MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Elementare und Analytische Zahlentheorie

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This conference on *"Elementary and Analytic Number Theory"* was organized by Hugh L. Montgomery (Ann Arbor), Wolfgang Schwarz (Frankfurt) and Eduard Wirsing (Ulm).
Forty-three mathematicians from eleven different countries followed the Institute's invitation. The thirty-four lectures presented during the week gave a stimulating survey of current progress in Analytic Number Theory. Some of the topics taken up during the meeting were:

> Arithmetic functions, arithmetic progressions, covering congruences, distribution of prime numbers, divisor problems, Goldbach representations, exponential sums, lattice points, moments of the Riemann and Hecke zeta-functions and L-functions, Littlewood's conjecture, multiplicative functions, quadratic and cubic forms, Ramanujan expansions, set addition, Waring's problem.

On the background of the unique atmosphere of the Institute we all had a great time of learning and exchanging ideas. The organizers and participants of this conference express their thanks to the Land Baden-Württemberg, the Director of the Institute, Prof. Kreck, and his staff for providing this enjoyable environment.

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The meeting was dedicated to the memory of

Prof. Dr. Hans-Egon Richert

** June 2, 1924 † November 25, 1993*

who had been an organizer of the conference on Elementary and Analytic Number Theory in Oberwolfach since 1975.

On the 25th of November 1993, Professor Dr. Hans-Egon Richert died in Blaustein near Ulm, Germany, after a long and severe illness. Richert held a chair of Mathematics at the University of Ulm from 1972 until his retirement as an emeritus professor in 1991.

Richert was born 1924 in Hamburg and was raised there. He had to complete high school at a private institution after being expelled from the public school in the period of the Third Reich for "anglophile leanings".

In 1946, at last back in Hamburg after the war and military service, he could begin his studies of mathematics. He obtained his diploma after eight terms and his PhD only one year later. When his mentor, Professor Max Deuring, accepted a position in Göttingen, the young Richert joined him as an assistant and obtained the venia legendi there in 1954. Soon after he was put in charge of one of the best mathematical libraries in Germany.

After holding a temporary chair in Göttingen he was offered a newly created chair at the University of Marburg. In that time of vigorous development of the entire university system, Richert contributed in an essential way to shaping the Marburg Mathematical Institute, and, of course, he devoted his special love and attention to the library. In 1972 he accepted a chair at the young University of Ulm. His professorship here was the second one in the Mathematics Department, after that of Alexander Peyerimhoff. There was again much need for development.

Science had priority for Richert, teaching as well as research. Those who knew him from committee meetings recall that Richert spoke only when it was really necessary. He used to cut short a tedious discussion with a few well chosen and constructive – but never offensive – words. He also participated in reasonable administrative activities: in 1974/75 he served his university as Prorektor, then as a member of the unbeloved room allocation board, which in effect had to allocate shortages, and he acted for almost twenty years as chairman of the examination committee, an office that he administered unbureaucratically and always in the interest of the students.

Richert's field of research was Analytic Number Theory. He made important contributions to additive prime number theory, Dirichlet series, Riesz summability, the multiplicative analog of the Erdős–Fuchs theorem, estimates of the number of non-isomorphic abelian groups and bounds for exponential sums for use, e. g. in estimating the error term of the prime number theorem. From about 1965 on he focussed his research increasingly on sieve methods. Motivated by their common interest in sieves an intensive research collaboration developed between Richert and Heini Halberstam. Among other things they put the proof of Chen's $p+P_2$ theorem into readable form. The monograph on Sieve Methods that they coauthored immediately became the indispensable basis for research of many number theorists. Also, Richert's name was long associated with the best estimate for the Dirichlet divisor problem.

The very high esteem he enjoyed in the mathematical community is reflected by the many invitations he received from mathematical institutions abroad, among others the University of Illinois at Urbana and the Tata Institute in Bombay.

His life was not confined to mathematics: Richert was also a stimulating conversational partner on many different subjects. He enjoyed exploring foreign countries and keeping records of his trips on film and tape, a pursuit that he followed with the same intensity as his science whenever time allowed.

In 1991 Professor Richert had to retire from his strenuous teaching duties. It is sad that he was not granted the long and fruitful period of retirement that he was looking forward to.

With the passing of Hans-Egon Richert we lose a treasured colleague and a researcher of international reputation.

Abstracts

The difference between consecutive primes

R. C. Baker, Provo

Let $y = x^{\theta}$. Find θ as small as possible such that $\pi(x) - \pi(x-y) > 0$ for all large x. In practice the only way to do this is to prove

$$\pi(x) - \pi(x-y) \;>\; c\,\frac{y}{\log x} \tag{1}$$

where c is a positive constant. Since 1979 sieve ideas have been used: the Rosser-Iwaniec sieve with bilinear error term (Iwaniec & Jutila, Heath-Brown & Iwaniec, Iwaniec & Pintz, Mozzochi, culminating in $\theta = 0.547\ldots$); and a sieve identity of Heath-Brown used by Lou and Yao to get $\theta = 6/11 + \varepsilon = 0.5454\ldots$. Much simpler than Lou & Yao is the approach of the speaker with G. Harman. Our sieve is the one first used by Harman to study fractional parts of αp, p prime, in 1983. Our result may be expressed in the form

$$\pi(x) - \pi(x-y) \;>\; (1+o(1))\,\frac{y}{\log x}\,(1 - \mathfrak{I}_2 - \mathfrak{I}_4)$$

where $\mathfrak{I}_j = \displaystyle\int_{\mathcal{B}_j} \frac{d\alpha_1}{\alpha_1} \cdots \frac{d\alpha_{j-1}}{\alpha_{j-1}}\,\frac{d\alpha_j}{\alpha_j^2}\,\omega\!\left(\frac{1-\alpha_1-\ldots-\alpha_j}{\alpha_j}\right)$, ω being Buchstab's function. $\mathcal{B}_j$ is a certain set depending on θ and increasing as θ decreases. We obtain (1) with $\theta = 0.535$. The other tools used are really not different, apart from a few tricks, from those of Heath-Brown and Iwaniec in the paper referred to above; in some places our approach is, indeed, simpler.

Central limit theorems for $\sum_{k=1}^{n} \{k\sqrt{2}\}$

J. Beck, Rutgers

THEOREM 1. Write $S(n) = \sum_{k=1}^{n}\{k\sqrt{2}\} - \frac{n}{2}$. $\{x\}$ denotes, as usual, the fractional part of x. Then there is a constant $c_1 > 0$ such that

$$\frac{1}{N}\,\#\left\{n \le N :\ \frac{S(n)}{c_1\sqrt{\log n}} \le \lambda\right\} \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-u^2/2}\,du$$

as $N \to \infty$ for all real λ.

THEOREM 2. Write $Z(n) = \sum_{k=1}^{n}\{k\sqrt{3}\} - \frac{n}{2} - c_2(\sqrt{3})\log n$. Then there is a constant $c_3 > 0$ such that

$$\frac{1}{N}\,\#\left\{n \le N:\ \frac{Z(n)}{c_3\sqrt{\log n}} \le \lambda\right\} \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-u^2/2}\,du$$

as $N \to \infty$ for all real λ.

THEOREM 3. Write $S(n;\beta) = \sum_{k=1}^{n}\{k\sqrt{2} - \beta\} - \frac{n}{2}$. Then there is a constant $c_4 > 0$ such that

$$\text{area}\left\{(\alpha,\beta) \in [0,1)^2:\ \frac{S([\alpha n];\beta)}{c_4\sqrt{\log n}} \le \lambda\right\} \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-u^2/2}\,du$$

as $N \to \infty$ for all real λ.

THEOREM 4. Let $((x)) = x -$ (nearest integer to x) $= \pm\|x\|$. Then

$$\sum_{n=1}^{N}\frac{1}{n\,((n\sqrt{2}))} = O(1) \qquad \text{but} \qquad \sum_{n=1}^{N}\frac{1}{n\,((n\sqrt{3}))} = c_5(\sqrt{3})\log N + O(1)$$

where $c_5(\sqrt{3}) \ne 0$.

