V_i there is associated a positive definite analytic function f_i such that:

   i) $V_i = f_i^{-1}(0)$,

   ii) $f/f_1 \cdots f_i$ is analytic and its zero set is contained in $\bigcup_{j>i} V_j$.

   iii) f_i is a sum of squares of p meromorphic functions defined in some neighbourhood of V_i, analytic outside of V_i.

Then f may be represented as a sum of $k \cdot 2^d$ squares of global meromorphic functions, where $d = \min\{\delta : 2^\delta \geq p + [\frac{p}{15}]\}$, $k = \min\{\gamma : \gamma \cdot 2^d \geq n + 1\}$.

The crucial step in the proof is the construction of the Riemannian vector bundle, whose transition functions are Pfister matrices of functions f_i.

Towards the maximal number of components of a surface of degree 5 in $\mathbb{R}P^3$

V. Kharlamov, Strasbourg

The problem of determining the maximal number of components of a surface of given degree m in $\mathbb{R}P^3$ was proved by D. Hilbert yet in 1900. The response is still not known, except m the trivial cases $m \leq 3$ and the case $m = 4$. In this last case the maximal number of components is equal to 10.

To determine the maximal number of components it is enough to consider non-singular surfaces: by a small variation, any singular surface can be replaced by a non-singular one having at least the same number of components.

A standard application of the Smith and Comesatti inequalities gives an estimate: the number of connected components of a non-singular surface of degree m in $\mathbb{R}P^3$ is less or equal to $\frac{5m^3 - 18m^2 + 25m}{12}$. In particular, it can not be more than 25 for $m = 5$.

In 1981 I constructed a surface of degree 5 in $\mathbb{R}P^3$ with 21 components. The proof was based on an appropriate equivariant version of Horikawa's theorem on the moduli space of numeric quintics.

Recently, Itenberg and I have constructed a real quintic with 22 components. It is homeomorphic to a non-connected sum of 21 spheres and a 1 sphere with 7 Möbius bands.

This example may serve, in particular, as a counter-example to one Arnold's conjecture concerning the maximal number of components of a surface of given degree. Arnold's conjecture bounds by 21 the number of components in the case of quintics.

According to one Viro's conjecture, this number should be less or equal to 23. This conjecture still remains open.

The homotopy groups of some spaces of real algebraic morphisms

Wojciech Kucharz, Albuquerque

(joint work with J. Bochnak, Amsterdam)

Let X and Y be affine real algebraic varieties. Denote by $\mathcal{R}(X,Y)$ the set of regular maps (that is, real algebraic morphisms) of X in Y. We consider $\mathcal{R}(X,Y)$ as a subspace of the space $\mathcal{C}(X,Y)$ of continuous maps of X in Y endowed with the compact open topology.

Let $\mathbb{F}$ denote one of the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions) and let $G_{n,p}(\mathbb{F})$ denote the Grassmann variety of p-dimensional $\mathbb{F}$-vector subspaces of $\mathbb{F}^n$. Here $G_{n,p}(\mathbb{F})$ is regarded as a real algebraic variety.

Theorem: Let X be a compact affine real algebraic variety. Let $i: \mathcal{R}(X, G_{n,p}(\mathbb{F})) \hookrightarrow \mathcal{C}(X, G_{n,p}(\mathbb{F}))$ be the inclusion map. Then for each f in $\mathcal{R}(X, G_{n,p}(\mathbb{F}))$, the induced map

$$i_*: \pi_k(\mathcal{R}(X, G_{n,p}(\mathbb{F})), f) \longrightarrow \pi_k(\mathcal{C}(X, G_{n,p}(\mathbb{F})), f)$$

is injective for $k = 0$ and a group isomorphism for $k \geq 1$.

Several examples in which i is a weak homotopy equivalence were also given.

W_f condition and Łojasiewicz inequlity

K. Kurdyka, Kraków & Chambery

(joint work with A. Parusinski, Sydney)

Let $f: X \to \mathbb{R}^h$ be a function, $X \subseteq \mathbb{R}^n$. Let $\mathcal{S}$ be a stratification of X. Suppose that $f|_S$ has constant rank for each $S \in \mathcal{S}$. We say that $\mathcal{S}$ is a W_f stratification of f iff for every $S, S' \in \mathcal{S}$, $S \subset \bar{S}'\backslash S'$, for every $x_0 \in S$ we have

$$\delta(T_x f^{-1} f(x) \cap S, T_{x'} f^{-1} f(x') \cap S') \leq C|x - x'|$$

for $x \in S$, $x' \in S'$ close to x_0, where C depends on x_0. δ is sine of the angle between corresponding tangent spaces.

We prove the following

Theorem: Let $f: X \to \mathbb{R}$ be a subanalytic (i.e. graph of f is subanalytic subset of $\mathbb{R}^n \times \mathbb{R}$), continuous function. Suppose also that f is locally bounded at each $x \in \overline{X}$). Then there exists subanalytic W_f stratification $\mathcal{S}$ of function f.

Moreover each such stratification leads to the following Lojasiewicz type inequality:

Proposition. If $S \in \mathcal{S}, x_0 \in S \subset f^{-1}(0)$, then there exists a constant $C > 0$ such that

$$|f(x)| \leq C \operatorname{dist}(x, S) \|\operatorname{grad} f(x)\|$$

for each $x \in X$ close to x_0.

Separation of clopen sets by étale cohomology

L. Mahé, Rennes

Let A be a ring, denote by $H^n(A)$ the group $H^n_{\mathrm{et}}(A, \mu_2)$. Colliot-Thélène and Parimala have defined a "signature map" $\Lambda: H^n(A) \to \operatorname{Cont}(\operatorname{Spec}_r A, \mathbb{Z}/2)$ which can be extended to $H^*(A) := \varinjlim H^n(A) \xrightarrow{\Lambda'} \operatorname{Cont}(\operatorname{Spec}_r A, \mathbb{Z}/2)$ (the limit being taken under the map $H^n(A) \xrightarrow{(-1)\vee} H^{n+1}(A)$). We give a proof that this Λ' is surjective. This has been already proved by Colliot-Thélène and Parimala in the case of a smooth variety over the real numbers, and in the general case by Scheiderer and Burési (independently). The proof presented here is a copy of the proof of the analog of the theorem for quadratic forms (The author):

1) Formal Mostowski's separation theorem (separation of clopen sets by "Nash functions").
2) Use of a "localization Theorem" (Burési) showing that $H^n(B) \to H^n(B_{1+s})$ is an isomorphism modulo the (-1)-torsion where s is a sum of squares.
3) Use of a trace formula to get control on the maps of type $H^n(B) \to H^n(B[X]/X^4 - d)$ where d is a unit totally positive on B.

Model theory and exponentiation

David Marker, University of Illinois, Chicago, USA

We survey recent work on O-minimal structures and exponentiation. O-minimality implies structural properties of definable sets analogous to those of semialgebraic sets. In particular we show that $\mathbb{R}$ with exponentiation and restricted analytic functions have quantifier elimination (once log is added). This is used to prove O-minimality. This is joint work with L. van den Dries and A. Macintyre.

Separating families and Bröcker's t-invariant

Murray Marshall, Univ. of Saskatchewan

Let $V \subseteq R^N$ be an algebraic set, R real closed, and let $S \subseteq V$ be a fixed semi-algebraic set. For $\underline{f} = (f_1, \ldots, f_p) \in R[t_1, \ldots, t_N]^p$ and $\underline{e} = (e_1, \ldots, e_p) \in \{-1, 0, 1\}^p$, let

$$U(\underline{f}, \underline{e}) := \{x \in V \mid \operatorname{sgn} f_i(x) = e_i \ , \ i = 1, \ldots, p\}.$$

Say $f_1, \ldots, f_p$ is a *separating family* for S if $S = \bigcup_{\underline{e} \in \Delta} U(\underline{f}, \underline{e})$ for some subset $\Delta \subseteq \{-1, 0, 1\}^p$. $p(S) =$ The least integer p such that S has a separating family $f_1, \ldots, f_p$. $t(S) =$ The least integer t such that S is the union of t basic sets. Fix a separating family $f_1, \ldots, f_p$ as above and let $s_i =$ the maximum of the stability indices of the basic open and closed sets $U_W(\underline{f}, \underline{e}) \subseteq \operatorname{Sper} R(W)$, $\underline{e} \in \Delta$, W running through the irreducible algebraic sets $W \subseteq V$, $i = 0, \ldots, d$, $d = \dim V$.

Theorem 1. $p(S) \leq \sum_{i=0}^{d} p(s_i)$, $t(S) \leq \sum_{i=0}^{d} \tau(s_i)$, where $p(0) = 1$, $p(s) = 4^{s-1} - 2^{s-1} + 1$ if $s \geq 1$, $\tau(0) = \tau(1) = 1$, $\tau(2) = 2$ and $\tau(s) = p(s)!/(\frac{p(s)+1}{2})!(\frac{p(s)-1}{2})!$ if $s \geq 3$.