On Waring's problem for cubes

J. Brüdern, Göttingen

Let $r(n)$ be the number of representations of n as the sum of four positive integer cubes. It is conjectured that

$$r(n) = \Gamma^3(\tfrac{4}{3})\,\mathfrak{S}(n)\,n^{1/3} + O\Big(\frac{n^{1/3}}{(\log n)^{1/4}}\Big) \qquad (1)$$

where $\mathfrak{S}(n) \gg 1$ is the familiar singular series. The best results currently available are due to Vaughan (1986) who showed that the asymptotic formula holds almost, and to the speaker (1991) who showed that $\#\{n \le N:\ r(n) = 0\} \ll N^{37/42+\epsilon}$. In this talk we describe joint work with N. Watt on short interval analogues of these results. In particular we have obtained:

THEOREM 1. Let $M = N^{\theta}$ with $\theta > 5/6$. Then the asymptotic formula (1) holds for all but $O(M(\log N)^{-1/5})$ integers n with $N < n \le N + M$.

THEOREM 2. In the same notation, with $\theta > 3/4$ one has $r(n) \gg n^{1/3}$ for all but $O(M^{1-\delta})$ integers n with $N < n \le N + M$; here $\delta = \delta(\theta) > 0$.

Mean values of the zeta-function

B. Conrey, Stillwater

We give a conjecture for the sixth power moment of the zeta-function:

$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^6 dt \;\sim\; 42\,T \sum_{n<T} \frac{d_3^2(n)}{n}\,.$$

Here $\zeta^3(s) = \sum_{n=1}^{\infty} d_3(n)\, n^{-s}$. This conjecture should be compared with the results of Hardy and Ingham:

$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^2 dt \;\sim\; T \sum_{n<T} \frac{d_1^2(n)}{n} \qquad \text{and} \qquad \int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^4 dt \;\sim\; 2T \sum_{n<T} \frac{d_2^2(n)}{n}\,.$$

Finally, we remark that

$$\sum_{n<T} \frac{d_k^2(n)}{n} \;\sim\; \frac{a_k}{\Gamma(1+k^2)} (\log T)^{k^2}$$

for any $k \geq 0$, where

$$a_k \;=\; \prod_p \left(1-\tfrac{1}{p}\right)^{k^2} \sum_{n=0}^{\infty} \frac{d_k^2(p^n)}{p^n} \;=\; \prod_p \left(1-\tfrac{1}{p}\right)^{k^2} \sum_{n=0}^{\infty} \frac{\Gamma^2(k+n)}{\Gamma^2(k)(n!)^2}\, p^{-n}\,.$$

It is the case that a_k is an entire function of k of order 2, $a_k = a_{1-k}$, and all zeros of a_k are on the $1/2$-line.

It should be mentioned that one might conjecture a similar behavior for the sum $\sum_{\chi \bmod k} |L(\frac{1}{2},\chi)|^6$, at least for prime moduli k. We indicate a connection between this problem and a generalized Dedekind sum.

On functions of the type f(an+b) g(cn+d) where f and g are multiplicative functions with absolute value ≤ 1

H. Delange, Orsay

We prove the following result

THEOREM 1. Let a,b,c,d be integers, a, c, $a+b$, $c+d \geq 1$, $ad-bc \neq 0$, let f and g be multiplicative functions satisfying $|f(n)| \leq 1$ and $|g(n)| \leq 1$ for all n. If $\sum_p (2-\mathrm{Re} f(p) - \mathrm{Re}\, g(p))/p \;<\; \infty$, then

$$\frac{1}{x}\sum_{n\leq x} f(an+b)\, g(cn+d) \;=\; C \prod_{\substack{p\leq x\\ p\nmid ac(ad-bc)}} \left(1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{r=1}^{\infty}\frac{f(p^r)+g(p^r)}{p^r}\right) + o(1)$$

as $x \to \infty$, where C is a constant depending on a,b,c,d,f,g.

COROLLARIES:

- If f and g have non-zero mean values, then $f(an+b)\,g(cn+d)$ has a mean value.
- Let F and G be real-valued additive functions. If F and G have limit-distributions, then $F(an+b)+G(cn+d)$ has a limit-distribution.

On the structure theory of set addition

J.-M. Deshouillers, Bordeaux

In this survey of some joint works with G. Freĭman, V. Sós, M. Tĕmkin and A. Yudin, we present some recent applications of the *structure theory of set addition*, introduced by G. Freĭman in the late 50's.

According to Erdős and Straus, we define an *admissible* subset $\mathcal{A}$ of $[1,N]$ to be such that whenever an integer can be written as a sum of s distinct elements from $\mathcal{A}$, then s is well defined. Improving on previous results, we show that the cardinality of such an admissible subset $\mathcal{A}$ is at most $(2+o(1))\sqrt{N}$. As shown by Straus, the constant 2 cannot be improved upon.

A subset $\mathcal{A}$ of $[1,N]$ is said to be *sum-free* if $\mathcal{A}$ and $\mathcal{A}+\mathcal{A}$ have no common element. We determine the structure of $\mathcal{A}$ when card $\mathcal{A}$ is larger that $0.4\,N-O(1)$.

Finally we show how to relate the concentration function of a sum of random variables to the concentration function of the summands by using the additive properties of the large values of the characteristic functions.

Multiplicative functions on the shifted primes

P. D. T. A. Elliott, Boulder

Let Q^* be the multiplicative group of positive rationals, Γ the subgroup generated by the shifted primes $p+1$, G the quotient group Q^*/Γ. Then G is finite. Let $|G|$ denote its order. Then:

THEOREM 1. There is a k such that every positive rational r has a representation

$$r^{|G|} = \prod_{i=1}^{k}(p_i+1)^{\varepsilon_i}, \qquad \varepsilon_i = \pm 1,\ p_i \text{ prime}.$$

THEOREM 2. $|G| \leq 3$.

Further argument, using a theorem of Ruzsa and the fact that G is known to be cyclic, shows that those integers for which a representation $r = \prod_{i=1}^{k}(p_i+1)^{e_i}$ holds, have a density δ, and $\delta \geq 1/3$.

On supersingular primes

E. Fouvry, Orsay

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{Q}$; $\mathcal{E}_{a,b}$ the elliptic curve with quasi-minimal equation: $\mathcal{E}_{a,b}: \; y^2 = x^3 + ax + b \qquad (a, b \in \mathbb{Z},\ 4a^3 + 27b^2 \neq 0)$ and

$$\pi_0(x, \mathcal{E}) = \#\{p \leq x;\ p \text{ supersingular prime for } \mathcal{E}\}\ .$$

Lang and Trotter conjectured the asymptotic formula

$$\pi_0(x,\mathcal{E}) \sim C_{\mathcal{E}} \frac{\sqrt{x}}{\log x} \qquad \text{as } x \to \infty,\ C_{\mathcal{E}} > 0,\ \mathcal{E} \text{ not CM}\ .$$

Improving the method of Elkies, we obtain the lower bound:

$$\pi_0(x,\mathcal{E}) \geq \frac{\log_3 x}{(\log_4 x)^{1+\delta}} \qquad \text{for any } \delta > 0 \text{ and } x > x_0(\mathcal{E},\delta)\ .$$

Several results on average are proved, e. g.

THEOREM 1.

$$\sum_{\substack{|a|<A \\ 4a^3+27b^2\neq 0}} \sum_{|b|<B} \pi_0(x,\mathcal{E}_{a,b}) \sim \frac{\pi}{3} \cdot 4AB \cdot \frac{\sqrt{x}}{\log x}$$

as $x \to \infty$, uniformly for $A, B > x^{1/2+\epsilon}$, $AB > x^{3/2+\epsilon}$.

THEOREM 2. Almost every elliptic curve $\mathcal{E}_{a,b}$ has its least supersingular prime less than $\sqrt{\max(|a|,|b|)}$.

We discuss on the number of supersingular primes common to two elliptic curves and on the average value of the constant $C_{\mathcal{E}_{a,b}}$.
These results were obtained in collaboration with R. Murty (Montreal) and E. Ullmo (Orsay).