Since $s_i \leq i$ by a result of Bröcker, this gives a global upper bound for $p(S)$, $t(S)$ depending only in d. Concerning lower bounds we have

Theorem 2. If $V = R^d$, there exists a semi-algebraic $S \subseteq V$ with $p(S) \geq \log_2(\beta^{d-1}(2)) + d$ where $\beta \colon \mathbb{N} \to \mathbb{N}$ is determined $\beta(n) = n(n+1)$.

Arrangements of topological planes: criteria of linearity

N. Mnev, University of Bern, Steklov Institute

In the terminology of oriented matroids, the following theorem is discussed:

All the obstructions for linear representability of an oriented matroid M (rank ≥ 4) live in homotopy groups of the space of one-element extentions of M.

Harnack's theorem for space curves

D. Pecker - Paris 6

Harnack's theorem for space curves is the following:

Theorem: If $k + c \leq C(d,n)$ (the classical Castelnuovo bound) then there exists a non degenerate irreducible algebraic curve of degree d in $P_n(\mathbb{R})$ with k singular isolated real points, $c+1$ smooth connected components homeomorphic to circles.

This generalizes classical results by Harnack ($n = 2, k = 0$) and Hilbert ($n = 3, k = 0$) and also more recent results on singular curves by Shustin ($n = 2$) and Tannenbaum (n arbitrary, but *complex* case).

The proof is simple, we first construct rational curves with many real isolated double points (like for example the curve $x = t^a$, $y = (t-1)^b$ $(a,b) = 1$). Then we simplify the double points of these curves using only elementary algebra.

The asymptotic values of a polynomial function on the real plane

M.J. de la Puente, Universided Complutense de Madrid

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a polynomial function and let $c(\lambda)$ denote the number of connected components of the level curve $V(f-\lambda)$, $\lambda \in \mathbb{R}$. It is well-known that if λ is a critical value for f then c may be discontinuous at λ. In this paper we define and study the other values λ, so called *restricted asymptotic values of f*, at which c may have discontinuities. They are related to unbounded subsets of $\mathbb{R}^2$ along which f tends to λ and the tangent plane to the graph of f tends to horizontal position.

First of all, we study certain properties satisfied by unbounded semialgebraic subsets of $\mathbb{R}^2$ over which f has a certain behaviour (is bounded, has a limit, etc.). Secondly, we define asymptotic classes, asymptotic values, restricted asymptotic classes and restricted asymptotic values of f and prove the finiteness of all of these families. Then the main result can be proved: for a given λ, which is not a restricted asymptotic value, we prove the existence of a compact subset B of $\mathbb{R}^2$ such that $c(\mu)$ equals the number of connected components of $V(f-\mu) \cap B$, where μ belongs to a neighbourhood of λ. Finally, we give some necessary conditions for λ to be an asymptotic value.

An invitation to computational real geometry through applications

Tomas Recio, Universided de Cantabria

We present some applications of computational real geometry in three fields: complexity, computational geometry and robotics. The work presented here is jointly with several people of the University of Cantabria. We comment briefly some points.

It is shown that in the Lie group $SE(3)$ of euclidean placements, the "natural" left invariant metric is not computable under the Blum-Shub-Smale model, when elements of $SE(3)$ are considered in $\mathbb{R}^{12}$. Even more, the topological shape of Voronoi diagrams under such distance, is not computable by the same model.

On a different topic, it is shown, that a generalized inverse kinematics problem in robotics (namely "tracking" a given semialgebraic path of a robot tip to the whole configuration space) is solved by means of algorithms for the Elimination of quantifiers.

Finally, let us mention that an algorithm is presented to parametrize a semialgebraic set in $\mathbb{R}^n$, with domain in $\mathbb{R}^d$, where d = dimension of S, and moreover keeping in the domain good properties of S from the connectivity point of view.

Smoothings of plane singularities

J.-J. Risler, DMI, ENS, Paris

Let $f = 0$ be a germ at $0 \in \mathbb{R}^2$ of a plane curve, f_ε a deformation by f such that $f_\varepsilon = 0$ be smooth. What can be the topological type of $\{f_\varepsilon = 0\}$?

If $\{f = 0\}$ has r real branches, $\{f_\varepsilon = 0\}$ has r "non-closed" components and p ovals.