A sieve identity and the roots of quadratic congruences

J. B. Friedlander, Toronto

Let $f \in \mathbb{Z}[x]$ be irreducible with degree ≥ 2. For p prime there are by a theorem of Lagrange no more than $\deg f$ solutions ν of the congruence $f(\nu) \equiv 0 \pmod{p}$. It is natural to wonder how these roots are distributed as p varies. We normalize by considering the fractional part $\{\nu/p\}$ and expect, for lack of evidence to the contrary, that these are uniformly distributed, that is, for arbitrary $0 \leq \alpha < \beta \leq 1$,

$$\#\{(\nu, p) : \; f(\nu) \equiv 0 \pmod{p}, \; p \leq x, \; \alpha \leq \{\nu/p\} < \beta\} \;\sim\; (\beta - \alpha)\,\pi(x)\,.$$

In the case where the sum is taken over all integers rather than just primes, the relevant equi-distribution was demonstrated by Hooley, about thirty years ago. Here we describe joint work with W. Duke and H. Iwaniec in which the required equi-distribution over primes is shown in the case that f is a quadratic polynomial with negative discriminant. We expect that the indefinite quadratic case can be settled by modifying the ideas, but higher degree polynomials offer much greater challenge. The proof involves, by the Weyl criterion a successful estimation of certain exponential sums over primes. A sieve identity, believed to be new, reduces the latter to certain linear and bilinear exponential sums. These can be related to certain Poincaré type series which can be estimated via the spectral theorem and known estimates for Bessel functions.

The circle method for quadratic and cubic forms

D. R. Heath-Brown, Oxford

An identity of Friedlander and Iwaniec gives the following form of the circle method: Let $w\colon\, \mathbb{R}^n \to \mathbb{R}$ be smooth, compactly supported, and let $\omega\colon\, \mathbb{R} \to \mathbb{R}$ be smooth and supported on $[1,2]$. Let F be a form of degree n. Then

$$\sum_{F(\mathfrak{x})=0} w(p^{-1}\mathfrak{x}) \;=\; \sum_{\mathfrak{c} \in \mathbb{Z}^n} \sum_{q=1}^{\infty} q^{-n-1} S(q;\mathfrak{c})\, I(q;\mathfrak{c})$$

where

$$S(q;\mathfrak{c}) \;=\; \sum_{\substack{a \,(\mathrm{mod}\, q)\\ (a,q)=1}} \; \sum_{\mathfrak{m}\,(\mathrm{mod}\, q)} e_q(aF(\mathfrak{m}) + \mathfrak{c}\cdot\mathfrak{m})\,,$$

$$I(q;\mathfrak{c}) \;=\; \int_{\mathbb{R}^n} w(p^{-1}\mathfrak{x})\, e_q(-\mathfrak{c}\cdot\mathfrak{x})\, h(q; F(\mathfrak{x}))\, d\mathfrak{x}\,.$$

Here

$$h(q;\theta) = \sum_{j=1}^{\infty} j^{-1}\left\{\omega\left(\frac{qj}{Q}\right) - \omega\left(\frac{|\theta|/qj}{Q}\right)\right\}$$

and $Q > 0$ is an arbitrary parameter.
The terms for $\mathfrak{r} = \mathfrak{o}$ lead to the singular series and integral. The formula already includes the "Kloosterman refinement" - the sum over a, but has the advantage that $a^{-1} (\bmod q)$ does not occur. Moreover it is easy to carry out the "second Kloosterman refinement", i. e. averaging over q.
One can prove asymptotic formulae when $d = 2$ and $n \geq 3$, establishing in particular the Hasse principle.
When $d = 3$ and $n \geq 4$ one can prove non-trivial upper bounds (at least for diagonal forms) providing one assumes the analytic continuation and Riemann Hypothesis for appropriate Hasse-Weil L-functions.

On B_{2k}-sequences

M. Helm, New York

A sequence $\mathcal{A}$ of natural numbers is called a B_r-sequence if the equation $n = a_1 + \ldots + a_r$, $a_1 \leq \ldots \leq a_r$, $a_i \in \mathcal{A}$, has at most one solution for all natural n. Let $A(n) = \sum_{a \leq n,\, a \in \mathcal{A}} 1$. We prove that every B_{2k}-sequence $\mathcal{A}$ satisfies

$$\liminf_{n\to\infty} \frac{A(n)}{n^{1/(2k)}} (\log n)^{1/(2k)} < \infty$$

provided that $A(n^2) \ll (A(n))^2$ for all sufficiently large n.
This proves in the case of even r a conjecture of P. Erdős that states that for every natural r every B_r-sequence $\mathcal{A}$ satisfies $\liminf_{n\to\infty} \frac{A(n)}{n^{1/r}} = 0$.

Sums of distinct squares

A. Hildebrand, Urbana

If $s \geq 5$, it is easy to see that every sufficiently large integer is representable as a sum of s distinct positive squares. Let $N(s)$ denote the largest integer not representable in this form. Halter-Koch [*Acta Arith.* 42 (1982), 11–20] computed the exact values of $N(s)$ for $5 \leq s \leq 12$ and gave inequalities which lead to an exponential upper bound for $N(s)$. On the other hand, since every integer representable as a sum of

s distinct positive squares is at least equal to the sum of the first s positive squares, i. e.

$$\sum_{i=1}^{s} i^2 = \tfrac{1}{6}\, s\,(s+1)\,(2s+1) \;\; =: \; P(s),$$

$N(s)$ is bounded from below by the polynomial $P(s)$.
In joint work with P. Bateman and G. Purdy we show that, in fact, $N(s)$ is asymptotically equal to this trivial lower bound. More precisely, we prove that

$$N(s) \;=\; P(s) + (2s)^{3/2} + \lambda_s\,(2s)^{5/4} \;+\; O(s^{9/8})$$

where $\lambda_s \;=\; \sqrt{2\max\left(\|\sqrt{2s}\|,\,\|\sqrt{2s}-\tfrac{1}{2}\|\right)}$ and $\|\cdot\|$ denotes the distance to the nearest integer.

Lattice points and exponential sums

M. N. Huxley, Cardiff

Two kinds of problems were discussed, estimating exponential sums and counting the number of plane lattice points within a contour, or lattice points close to the contour.
New results of Sargos, Filaseta and Trifonov, and an argument of Huxley and Kolesnik that iterates between the two types of problems, were mentioned. Two particular results are:

THEOREM. (Huxley). The error term $E(t)$ in the mean square of the zeta-function on the critical line is $O\left(T^{72/227+\epsilon}\right)$.

THEOREM. (Huxley & Watt). The error term $\Delta(x)$ in the number of integer ideals in a quadratic number field $\mathbb{Q}(\sqrt{D})$ with norm up to x is $O\left((D^2x)^{23/73+\epsilon}\right)$ for x large compared with $|D|$.

The mean square of the zeta-function in the critical strip

A. Ivić, Belgrade

For fixed σ such that $1/2 < \sigma < 1$ and $T \geq 2$ let

$$E_\sigma(T) \;=\; \int_0^T \left|\zeta(\sigma+it)\right|^2 dt \;-\; \zeta(2\sigma)\,T \;-\; (2\pi)^{2\sigma-1}\,\frac{\zeta(2-2\sigma)}{2-2\sigma}\,T^{2-2\sigma}$$

denote the error term in the mean square formula for the Riemann zeta-function $\zeta(s)$ in the critical strip $1/2 < \sigma < 1$. Recently much work has been done on this important function, and the aim of this talk is to present these results. In particular, joint results of K. Matsumoto and the speaker are presented. These include for $1/2 < \sigma < 3/4$ and $C > 0$

$$E_\sigma(T) = \Omega_-\left(T^{\frac{3}{4}-\sigma}\exp\left(C(\log\log T)^{\sigma-\frac{1}{4}}(\log\log\log T)^{\sigma-\frac{5}{4}}\right)\right)$$

and

$$E_\sigma(T) \ll \begin{cases} T^{\frac{\kappa-\sigma+3/4}{\kappa+1}}(\log T)^{\frac{\kappa}{\kappa+1}} & \frac{1}{2} < \sigma < \frac{3}{4}, \\ T^{\frac{\kappa(1-\sigma)}{\kappa(1-\sigma)+1/4}}\log T & \frac{3}{4} \le \sigma < 1. \end{cases}$$

where (κ, λ) is an exponent pair such that $\lambda = \kappa + \frac{1}{2}$.

Class group L-functions

H. Iwaniec, Rutgers

This is joint work with W. Duke and J. Friedlander.
We consider the imaginary quadratic field $\mathfrak{K} = \mathbb{Q}(\sqrt{-D})$ of discriminant $-D$. Let χ be a character of the ideal class group $\mathcal{H}$, to which is attached the L-function

$$L_{\mathfrak{K}}(s,\chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})(N\mathfrak{a})^{-s}$$

where $\mathfrak{a}$ ranges over non-zero integral ideals and $N\mathfrak{a}$ is the norm. Various estimates and mean value asymptotics are established for these L-functions which are uniform with respect to the discriminant. In particular we get the following

THEOREM. If $\mathrm{Re}\, s = 1/2$ we have

$$\frac{1}{h}\sum_{\chi\in\widehat{\mathcal{H}}} |L_{\mathfrak{K}}(s,\chi)|^2 = \lambda_D(s) + O(D^{-1/28+\varepsilon})$$

where the main term $\lambda_D(s)$ is given explicitly as the sum of residues of a certain meromorphic function;

$$\lambda_D(\tfrac{1}{2}) \sim 6\pi^{-2}w^{-1}L(1,\chi_D)\log^3 D, \qquad \text{if } s = 1/2$$

$$\lambda_D(\tfrac{1}{2}+it) \sim \tfrac{1}{2}\pi^{-2}w^{-1}|\zeta(1+2it)|^2 L(1,\chi_D)\log D, \qquad \text{if } s \ne 1/2.$$

The error term is estimated by means of spectral theory of automorphic forms. The equidistribution of Heegner points due to W. Duke is used. An estimate for the individual $L_{\mathfrak{K}}(s,\chi)$ is also given which is sharper than the convexity bound subject to certain natural conditions.

The mean twelfth power of Dirichlet L-functions

M. Jutila, Turku

In collaboration with Y. Motohashi, the following mean value estimate was recently obtained:

THEOREM.

$$\sum_{D\le X}\ \sum_{\chi \bmod D}^{*} \int_0^T \left|L(\tfrac{1}{2}+it,\chi)\right|^{12} dt \ \ll\ X^3T^2(XT)^{\epsilon}$$

where * indicates that the sum is over primitive characters.

This includes a well-known mean value theorem of D. R. Heath-Brown for the Riemann zeta-function as well as an analogous estimate due to T. Meurman for L-functions to a single modulus. The proof is essentially based on spectral methods. The same ideas apply even to cusp form L-functions twisted with Dirichlet characters.

Lattice points in three-dimensional convex bodies

E. Krätzel, Jena

Let $\mathcal{K}$ be a three-dimensional convex body, where the origin is an inner point. Suppose that the boundary of $\mathcal{K}$ is an analytic map of the unit sphere. We obtain for the number of lattice points $R(x)$ in the blown-up body $x\mathcal{K}$

$$R(x) \ = \ \mathrm{vol}(\mathcal{K})\,x^3 + \Delta(x)$$

with the trivial estimation for the remainder $\Delta(x) \ll x^2$. Assuming the Gaussian curvature to be positive at all points of the boundary, in 1950 Hlawka improved this to $\Delta(x) \ll x^{3/2}$, and in 1992 W. G. Nowak and the author proved

$$\Delta(x) \ \ll\ x^{25/17}(\log x)^{10/17}\,.$$

When the boundary of $\mathcal{K}$ has isolated points with Gaussian curvature zero a new situation arises. The estimation of the remainder depends on the order of these zeros. Furthermore, the position of the lattice with respect to a point with Gaussian curvature zero plays an important role. If the tangential plane at this point has rational slope with respect to the coordinate planes even a new main term arises with an order which is greater than $3/2$. If one or two slopes are irrational, the contribution of this point to the estimation of the remainder is much smaller.

Ramanujan expansions revisited

L. Lucht, Clausthal

In 1918 Ramanujan investigated the sums $c_n(a)$ of the ath powers of all nth primitive roots of unity and proved that numerous arithmetical functions $g : \mathbb{N} \to \mathbb{C}$ possess an expansion of the form

$$g(a) = \sum_{n=1}^{\infty} \widehat{g}(n)\, c_n(a) \qquad (a \in \mathbb{N})$$

with suitable coefficients $\widehat{g}(n)$. A general approach is based on certain orthogonality relations between the Ramanujan sums by which the consideration is restricted to arithmetical functions g having a non-zero mean value. The observation that all Ramanujan sums $c_n(a)$ are closely related to $c_n(1) = \mu(n)$ where μ is the Möbius function yields a different approach. In particular for multiplicative functions having no mean value a conceptual explanation for the existence of Ramanujan expansions is given. The main tools are inversion theorems for Dirichlet series and relationship theorems with logarithmic weights. The divisor functions d_{k+1} $(k \in \mathbb{N})$ may serve as concrete examples: They have pointwise convergent Ramanujan expansions with coefficients

$$\widehat{d}_{k+1}(n) = \frac{(-1)^k}{k!} \frac{\log^k n}{n} \prod_{p^\nu \| n} \left(\left(1 - \frac{1}{p}\right)^k \sum_{\rho \geq \nu} \binom{k+\rho-1}{k-1} \frac{1}{p^{\rho-\nu}} \right)^{-1}.$$

The case $k = 1$ is contained in Ramanujan's original paper.

Short intervals with very few prime numbers

H. Maier, Ulm

We prove that for $\varepsilon(x) > 0$, $\lim_{x\to\infty} \varepsilon(x) = 0$, we have that

$$\liminf_{x\to\infty} \frac{\pi\left(x + (\log x)^{1+\varepsilon(x)}\right) - \pi(x)}{(\log x)^{\varepsilon(x)}} = 0$$

The proof uses ideas of Erdős and Rankin applied in their papers on large gaps between consecutive primes and the matrix method of the aüthor from his irregularity results in the local distribution of prime numbers. A novel feature is the use of a result of Friedlander on the number of integers free of small and large prime factors.

Group valued arithmetical functions

J.-L. Mauclaire, Paris

Let $\mathcal{G}$ be an abelian group, $q \in \mathbb{N}_0$, $q \geq 2$. A $\mathcal{G}$-valued q-additive function f is a function $f : \mathbb{N}_0 \to \mathcal{G}$ satisfying the relation $f(aq^r + b) = f(aq^r) + f(b)$, if $0 \leq b \leq q^r - 1$, $a \geq 0$ and $r \geq 0$.
In the case $\mathcal{G} = \mathbb{T}$, the spectrum of f provides information on the harmonic measure determined by f, and its existence depends on the convergence of the product

$$P(q, f, \alpha, k; N) = \left| \prod_{0\leq r\leq N} \frac{1}{q} \sum_{0\leq a\leq q-1} f(aq^r)^k e^{ik\alpha aq^r} \right|$$

for $k = 1$ and α a real number. This product if convergent for $k = 1$ is convergent for all $k \in \mathbb{Z}$. A consequence is that if $f(n) = e^{ig(n)}$, then there exists a probability measure ν on $\mathbb{R}/\mathbb{Z}$, $A(n)$ a sequence in $\mathbb{R}/\mathbb{Z}$, such that the sequence of probability measures

$$\frac{1}{N} \sum_{n\leq N} \delta_{(g(n) - \alpha n - A(\log_q N))},$$

where $\log_q N = [\log N / \log q]$, converges weakly to ν. A similar result is true in any locally compact abelian group. In fact we have:

THEOREM. Let $\mathcal{G}$ be a locally compact abelian group, f a $\mathcal{G}$-valued q-additive function. If for every continuous character χ of $\mathcal{G}$, there exists a real number $\alpha(\chi)$ such that the product $P(q, \chi \circ f, \alpha, 1; N)$ converges, then, there exists $r_0 \in \mathbb{N}$, $c \in \mathcal{G}$, a sequence $A(N)$ and a probability measure ν on $\mathcal{G}$, such that the sequence

$$\frac{1}{N}\sum_{n\leq N}\delta_{(f(q^{r_0}n)-cn-A(\log_q N))}$$

converges weakly to ν.

Recurrent sequences with locally uniformly bounded numbers of prime divisors

C. Methfessel, Clausthal

The talk presents a proof of the following result:

THEOREM. Let $m \in \mathbb{N}$ and $g: \mathbb{N} \to \mathbb{Z}$ be recurrent with $\omega(|g(n)|) \leq m$ for all $n \in \mathbb{N}$ with $g(n) \neq 0$. Then the set

$$P_g := \{p \in \mathbb{P}: \ \exists\, n \in \mathbb{N} \text{ such that } g(n) \neq 0,\ p \mid g(n)\}$$

is finite.