Theorem I $p \leq \frac{\mu - r + 1}{2}$ ovals, where $\mu :=$ Milnor number.

Theorem II This bound is sharp when $n = 1$ and f (locally) irreducible.

Theorem III When the branches have distinct tangents, there exists a, with $0 \leq a \leq r-1$ and a smoothing with $\frac{\mu - r + 1}{2} - a$ ovals.

Theorem IV In the general case there exists a smoothing with $\frac{\mu - r + 1}{2} - b$ ovals, for some number b, $0 \leq b \leq 5(r-1)$.

The method is by blowing-ups.

Purity for real spectra; applications

C. Scheiderer, Regensburg

Let X be an excellent regular scheme and $Z \subset X$ a closed regular subscheme of constant codimension c. Let F be a locally constant sheaf on $X_r :=$ real spectrum of X. Let $\underline{H}^q_{Z_r}(X_r, F)$ be the sheaf on X_r associated to the presheaf $U \mapsto H^q_{U \cap Z_r}(U, F)$ ($U \subset X_r$ open.)

Theorem (Purity). $\underline{H}^q_{Z_r}(X_r, F) = 0$ for $q \neq c$, and $\underline{H}^c_{Z_r}(X_r, F)$ is Zariski locally isomorphic to $F|_{Z_r}$.

The essential part of the proof is concerned with the case $c = 1$, $X = \mathrm{spec} A$ with A a strictly real local ring. First one does the complete case using Čech cohomology and the

Weierstraß preparation theorem. The general case is inductively reduced to the complete case, applying resolution of singularities and Ruiz's theorem on real spectrum formal fibres.

In the second part some applications were sketched. For any (noetherian) scheme X as above one constructs a complex

$$0 \to \bigoplus_{x\in X^{(0)}} H^0(\operatorname{sper}\kappa(x),\mathbb{Z}/2) \to \bigoplus_{x\in X^{(1)}} H^0(\operatorname{sper}\kappa(x),\mathbb{Z}/2) \to \bigoplus_{x\in X^{(2)}} \cdots$$

whose cohomology is canonically identified with $H^*(X_r,\mathbb{Z}/2)$. This complex is new even for algebraic varieties $/\mathbb{R}$, and is quite useful. From it cycle maps $\mathrm{cl}^q\colon CH^q(X) \to H^q(X_r,\mathbb{Z}/2)$ are constructed which in the classical cases ($\mathbb{R}$-varieties, compact real analytic manifolds) coincide with the Borel-Haefliger maps. This construction permits to give a new proof of Bröcker's theorem, saying that on a smooth complete R-variety X every divisor D with $\mathrm{cl}^1(D) = 0 \in H^1(X(R),\mathbb{Z}/2)$ is rationally equivalent to a divisor D' with $|D'|(R) = \emptyset$. Here R is an archimedean real closed field. The proof is purely algebraic and uses no approximation techniques. Instead fan theory is applied, which again uses crucially the fact that R is archimedean.

Separating ideals

Niels Schwartz, Passau

In his investigation of the Pierce-Birkhoff Conjecture J. Madden introduced the notion of the separating ideal $\langle\alpha,\beta\rangle$ of $\alpha,\beta \in \operatorname{Sper}(A)$ (real spectrum of the ring A). In a certain sense $\langle\alpha,\beta\rangle$ measures the distance between α and β. Similarly it is possible to associate a separating ideal $\langle v,w\rangle$ with valuations v and w of A. To every $\alpha \in \operatorname{Sper}(A)$ there belongs a valuation v_α in a canonical way. It is trivially true that always $\langle v_\alpha,v_\beta\rangle \subseteq \langle\alpha,\beta\rangle$. The geometric meaning of separating ideals and of the connections between $\langle v_\alpha,v_\beta\rangle$ and $\langle\alpha,\beta\rangle$ can be made visible very clearly in the real plane. The separating ideals can be nontrivial only if α and β (or v and w) have a common center. If this common center is the origin then examples suggest that, informally speaking, the separating ideals are small if the degree of tangency of the real sepctrum points is high. This can be made precise by introducing tangent spaces for valuations and, additionally, tangent directions in the tangent spaces for points of the real spectrum. (The new results reported in the lecture were obtained in collaboration with J. Madden.)

Riemann surfaces, algebraic curves and their period matrices

Mika Seppälä, Helsinki

Recall that the following categories are equivalent:

(a) compact genus g Riemann surfaces;

(b) smooth complex algebraic curves of genus g;

(c) period matrices ($g \times g$).