This extends the corresponding result on recurrent sequences of prime powers (i.e. the case $m = 1$), which was obtained by Erdős, Maxsein and Smith in 1990.

Sums of numbers with many divisors

H. L. Montgomery, Ann Arbor

This is joint work with P. Erdős. Recently I was asked for representations of n in the form $n = m_1 + \ldots + m_k$ (1) where the m_i are positive integers with $d(m_i)$ as large as possible. To address this, one must first determine how often $d(m)$ is large. Put

$$D(x) = \exp\left(\frac{\log x \log 2}{\log\log x}\right).$$

Thus $d(m) > D(m)$ infinitely often, but $d(m) < (D(m))^{1+\varepsilon}$ for $m > m_0(\varepsilon)$. By extending a result of Erdős and Nicolas (1981), it is shown that if $0 < \alpha < 1$ then the number of $m \leq x$ for which $d(m) > (D(x))^{\alpha}$ is $x^{1-\alpha+o(1)}$. Thus it is clear that there exist infinitely many n with no representation (1) with $d(m_i) > (D(n))^{\alpha}$, if $\alpha > 1 - 1/k$. We show, conversely, that if $\alpha < 1 - 1/k$ then all sufficiently large n can be written in the form (1) with $d(m_i) > (D(n))^{\alpha}$.

The mean values of the zeta- and Dirichlet L-functions

Y. Motohashi, Tokyo

The first part of the talk is on the fourth power moment of Dirichlet L-functions. This is joint work with Matti Jutila.

A relation between this classical problem and the Ramanujan conjecture on the size of Fourier coefficients of non-homomorphic cusp-forms/SL(2, $\mathbb{Z}$) is pointed out in the following form:

THEOREM. Let $t_j(n)$ be the Hecke eigen-value/SL(2, $\mathbb{Z}$). Let α be such that $|t_j(n)| \ll_\varepsilon n^{\alpha+\varepsilon}$ for all $j \geq 1$ and $n \geq 1$ with an arbitrary small $\varepsilon > 0$. Then we have

$$\sum_{\chi \bmod D}^{*} \int_T^{T+T_0} |L(\tfrac{1}{2}+it,\chi)|^4 \, dt \ll D(T_0 + T^{2/3} D^{(2\alpha-1)/3}) (DT)^\varepsilon$$

provided $T^\varepsilon \leq T_0 \leq T^{2/3}$.

The second part is on an approach to the eighth power moment of the zeta-function. By an application of the Fourier-Hermite expansion and the Parseval formula one may approach to the eighth power moment starting from the fourth power moment. More precisely, let

$$|\zeta(\tfrac{1}{2} + i(T+Gt))|^4 = \sum_{n=0}^{\infty} h_n(T,G) H_n(t)$$

be the Fourier-Hermite expansion. Then, by the Parseval formula we have

$$\frac{1}{G\sqrt{\pi}} \int_{-\infty}^{+\infty} |\zeta(\tfrac{1}{2} + i(T+t))|^8 e^{-(t/G)^2} dt = \sum_{n=0}^{\infty} n!\, 2^n \, |h_n(T,G)|^2 .$$

where the Fourier coefficients $h_n(T,G)$ are explicitly given in terms of the spectrum of the non-Euclidean Laplacian. We have

$$\sum_{n=0}^{\infty} h_n(T,G)\, \xi^n = \frac{1}{G\sqrt{\pi}} \int_{-\infty}^{+\infty} |\zeta(\tfrac{1}{2} + i(T+\tfrac{1}{2}G\xi+t))|^4 e^{-(t/G)^2} \, dt ,$$

and we have already developed a theory which yields an explicit formula for the last integral. It is to be seen whether or not this approach to $\int |\zeta|^8$ will yield any non-trivial estimate for this highly important mean value of the zeta-function.

On zeros of polynomials with 0,1-coefficients

A. Odlyzko, Murray Hill

Zeros of polynomials with 0,1-coefficients exhibit interesting fractal properties. If W is the set of such zeros, and $\overline{W}$ its closure, then (together with B. Poonen) I have shown that $\overline{W}$ is path connected, $\overline{W}$ contains an open set that contains $\{z \in \mathbb{C}: |z| = 1, z \neq 1\}$. $\overline{W}$ contains the segment $[-q, -1/q]$, where $q = (1+\sqrt{5})/2$, and the only points $z \in \overline{W}$, such that $\mathrm{Re}(z) < -1.5$, are in this line segment.

On the prime ideal theorem and applications

A. Perelli, Genova

For $k, d, Q \in \mathbb{N}$, let $f(x) = x^k + Qd$. Let $\nu(d)$ denote the number of irreducible factors of $f(x)$ and $\rho(d,p)$ the number of roots $(\bmod p)$ of $f(x)$. We obtain the following

THEOREM. (Nair & Perelli). Let $X, Y \geq 2$; $\varepsilon, A > 0$, and $Y^{1/2+\varepsilon} \leq H \leq Y$. Then we have

$$\sum_{Y \leq d \leq Y+H} \Big| \sum_{X \leq p \leq 2X} (\rho(d,p) - \nu(d)) \Big| \ll_{k,A,\varepsilon} \frac{HX}{\log^A X}$$

uniformly for $Q \leq Y^A$.

The independence of the ranges for X and Y comes from the possibility of considering both d and p as "modulus" in an appropriate sense.

The above theorem has several applications. We mention three of them:

Application to the irregularity in the distribution of primes represented by polynomials, to the representation of integers n as $n = p + m^k$, p prime and m integer and to the distribution on average of primes in short intervals.

On sequences containing no arithmetic progressions

R. A. Rankin, Glasgow

A strictly increasing sequence $\{a_i\}$ of non-negative integers is called an $\mathcal{A}_n$-sequence if it contains no n distinct terms in arithmetic progression ($n \geq 3$). For an $\mathcal{A}_n$-sequence we write $A_n(x) = \sum_{a_i \leq x} 1$. The densest $\mathcal{A}_3$-sequence so far constructed is due to F. Behrend [*Proc. Nat. Acad. Sci. U.S.A.* 32 (1946), 331-332] who obtained for any $\varepsilon > 0$

$$A_3(x) \; > \; x \exp\Big(-2\sqrt{2\log 2}\,(1+\varepsilon)\sqrt{\log x}\Big)\,. \qquad (1)$$

In 1962 using an explicit formula for $r(s, P; N)$, the number of solutions of the equation $x_1^2 + \ldots + x_s^2 = N$ where $0 \le x_i \le P$, $i = 1, \ldots, s$, I was able to construct an $\mathcal{A}_n$-sequence for $n \ge 3$ [*Proc. Royal Soc. Edinburgh* 65A (1962), 333–344] with:

$$A_n(x) \; > \; x \exp\Big(-c_n(1+\varepsilon)(\log x)^K\Big)\,, \qquad (2)$$

where c_n is a known positive constant ($c_3 = c_4 = 2\sqrt{2\log 2}$) and $K = 1/(k+1)$ where $n > 2^k$. This gives the same result for $n = 4$ as it does for $n = 3$. My efforts to get an estimate of larger order for $n = 4$ have been unsuccessful although I can get rid of the ε in (1) replacing the right side for $n = 3, 4$ by $x \exp\big(-c\sqrt{\log x}\big)(\log x)^{-1/4}$ and similarly for higher values of n.

Power moments of Hecke zeta-functions

U. Rausch, Ulm

Let K be an algebraic number field of degree $[K : \mathbb{Q}] = n = r_1 + 2r_2$ (in the standard notation), $e_p = 1$ for $p = 1, \ldots, r_1$ and $e_p = 2$ for $p = r_1 + 1, \ldots, r+1$, $r = r_1 + r_2 - 1$. For $x = (x_1, \ldots, x_{r+1})$, $x_p > 0$, $\varepsilon > 0$, and $k \in \mathbb{N}$ we consider the function

$$F_\varepsilon(x) \; = \; \frac{1}{2\pi} \sum_\lambda \int_{-\infty}^{\infty} \left|\zeta(\tfrac{1}{2} + it; \lambda)\right|^{2k} \prod_{p=1}^{r+1} x_p^{-ie_p(t-b_p)} \cdot \exp\Big(-\varepsilon \sum_{p=1}^{r+1} e_p^2 (t-b_p)^2\Big)\, dt$$

where the summation is over all Grössencharacters λ of the form $\lambda(\alpha) = \prod_{p=1}^{r+1} |\alpha^{(p)}|^{ie_p b_p}$ with $b_p \in \mathbb{R}$ such that $\sum_{p=1}^{r+1} e_p b_p = 0$ and $\lambda(\eta) = 1$ for every unit $\eta \in K$. ζ denotes Hecke's zeta-function.

$F_\varepsilon(x)$ is expanded into a series similar to that occurring in the Voronoï formula associated with the divisor problem. In particular it follows that

$$F_\varepsilon(x) \; \ll \; \varepsilon^{-(nk+r+1)/4}\, X^{-1/2}\big(|\log(\varepsilon^{nk/2}X)| + 1\big)^{2k-1}$$

for $X := \prod_{p=1}^{r+1} x_p^{e_p} \ge 1$ and $0 < \varepsilon \le 1/8$.

For $k = 1$, $X = 1$ and totally real fields K, this is just about what an analogue of the Lindelöf Hypothesis would give.

Structure of sumsets

I. Ruzsa, Budapest

By Szemerédi's theorem, every set of integers $\mathcal{A} \subset [1..N]$, $|\mathcal{A}| > \alpha N$, contains a long arithmetical progression, but its length may be $\ll \log N$. In contrast, a triple sumset $\mathcal{A}+\mathcal{A}+\mathcal{A}$ contains an arithmetical progression of length $\gg N^{\gamma}$, $\gamma = \gamma(\alpha) > 0$ (Freĭman, Halberstam, Ruzsa), while for $\mathcal{A}+\mathcal{A}$ the maximal length lies in

$$\left(\exp(\log N)^{1/3-\varepsilon},\ \exp(\log N)^{2/3+\varepsilon}\right) \qquad \text{(Bourgain, Ruzsa).}$$

Better than simple arithmetical progressions are k-dimensional arithmetical progressions of the form $\mathcal{P} = \{a + q_1 x_1 + \ldots + q_k x_k :\ 0 \le x_i \le \ell_i - 1\}$ and Bohr sets, finite analogues of Bohr neighbourhoods. The connection between them is explored, a source of which is a theorem of Bogolyubov (1937): If $\mathcal{A} \subset \mathbb{Z}$, $\bar{d}(\mathcal{A}) > 0$ then $\mathcal{A}+\mathcal{A}-\mathcal{A}-\mathcal{A}$ is a Bohr neighbourhood of 0.

On Littlewood's conjecture

J. W. Sander, Hannover

In this talk we describe joint work with with S. Schäffer.
In 1948 Littlewood made the following conjecture (LC): For all $\alpha, \beta \in \mathbb{R}$ and all $\varepsilon > 0$ there is some n such that

$$n\,\|\alpha n\|\,\|\beta n\| < \varepsilon$$

where $\|x\|$ denotes the distance between the real number x and the nearest integer.

In 1955 Cassels and Swinnerton-Dyer proved that this is true if α, β are elements of a fixed real cubic number field. Apart from that not much is known. A "proof" by B. F. Skubenko [*J. Soviet Math.* 53, no. 3 (1991), 302-321] seems to have gaps. LC would be wrong if for some $\alpha, \beta \in \mathbb{R}$ and some $\varepsilon > 0$ $\quad N\,\|\alpha n\|\,\|\beta n\| \ge \varepsilon$ for all N and all $1 \le n \le N$.

THEOREM 1. For $\alpha = \frac{3-\sqrt{5}}{2}$ and all N there is some β such that for all $1 \le n \le N$

$$N\,\|\alpha n\|\,\|\beta n\| \ \ge\ \frac{3-\sqrt{5}}{4} \ \approx\ 0,19.$$

THEOREM 2. For all $\alpha, \beta \in \mathbb{R}$ and all $N \ge 2$ there is some $n \le N$ such that

$$N\,\|\alpha n\|\,\|\beta n\| \ <\ \frac{2N}{5N+4} \ <\ \frac{2}{5}.$$

Integers not of the form $n - \varphi(n)$

A. Schinzel, Warszawa

W. Sierpiński asked in 1959 whether there exist infinitly many positive integers not of the form $n - \varphi(n)$, where φ is Euler's function. The following theorem – proved jointly with J. Browkin using covering congruences – gives an affirmative answer to this question:

THEOREM. For every positive integer k we have $2^k \cdot 509203 \neq n - \varphi(n)$.

On a certain nonary cubic form

R. C. Vaughan, London

In joint work with T. D. Wooley we investigate the number of solutions S of the diophantine system

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 &= y_1^3 + y_2^3 + y_3^3 \\ x_1 + x_2 + x_3 &= y_1 + y_2 + y_3 \end{aligned} \qquad (1)$$

with $x_i \le P$, $y_i \le P$, $i = 1, 2, 3$.
Heath-Brown in 1989 in important work on Waring's problem used the estimate $S = O(P^{3+\varepsilon})$, and Hua (1948) had shown that $S = O(P^3 \log^9 P)$.
By suitable transformations we replace the system (1) by the nonary cubic form

$$d_1 e_1 f_1 + d_2 e_2 f_2 + d_3 e_3 f_3 = d_1 d_2 d_3 + e_1 e_2 e_3 + f_1 f_2 f_3 \qquad (2)$$

subject to sundry side conditions. In this way we are able to show that the number U of non-trivial solutions of (1) is $O\big(P^{7/3}(\log 2P)^{11}\big)$. Thus

THEOREM 1. We have $S = 6P^3 + O\big(P^{7/3}(\log 2P)^{11}\big)$.

By further refinement we can establish

THEOREM 2. The number U satisfies $U \asymp P^2 \log^5 P$.

Since the contribution from the "Major Arcs" in this problem is also $\asymp P^2 \log^5 P$ this suggests a relationship between "Non-trivial" solutions and "Major Arcs" and we attempt to place this idea on firmer ground.

Prime numbers in almost all short intervals

N. Watt, Göttingen

Assuming the Riemann Hypothesis Selberg showed that almost all of the intervals $[n, n+f(n)\log^2 n]$ will contain primes, provided that $f(n) \to \infty$. He also showed, unconditionally, that, for $\theta > 19/77$, almost all of the intervals $[n, n + n^\theta]$ will contain primes. Later results are: Montgomery $\theta > 1/5$, Huxley $\theta > 1/6$, Harman $\theta > 1/10$. The last result uses sieve methods which were first introduced for the problem of an individual short interval by Iwaniec and Jutila in 1977. More recently Harman and Heath-Brown have independent unpublished proofs for $\theta > 1/12$. Using the following result:

$$\int_0^T \left|\zeta(\tfrac{1}{2} + it)\right|^4 \left|\sum_{n=1}^{N} a_n n^{it}\right|^2 dt \ll_\varepsilon T^{1+\varepsilon} N$$

for $|a_n| \le 1$ $(n \in \mathbb{N})$, $T \ge 1$, $N \ge 1$ and $\varepsilon > 0$, we can show that almost all intervals $[n, n + n^\theta]$ contain primes provided $\theta > 1/14$.

Multiple prime divisors of binomial coefficients

E. Wirsing, Ulm

Generalizing work of Sárközy and Sander we count (multiple) prime divisors of $\binom{n}{r}$ weighted in the traditional way.

THEOREM 1. For every $k \in \mathbb{N}$

$$\sum_{p^k | \binom{2n}{n}} \log p \sim c_k n^{1/k},$$

as $n \to \infty$, where the positive constants c_k are explicitly given.

THEOREM 2. Let $k \in \mathbb{N}$, $\varepsilon > 0$ and $n^\varepsilon \le r \le n/2$. Then we have

$$\sum_{p^k | \binom{n}{r}} \frac{\log p}{p} \sim d_k \log r,$$

for $n \to \infty$, where $d_k = \frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+2} - + \ldots$, in particular $d_1 = \log 2$.

As in the papers quoted above the essential tool is an estimate for certain exponential sums. The estimate used by Sander suffices for theorem 1 but the proof of theorem 2 requires more:

THEOREM 3. Let $h(x) = h_1 x + \ldots + h_J x^J$ be a real polynomial and $\delta > 0$. Then with a positive constant $c = c(J, \delta)$ provided that

$$\max_{j=1,\ldots,J} (|h_j|^\delta) \le N \le \max_{j=1,\ldots,J} (|h_j|^{\frac{1}{j}-\delta})$$

we have

$$\sum_{p \text{ prime}, p \le N} e\left(h\left(\tfrac{1}{p}\right)\right) \ll N^{1-c}$$

where the O-constant depends only on J and δ.