It is our intention to develop computer programs that can be used to study the above equivalence of categories.

For real algebraic curves (with real points) the above problem can be attacked directly:

(1) Real algebraic curves with real points correspond to Klein surfaces with a non-empty boundary.

(2) Such Klein surfaces X can be presented in turns of a Fuchsian groups G of the 2^{nd} kind acting on the unit disk D. For an orientable X we may choose G in such a way that $D/G = \operatorname{Int} X$ (interier of X). Non-orientable Klein surfaces X can be treated by first passing to their orientable double coverings.

(3) G being of the 2^{nd} kind, the space of holomorphic 1-forms on X has a basis that can be expressed in turns of certain Poincaré series of weight 1. This allows one to compute (an approximation of) the period matrix of X.

The algorithm sketched above has been coded and implemented in Helsinki.

On a result of B. Segre for real cubic surfaces

R. Silhol, Montpellier

A smooth real cubic surface in $\mathbb{P}^3(\mathbb{R})$ can have 27, 15, 7 or 3 real lines. This is well known since the 19^{th} century. The result of Segre is more subtle. Let D be a line in the cubic surface X, and let P be a plane containing D then $P \cap X = D \cup C$ where C is an, eventually degenerated, conic. It can easily be shown that P is tangent to X at the 2 points of intersection of D and C. Conversely if $x \in D$ then the tangent plane to X at x contains D. This defines an involution σ on D. But since a real line is holomorphic to S^1, it can have 0 or 2 fixed points. Segre calls the line elliptic in the first case, hyperbolic in the second, and he proves.

Theorem (Segre): If X is a smooth real cubic with n lines then,

— 12 are elliptic and 15 hyperbolic if $n = 27$

— 6 are elliptic and 9 hyperbolic if $n = 15$

— 2 are elliptic and 5 hyperbolic if $n = 7$

— 0 are elliptic and 3 hyperbolic if $n = 3$

In modern terms this result can be explained as follows. Fixing a spin structure on $\mathbb{P}^3$ allows to select a class of even framings of the normal bundle to circles embedded in $\mathbb{P}^3$. Even and odd framings differ by a full twist.

Let X be a surface embedded (or immersed) in $\mathbb{P}^3$ and let C be an embedded circle on X. Then one can define $\hat{q}(C)$ to be the number mod. 4 of left half turns the normal bundle to X restricted to C does when moving along C with respect to an even framing. This defines a map $q\colon H_1(X,\mathbb{Z}/2) \to \mathbb{Z}/4$ such that $q(\alpha+\beta) = q(\alpha)+q(\beta)+2(\alpha,\beta)$, where $(\ \cdot\ )$ is the intersection form. For embedded surfaces in $\mathbb{P}^3$ the isometry class of such a form is uniquely determined. Returning to the result of Segre it is easy to show that a line is elliptic (resp. hyperbolic) iff $q(D) = 1$ (resp. $q(D) = -1$). From this the theorem of Segre reduces to an easy computation.

Counting real zeros with multiplicity

Gilbert Stengle, Lehigh University, USA

Let R be a real closed field, $f(x)$ an element of $R[x]$. To count the real zeros of f we can either manipulate f itself or dually use the algebra $A_f = R[x]/(f)$ as the basic datum. In the former case the classic method of Sturm gives an algorithmic procedure for counting the number of real zeros. In the latter case the method of Sylvester associates to A_f a quadratic form with signature equal to the number of real zeros. Each of these counts $\nu(f)$, the number of topological zeros, that is, zeros are counted without multiplicity. We give a formula for $\mu(f)$, the number of real zeros counted with multiplicity. This formula is semialgebraic in character, depending on numerical attributes of A_f as embodied in its lattice of preorderings. (A preordering is a subsemiring containing the squares. We also count A_f as an improper ordering.) Our main result is the following.

Theorem: Let Λ_f be the lattice of preorderings of A_f. Let $\lambda(f)$ be the length of Λ_f. Then $\mu(f)$, the number of real zeros of f counted with multiplicity, is given by

$$\mu(f) = \frac{\lambda(f^3) - \lambda(f)}{3}.$$

This is a consequence of the more detailed formula:

Proposition: Let $\nu_{\mathrm{odd}}(f)$ be the number of real zeros of f of odd degree counted without multiplicity. Then

$$\lambda(f) = \frac{3}{2}\mu(f) - \frac{1}{2}\nu_{\mathrm{odd}}(f).$$

This in turns follows from the following facts.