Goldbach's problem with restrictions

D. Wolke, Freiburg

By using a combinatorial method of Erdős [*Proc. Amer. Math. Soc.* 5 (1954), 847–853] and various results on Goldbach representations the following was proved partly joint with G. Dufner.

THEOREM 1. Let $\alpha_1, \alpha_2, \alpha_3 > 0$; $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$; $\alpha_1+\alpha_2+\alpha_3 = 1$; $\beta_1+\beta_2+\beta_3 = 1$. Then there exist sets of primes $\mathbb{P}_j$ with $\#\{p \in \mathbb{P}_j : p \le x\} \ll x^{\alpha_j}(\log x)^{\beta_j}$, $j = 1, 2, 3$, and $\forall\, n \ge n_0$: $2n+1 = p_1 + p_2 + p_3$, $p_j \in \mathbb{P}_j$ $(j = 1, 2, 3)$.

This generalizes a theorem of E. Wirsing [*Analysis* 6 (1986), 285–308].

THEOREM 2. There is a set $\mathbb{P}' \subseteq \mathbb{P}$ with $P'(x) \ll \log^2 x$ and $\forall\, n \ge n_0$: $2n+1 = p + p_1' + p_2'$, $p_1', p_2' \in \mathbb{P}'$.

Reported by: P. Bauer (Frankfurt) & U. Vorhauer (Ulm).

PROBLEMS POSED

Oberwolfach, 16 March 1994

1. ***(Paul Erdős)*** Suppose that $m_1 < m_2 < \ldots < m_k$, and that numbers $a_1, a_2, \ldots, a_k$ are chosen so that every integer n satisfies at least one of the congruences $n \equiv a_i (\bmod m_i)$. Then we say that the arithmetic progressions $a_i (\bmod m_i)$ form a system of *covering congruences*. Do there exist systems of covering congruences in which all the moduli are large? Choi achieved $\min m_i = 20$, and 24 has been claimed. Prize: $1,000.

2. ***(Paul Erdős)*** Can every odd integer n be expressed as a sum of a prime and a bounded number of powers of 2? Erdős used covering congruences to show that there are arithmetic progressions free of odd numbers of the form $p + 2^k$. Crocker (Pacific J. Math. 1971) showed that there are infinitely many odd n not of the form $p + 2^j + 2^k$, but the set of numbers constructed is rather sparse. Gallagher proved that the sum of a prime and k powers of 2 has density tending to 1 as $k \to \infty$. It may be that three powers of 2 suffice to represent every sufficiently large odd number.

3. ***(Paul Erdős)*** Suppose that $a_1 < a_2 < \ldots$, and that every positive integer n can be written in the form $n = p + a_i$. Let $A(x)$ denote the number of $a_i \le x$. Erdős proved that there exists such a sequence with $A(x) \ll (\log x)^2$. Can this be improved? Ruzsa proved that such a set must have $A(x) \ge (e^\gamma + o(1)) \log x$.

4. ***(Paul Erdős)*** Call a prime "good" if every even number $\le p - 3$ can be written as $q - r$ where q and r are primes, $p \ge q > r$. Do there exist infinitely many good primes? In this connection, do there exist arbitrarily large numbers x such that $\pi(x) - \pi(x - t) > ct/\log t$ uniformly for $3 \le t \le x$?

5. ***(Imre Ruzsa)*** Let $\mathcal{A}$ be a set of integers in $[1, N]$ such that the equation $x + 3y = 2u + 2v$ has no solution in $\mathcal{A}$. How large can $\mathcal{A}$ be? It is known that the maximum size lies between $\sqrt{N}$ and $N/(\log N)^c$. Expect that the answer is $N^{1-\varepsilon}$.

6. ***(Imre Ruzsa & Vera Sós)*** Let $\mathcal{A}$ be a set of positive integers, and let $r(n)$ be the number of pairs (i, j) such that $a_i + a_j = n$. Put $S(x) = \sum_{n \le x} r(n)$. The Erdős-Fuchs theorem, as strengthened by Jurkat, asserts that $S(x) - cx = \Omega(x^{1/4})$. In contrast to this, if one instead counts solutions of $a_i + 2a_j = n$, similarly for $a_i + 3a_j = n$ etc., then one can get $S(x) = x$. What about $2a_i + 3a_j$?

7. ***(Aleksandar Ivić)*** Let $14 < \gamma_1 \le \gamma_2 \le \ldots$ be the ordinates of the zeros of the zeta function on the critical line, counted according to multiplicity. Let $A(T)$

denote the number of n such that $\gamma_n \le T$ and

$$\max_{\gamma_n \le t \le \gamma_{n+1}} |\zeta(\tfrac{1}{2} + it)| \ge \gamma_{n+1} - \gamma_n.$$

How does $A(T)$ compare with the total number $N_0(T)$ of zeros on the critical line up to T? Prize: 500 Billion Dinars.

*8. (**Jörg Brüdern**)* Let $r(n)$ denote the number of representations of n as a sum of three positive cubes. Determine the density of the set of n for which $r(n) > 0$. It is conjectured that this density is > 0. The best known result in this direction is due to Vaughan (1989), who showed that the number of such n not exceeding N is $> N^{11/12-\epsilon}$. Let $\rho(n)$ denote the number of representations $n = x^3 + y^3 + z^3$ with y, z smooth in the sense that $p|yz$ implies that $p \le n^{\epsilon}$. Show that

$$\sum_{n \le x} \rho(n)^2 \ll N^{\beta}$$

with $\beta < 13/12$. Vaughan proved this with $\beta = 13/12 + \varepsilon$. Let C denote the set of numbers for which $r(n) > 0$.

Problem 1. Gaps. Let $c_1 < c_2$ be consecutive members of C. Trivially $c_2 - c_1 \ll c_1^{(2/3)^3}$. Do better. Prize: \$100. Weaker: Do better for almost all short intervals.

Problem 2. Show that

$$\sum_{c_i \le x} (c_{i+1} - c_i)^2 \ll N^{\gamma}$$

with $\gamma < 1 + (2/3)^3$.

Problem 3. In joint work with A. Balog it was shown that the number of solutions of $c_1 + c_2 = 2c_3 \le N$ with the $c_i \in C$ is $> N^{2-1/4-\epsilon}$. Let $R(N)$ denote the number of solutions of $x_1^3 + x_2^3 + x_3^3 + y_1^3 + y_2^3 + y_3^3 = 2(z_1^3 + z_2^3 + z_3^3) \le N$. It is easily seen that $R(N) \sim \gamma N^2$. To obtain a simple proof of the above results, one would like to have the following inequality:

$$\sum_{c_1 + c_2 = 2c_3 \le N} \rho(c_1)^2 \rho(c_2)^2 \rho(c_3)^2 < N^{2+1/4+\epsilon}.$$

*9. (**Yoichi Motohashi**)* In work with A. Ivić it was shown that

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt = T P_4(\log T) + E_2(T)$$

where P_4 is a polynomial of degree 4 and $E_2(T) \ll T^{2/3}(\log T)^C$. Improve on this. Prize: A special dinner in my home.

10. (Roger Heath-Brown) Let p_n be the nth twin prime. Thus $p_{n+1} - p_n = \Omega((\log p_n)^2)$. Improve on this. Thus, or otherwise, improve on the trivial relation

$$\pi_2(x) - c\int_2^x \frac{dt}{(\log t)^2} = \Omega(1).$$

11. (Roger Heath-Brown) Is it true that

$$\int_0^T \Big|\sum_{n=1}^N a_n n^{-it}\Big|^2 dt \gg T|a_1|^2$$

if $T \geq T_0$? (Here the a_n are arbitrary complex numbers.) This is a question of Ramachandra; he conjectures that the answer is yes, I conjecture that it is no. Certainly $T_0 \geq 2\pi/\log 2$.