1. $\lambda(x^m) = \left[\frac{3m}{2}\right]$
2. $\lambda((x^2+a^2)^m) = 0 \quad$ if $a \neq 0$
3. $\Lambda(fg) = \Lambda(f) \oplus \Lambda(g) \quad$ if $(f,g) = 1$
4. λ is additive on direct sums.

These imply that $\lambda(f)$ is a sum $\sum \lambda(x^{m_i})$. The theorem follows from the proposition, since $\nu_{\text{odd}}(f^{3m}) = \nu_{\text{odd}}(f^m)$ so that

$$\lambda(f^3) - \lambda(f) = \sum \left[\frac{9}{2}m_i\right] - \left[\frac{3}{2}m_i\right] = 3\sum m_i = 3\mu(f).$$

A real version of the Greuel-Lê formula

Zbigniew Szafraniec, Gdańsk

Let $F = (f_1, \ldots, f_k)\colon \mathbb{R}^n \to \mathbb{R}^k$, $k < n$, be a polynomial mapping. Let $W = F^{-1}(0)$, $W_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$, let $I \subset \mathbb{R}[x_1, \ldots, x_n]$ be the ideal generated by $f_1, \ldots, f_k$ and all minors $\frac{\partial(\|\cdot\|^2, f_1, \ldots, f_k)}{\partial(x_{i_1}, \ldots, x_{i_{k+1}})}$, where $\|\cdot\|$ is the norm in $\mathbb{R}^n$. Let $Q(F) = \mathbb{R}[x_1, \ldots, x_n]/I$.

Theorem. Assume that $\dim_{\mathbb{R}} Q(F) < \infty$ and that rank $DF_{\mathbb{C}}(X) = k$ for every $x \in W_{\mathbb{C}}$. Then there is an explicitley defined linear functional $\varphi\colon Q(F) \to \mathbb{R}$ such that the quadratic form Φ on $Q(F)$ defined by $\Phi(g) = \varphi(g^2)$ is non-singular and $\chi(W) =$ signature Φ, where $\chi(W)$ is the Euler characteristic of W.

Approximation theorems for C^∞ solutions of systems

A. Tognoli, Trento, Italy

Let U be an open set of $\mathbb{R}^n$, by $C^\infty(U)$, $C^\omega(U)$ we shall denote the ring of C^∞ and analytic real function on U. In the following we shall consider the Whitney strong topology on $C^\infty(U)$. Let (1) $\sum_{j=1}^{n} \alpha_{ij}(x)t_j = \rho_i(x)$, $i = 1, \ldots, q$, $\alpha_{ij} \in C^\omega(U)$ $\rho_i \in C^\omega(U)$ be a linear system. Then we have the following

Theorem 1: Any C^∞ solution $(f_1 \ldots f_n)$ of (1) can be approximated by analytic solution $(g_1 \ldots g_n)$. Moreover if $f_1, \ldots, f_p \in C^\omega(U)$ $p \le n$, then we may assume $g_i = f_i$, $i = 1, \ldots, p$. Let $X \subseteq U$ be a coherent analytic set, if $f_i|_X \in C^\omega(X)$, then we may assume $g_i|_X = f_i|_X, i = 1, \ldots, n$.

Theorem 2: Let V be a paracompact real analytic manifold, then every closed (exact) differential form can be approximated by analytic closed (exact) differential forms, that are in the same cohomological class.

Real algebraic geometry of control problems

Y. Yomdin, Rehovot

I presented some finiteness results for the "reachable set" of a polynomial control problem $(*)$ $\dot{x} = f(x, u)$, $x, u \in \mathbb{R}^2$. Certain semialgebraic tricks, bounding the behaviour of the trajectories of $(*)$ have been presented. In particular, the following conjecture has been discussed: the rotation of an algebraic vector field around any algebraic submanifold (in a given time) is "algebraically" bounded.

Approximation by abstract Nash functions

Digen Zhang, Regensburg

Let A be a commutative ring with 1. There are two important sheaves on the real spectrum Sper A of A. If $A = \mathbb{R}[X_1, \cdots, X_n]$, then $\mathcal{C}_A$ is the sheaf of semialgebraic functions on $\mathbb{R}^n$ and $\mathcal{N}_A$ is the sheaf of Nash functions on $\mathbb{R}^n$. It is well known, that a semialgebraic function on $\mathbb{R}^n$ is piecewise Nash functions, and is approximated by Nash functions on $\mathbb{R}^n$. In fact it holds also for following abstract functions.

Let U be a constructible open subset of Sper A.

Definition 1: An element $f \in \mathcal{C}_A(U)$ ($\mathcal{N}_A(U)$, respectively) is called an ***abstract semialgebraic (Nash,*** respectively) function on U.

Proposition 2: Let f be an Nash function on U. Then $f(.) \colon U \longrightarrow \coprod_{\alpha \in U} k(\alpha)$, $\alpha \longmapsto f(\alpha)$, where $f(\alpha)$ is the image of f_α in the residue field $\mathcal{N}_{A,\alpha}/m_{A,\alpha} = k(\alpha)$, is an abstract semialgebraic function on U.

Proposition 3: Let X be a (pro)constructible subset of Sper A and $f \in \mathcal{C}_A(X)$. Then there are a covering $\{X_i\}_{i=1}^n$ of X by (pro)constructible subsets, open subsets U_i of Sper A and Nash functions f_i on U_i for $i = 1, \cdots, n$ such that $X_i \subseteq U_i$ and $f(\alpha) = f_i(\alpha)$ for any $\alpha \in X_i$.

The main result of my talk is the following:

Theorem 3: Let f, ε be abstract semialgebraic functions on U with $\varepsilon > 0$ on U (i.e. $\varepsilon(\alpha) > 0, \forall \in U$). Then there is an abstract Nash function g on U such that $|f - g| < \varepsilon$ (i.e. $|f(\alpha) - g(\alpha)| < \varepsilon(\alpha)$, $\forall \alpha \in U$).

To prove the theorem, we use the following abstract Mostowski separation theorem.

Theorem 4: Let F, G be disjoint closed subsets of Sper A. Then there is an abstract Nash function on Sper A in the following form,

$$f = \sum_{i=1}^{m} a_i \sqrt{1 + b_{i1}^2 + \cdots + b_{il_i}^2},$$

where $a_i, b_{ij} \in A$, such that $f > 0$ on F and $f < 0$ on G.

Berichterstatter: Digen Zhang

Tagungsteilnehmer

Prof.Dr. Francesca Acquistapace
Dipartimento di Matematica
Università di Pisa
Via Buonarrotti, 2

I-56127 Pisa

Prof.Dr. Ludwig Bröcker
Mathematisches Institut
Universität Münster
Einsteinstr. 62

D-48149 Münster

Prof.Dr. Carlos Andradas
Facultad de Matematicas
Depto. de Algebra
Universidad Complutense de Madrid

E-28040 Madrid

Prof.Dr. Fabrizio Broglia
Dipartimento di Matematica
Università di Pisa
Via Buonarrotti, 2

I-56127 Pisa

Prof.Dr. Eberhard Becker
Fachbereich Mathematik
Universität Dortmund

D-44221 Dortmund

Prof.Dr. Michael A. Buchner
Dept. of Mathematics and Statistics
University of New Mexico

Albuquerque , NM 87131
USA

Dr. Ralph Berr
Fachbereich Mathematik
Universität Dortmund

D-44221 Dortmund

Dr. Jérôme Burési
U. E. R. Mathématiques
I. R. M. A. R.
Université de Rennes I
Campus de Beaulieu

F-35042 Rennes Cedex

Prof.Dr. Jacek Bochnak
Mathematisch Instituut
Vrije Universiteit
De Boelelaan 1081

NL-1081 HV Amsterdam

Prof.Dr. Jean Louis Colliot-Thelene
Mathematiques
Université de Paris Sud (Paris XI)
Centre d'Orsay, Bâtiment 425

F-91405 Orsay Cedex

Prof.Dr. Charles N. Delzell
Dept. of Mathematics
Louisiana State University

Baton Rouge , LA 70803-4918
USA

Prof.Dr. Margherita Galbiati
Dipartimento di Matematica
Universita di Pisa
Via Buonarrotti, 2

I-56127 Pisa

Prof.Dr. Max A. Dickmann
U. F. R. de Mathématiques
Case 7012
Universite de Paris VII
2, Place Jussieu

F-75251 Paris Cedex 05

Prof.Dr. Zbigniew Hajto
Depto. de Algebra, Geometria y
Topologia
Universidad

E-47005 Valladolid

Prof. Dr. Thomas Fiedler
Mathématiques
Universite Paul Sabatier
118, route de Narbonne

F-31062 Toulouse Cedex

Prof.Dr. Johan Huisman
Mathematisch Instituut
Rijksuniversiteit te Utrecht
P. O. Box 80.010