12. (Antal Balog) It is known that

$$\sum_{h\leq x}\Big|\frac{1}{x}\sum_{n\leq x}\Lambda(n)\Lambda(n+h) - \mathfrak{S}(h)\Big|^2 \ll \frac{x}{(\log x)^A}.$$

Can it be shown that

$$\sum_{h\leq x}\max_{z\leq y\leq x}\Big|\frac{1}{y}\sum_{n\leq y}\Lambda(n)\Lambda(n+h) - \mathfrak{S}(h)\Big|^2 \ll \frac{x}{(\log x)^A}$$

where $z = (\log x)^A$. Easier:

$$\sum_{h\leq x}\max_{y\leq x}\Big(\frac{1}{y}\sum_{n\leq y}\Lambda(n)\Lambda(n+h)\Big) \ll x.$$

This would be easy if y were restricted to the range $x^\epsilon \leq y \leq x$, since then it would be enough to note that the inner sum is uniformly $\ll \mathfrak{S}(h)y(\log y + h)/\log y$.

13. (Brian Conrey) Is there a sequence a_n and a constant β such that

$$f(x,y) = y^{1/2}\sum_{n=1}^{\infty} a_n\Big(\cos(2\pi nx) - \frac{\sin(2\pi nx)}{2\pi nx}\Big)K_{i\beta}(2\pi ny)$$

satisfies the functional equation $f(z) = f(-1/z)$? Here K is the Bessel function. If so, then the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ satisfies a functional equation with two gamma factors. Note that if $(\sin 2\pi nx)/2\pi nx$ were omitted from the above, then the question would reduce to the question of the existence of even Maass wave forms.

14. (Hugh Montgomery) Assume RH. In 1935, Wintner proved that the normalized error term $g(y) = (\psi(e^y) - e^y)/e^{y/2}$ in the prime number theorem has an asymptotic distribution function $F(x)$. Wintner also proved that $0 < F(x) < 1$ for any finite x, and that the distribution has moments of all orders. Show that F is strictly increasing. That is, there is no interval of positive length on which F is constant. Also, show that F is continuous. These assertions are trivial if the ordinates $\gamma > 0$ of the zeros of $\zeta(s)$ are linearly independent over $\mathbb{Q}$.

15. ***(Hugh Montgomery)*** Show that

$$\sum_{|d|\le D}^{*}\Big|\sum_{0<n\le X}\mu(n)^2 a_n\Big(\frac{d}{n}\Big)\Big|^2 \ll (X+D)(XD)^{\varepsilon}\sum_{0<n\le X}\mu(n)^2|a_n|^2.$$

Here the outer sum on the left is restricted to primitive quadratic discriminants.

16. ***(Paul Erdős)*** The distribution function $F(x)$ of $\phi(n)/n$ is singular. Prove that there is no real number x such that $F'(x)$ exists and has a positive finite value. It may even be that there is no x for which the one-sided derivative exists and has a positive finite value.

17. ***(Andrew Odlyzko)*** Suppose that $f(z)$ is a polynomial of the form $f(z) = 1+\sum_{j=1}^{d} a_j z^j$ where $a_j = 0$ or 1. (a) Show that f cannot have a multiple zero in the complex plane, other than on the unit circle $|z| = 1$. (b) What is the maximal multiplicity of a zero of $f(z)$ at -1, as a function of d? (c) Are most of the polynomials $f(z)$ of this form irreducible?

18. ***(Andrew Odlyzko)*** Write $d_i = p_{i+1} - p_i$. The value of d_i that occurs most frequently, for $p_i \le x$, is called a *champion*. Show that the champions tend to infinity with x. Assuming a quantitative form of the prime k-tuple conjecture, it can be shown that the champions are 4, 2, 6, 30, $210, \ldots$, and subsequently $\prod_{p\le y} p$.

19. ***(Gérald Tenenbaum)*** Suppose that $\mathcal{A}$ is a set of positive integers. Let $\delta(\mathcal{A})$ denote the logarithmic density of $\mathcal{A}$. Let $\mathcal{M}(\mathcal{A}) = \{ma : m \ge 1, a \in \mathcal{A}\}$. The set $\mathcal{A}$ is called a *Behrend set* if $\delta\mathcal{M}(\mathcal{A}) = 1$. A Behrend set $\mathcal{A}$ is called a *witness* if $\delta(\mathcal{A}\setminus\mathcal{A}') = 0$, $\mathcal{A}'\subseteq\mathcal{A}$ implies that $\mathcal{A}'$ is Behrend. Suppose that $\mathcal{W}$ is a witness. It is easy to show that if $\delta(\mathcal{W})$ exists and $\delta\mathcal{M}(\mathcal{A})\cap\mathcal{W} = \delta\mathcal{W}$ then $\mathcal{A}$ is Behrend. It is also easy to see that if $(a,q) = 1$ then the arithmetic progression $a \pmod q$ is a witness. The set $\mathcal{A} = \{n : \omega(n) \le \log\log n\}$ is Behrend. Is it a witness?

20. ***(Andrzej Schinzel)*** Let $q \in \mathbb{Z}[x]$ have degree 2. Do there exist infinitely many integers x such that the largest prime factor of $q(x)$ is $< x^{\sqrt{5}/4}$. The expected answer is x^{ε}. In particular, what if $q(x) = 4x^2 + 4x + 9$.

21. (Andrzej Schinzel) Let $f \in \mathbb{Z}[x]$, with at least three distinct zeros in $\mathbb{C}$. Is it true that $f(x)$ has a prime factor of multiplicity exactly 1, for all sufficiently large x.

22. (Andrzej Schinzel) Show that the set of numbers not of the form $n - \varphi(n)$ has positive lower density. See J. Browkin and A. Schinzel, "On integers not of the form $n - \varphi(n)$."

23. (Andrzej Schinzel) What is the least positive number n with the property that $n2^k - 1$ is composite for all $k \geq 0$? It is known that 509203 is a number with this property.

24. (John Friedlander) Let α and β be fixed, $0 \leq \alpha < \beta \leq 1$. Let $f \in \mathbb{Z}[x]$ be irreducible over $\mathbb{Q}$. Let $r(m)$ denote the number of solutions of $f(a) \equiv 0 (\bmod m)$ with $\alpha m \leq a \leq \beta m$. Hooley proved that $\sum_{m \leq x} r(m) \sim c_f(\beta - \alpha)x$ where $c_f > 0$ is a certain constant. Duke, Friedlander and Iwaniec showed that $\sum_{p \leq x} r(p) \sim (\beta - \alpha)\pi(x)$ when $f(x) = ax^2 + bx + c$ and $b^2 - 4ac < 0$. Extend this to all irreducible f.

25. (Jürgen W. Sander) Let $\mathcal{H}(x,y) = \{(u,v) \in \mathbb{R}^2 : |u-x||v-y| < 1\}$, and put $\mathcal{B}(r) = [0,r]^2$. A set $\mathcal{M}$ of points (x_i, y_i) in $\mathcal{B}(r)$ is called *r-admissible* if $(x_i, y_i) \notin \mathcal{H}(x_j, y_j)$ whenever $i \neq j$. Show that

$$\lim_{r\to\infty} \frac{1}{r^2} \max_{r-\text{admis}\mathcal{M}} |\mathcal{M}| = \frac{1}{\sqrt{5}}.$$

That the liminf is at least this large can be seen by taking the (x_i, y_i) to be points of an appropriate lattice.

26. (Peter Elliott) Show that every positive rational number r can be written in the form $r = \frac{p+1}{q+1}$ where p and q are primes. Easier: Show that there is a number k such that every positive rational number r can be expressed in the form $r = \prod_{i=1}^{k}(p_i + 1)^{\varepsilon_i}$ where $\varepsilon_i = \pm 1$, p_i prime. Easier still: Allow k to depend on r. This last problem is equivalent to showing that the numbers $p+1$ generate the multiplicative group Q^* of positive rational numbers. Let Γ denote the subgroup of Q^* generated by the numbers $p+1$. Elliott has shown that the index of Γ in Q^* is at most 3.

27. (R. A. Rankin) Let $\mathcal{A}$ be a set of positive integers containing no three members in arithmetic progression, and let $A_3(x)$ denote the number of members of $\mathcal{A}$ not exceeding x. It is known that $A_3(x)$ can be as large as $x\exp(-c\sqrt{\log x})$ where $c = 2\sqrt{2\log 2}$. Improve on this. Simply a reduction in the value of c would already be interesting.

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