NL-3508 TA Utrecht

Prof.Dr. Elisabetta Fortuna
Dipartimento di Matematica
Universita di Pisa
Via Buonarrotti, 2

I-56127 Pisa

Prof.Dr. Friedrich Ischebeck
Mathematisches Institut
Universität Münster
Einsteinstr. 62

D-48149 Münster

Prof.Dr. Jean-Pierre Françoise
U.F.R. 920
Université de Paris VI, Mathem.
Couloir 46-56, 5eme etage
4, place Jussieu

F-75252 Paris

Prof.Dr. Ilia Itenberg
Departement de Mathematiques et
Informatiques
Ecole Normale Superieure
45, rue d'Ulm

F-75005 Paris Cedex 05

Prof.Dr. Piotr Jaworski
Institute of Mathematics
University of Warsaw
ul. Banacha 2

02-097 Warszawa
POLAND

Prof.Dr. Louis Mahe
U. E. R. Mathematiques
I. R. M. A. R.
Universite de Rennes I
Campus de Beaulieu

F-35042 Rennes Cedex

Prof.Dr. Vlatcheslav Kharlamov
Institut de Mathematiques
Université Louis Pasteur
7, rue René Descartes

F-67084 Strasbourg Cedex

Prof.Dr. David Marker
Dept. of Mathematics
University of Illinois at Chicago
Box 4348

Chicago , IL 60680
USA

Prof.Dr. Manfred Knebusch
Fakultät für Mathematik
Universität Regensburg

D-93040 Regensburg

Prof.Dr. Murray Marshall
Dept. of Mathematics
University of Saskatchewan

Saskatoon Sask.. S7N 0W0
CANADA

Prof.Dr. Wojciech Kucharz
Dept. of Mathematics and Statistics
University of New Mexico

Albuquerque , NM 87131
USA

Prof.Dr. Nikolai E. Mnev
Mathematisches Institut
Universität Bern
Sidlerstr. 5

CH-3012 Bern

Prof.Dr. Krzysztof Kurdyka
Instytut Matematyki
Uniwersytet Jagiellonski
ul. Reymonta 4

30059 Krakow
POLAND

Prof.Dr. Daniel Pecker
33, rue Julien-Lacroix

F-75020 Paris

Prof.Dr. Alexander Prestel
Fakultät für Mathematik
Universität Konstanz
D 197
Postfach 5560

D-78434 Konstanz

Prof.Dr. Maria Jesus de la Puente
Facultad de Matematicas
Depto. de Algebra
Universidad Complutense de Madrid

E-28040 Madrid

Prof.Dr. Tomas Recio
Departamento de Matematicas,
Estadistica y Computacion
Universidad de Cantabria

E-39071 Santander

Prof.Dr. Jean-Jacques Risler
Departement de Mathematiques et
Informatiques
Ecole Normale Superieure
45, rue d'Ulm

F-75005 Paris Cedex 05

Dr. Claus Scheiderer
Fakultät für Mathematik
Universität Regensburg

D-93040 Regensburg

Prof.Dr. Niels Schwartz
Fakultät für Mathematik
und Informatik
Universität Passau

D-94030 Passau

Prof.Dr. Mika Seppälä
Dept. of Mathematics
University of Helsinki
P.O. Box 4
Hallituskatu 15

SF-00014 Helsinki 10

Prof.Dr. Robert Silhol
Mathematiques
Universite des Sciences et
Techniques du Languedoc
Place Eugene Bataillon

F-34060 Montpellier Cedex

Prof.Dr. Gilbert Stengle
Dept. of Mathematics
Lehigh University
Christmas-Saucon Hall 14

Bethlehem, PA 18015
USA

Prof.Dr. Zbigniew Szafraniec
Institute of Mathematics
University of Gdansk
ul. Wita Stwosza 57

80-952 Gdansk
POLAND

Prof.Dr. Alberto Tognoli
Dipartimento di Matematica
Università di Trento
Via Sommarive 14

I-38050 Povo (Trento)

Prof.Dr. David Trotman
U. E. R. de Mathématiques
Université de Provence
3, Place Victor Hugo

F-13331 Marseille Cedex 3

Prof.Dr. Yosef N. Yomdin
Dept. of Mathematics
The Weizmann Institute of Science
P. O. Box 26

Rehovot 76 100
ISRAEL

Dr. Digen Zhang
Fakultät für Mathematik
Universität Regensburg

D-93040 Regensburg