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Tagungsbericht - 12/1996
Funktionentheorie: konforme und quasikonforme Abbildungen
24.03-30.03.1996

The meeting was organized by

F. W. Gehring (Ann Arbor), R. Kühnau (Halle), St. Ruscheweyh (Würzburg).

Forty-nine mathematicians from eleven countries took part in the meeting. The scientific activities consisted of three different parts:

Expository Lectures, Specialized Talks, Informal Seminars.

The Expository Lectures were intended to give participants a survey of areas where there had been recent important advances or new fields which could be expected to play an important role in complex analysis in the future. The four areas selected for the meeting are as follows

1. **Classical Geometric Function Theory** - C. Pommerenke lectured on recent surprising developments in the global integrability of the derivatives of conformal mappings and the role the computer has had in this work.

2. **Quasiconformal Geometry** - K. Astala explained the close connection between quasiconformal mappings and holomorphic motions and showed how the latter method was used to solve a 40 year old problem. J. Heinonen then sketched a radically new way for deriving global properties for quasiconformal mappings from the local dilatation condition, a method which will permit researchers to extend the theory to general metric spaces and, in particular, the Heisenberg and Carnot groups.

3. **Potential Theory and Harmonic Measure** - A. Solynin discussed some classical extremal problems for harmonic measure and explained a general method by which these and other long outstanding questions could be settled.

4. **Circle Packing** - K. Stephenson discussed analogues between circle packing and complex analysis and illustrated how circle packing has been used to make substantial progress on certain classical conjectures such as the Kreisnormierungsproblem of Koebe. The ideas and methods described in these lectures are very new and were of considerable interest to most participants.

The shorter Specialized Talks were in large part focussed on topics in the three of the four areas mentioned above which were already familiar to most participants.

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1. **Classical Geometric Function Theory** - M. Bonk gave a new way to derive an inequality for the euclidean length of a hyperbolic geodesic using weights. J. Garnett outlined recent progress on a problem concerning interpolating Blaschke products. A. Grinshpan discussed Grunsky operators and quasiconformal extendability. K. Øyma considered extensions of interpolation theorems. D. Prokhorov pointed out the relation between the equations of Hele-Shaw in mechanics and of Loewner-Kufarev in function theory. S. Rohde and A. Volberg spoke on the Julia sets of complex dynamics.

2. **Quasiconformal Geometry** - V. Andrievskii spoke on Erdös-Turán theorems for quasiconformal circles and arcs. W. Bergweiler gave a new application of quasiconformal mappings to iteration theory and value distribution. C. Earle applied the frame criterion to answer an open problem for Riemann surfaces. S. Krushkal, J. Krzyz and E. Reich spoke on various aspects of quasiconformal reflection and extension. S. Rickman exhibited a quasiregular mapping with branching on a wild Cantor set and K. Strebel talked on extremal mappings with given boundary values.

3. **Potential Theory and Harmonic Measure** A. Baernstein used his well known inequality to discuss the sets where univalent functions are large. P. Duren discussed an extension of the notion of capacity. W. Hengartner and T. Suffridge explained similarities and differences between the theory of injective analytic and harmonic mappings. R. Laugesen compared the eigenvalues of the Laplacian for various domains and surfaces while O. Schramm discussed the problem of percolation, currently of interest to physicists.

The two Informal Seminars consisted of a number of short, sometimes spontaneous, presentations of results and/or problems together with a free and completely unstructured discussion of issues related to topics which had been raised in earlier talks. The purpose of these seminars was to

1. give all participants an opportunity to speak,
2. encourage participants to present open problems,
3. generate free and informal discussion.

These seminars were instituted by the Organizers as experiments. We believe that they were relatively successful in achieving these goals. For example, a presented problem related to numerical analysis was answered during the same session by somebody in Teichmüller theory. This could not have happened except during a free and informal discussion in a group of mathematicians with a broad range of interests and backgrounds.

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Vortragsauszüge

V. ANDRIEVSKII:

Erdös-Turán type theorems on quasiconformal curves and arcs

The theorems of Erdös and Turán mentioned in the title are connected with the distribution of zeros of a monic polynomial with known uniform norm along the segment or the unit disk. We extend these results to a polynomial with known uniform norm along an arbitrary quasiconformal curve or arc. As applications, estimates for the distribution of the zeros of best uniform approximants and of the values of orthonormal polynomials are obtained.

K. ASTALA:

Planar quasiconformal maps, extremal problems and applications

We give an exposition of planar quasiconformal (qc) mappings from the analytic point of view. Of the great variety of connections of qc maps to other areas we recall e.g. how recent qc distortion results give optimal regularity and removability estimates for solutions of general elliptic equations (in plane), optimal bounds of distortion of dimension under holomorphic deformations and precise bounds for the Beurling transform. The main emphasis is, however, on new extremal problems (weighted area estimates) for qc maps which have arisen from and have applications to the theory of homogenization and conductivity properties of composite materials.

A. BAERNSTEIN II:

The size of the set where a univalent function is large

Let S denote the usual class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are univalent in the unit disk. We prove, with $|E|$ denoting 1-dimensional Lebesgue measure,

Theorem 1 *For* $\frac{1}{4} \leq t \leq 1$, *and* $f \in S$,

$$|\{e^{i\Theta} : |f(e^{i\Theta})| > t\}| \geq 4 \sin^{-1}(t^{-\frac{1}{2}} - 1).$$

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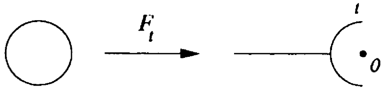
Equality holds $\iff f(z) = e^{-i\alpha}F_t(e^{i\alpha}z)$, *where* $F_t \in S$ *is the map of the disk onto the complement of* $(-\infty, -t) \cup$ *(symmetric circularly fork slit on* $|w| = t$*).*



This result had been conjectured by A. Solynin. It is derived from **Theorem 2**, which asserts that if Ω_1, Ω_2 are domains in $\mathbb{C}$ with $0 \in \Omega_1 \cap \Omega_2, \Omega_2$ circularly symmetric, and Ω_1, Ω_2 satisfying certain conditions, then for the Green functions one has an inequality

$$\int_{-\pi}^{\pi} \Phi(g(re^{i\Theta}, 0, \Omega_1,))d\Theta \leq \int_{-\pi}^{\pi} \Phi(g(re^{i\Theta}, 0, \Omega_2))d\Theta, \; \forall r \in (0, \infty), \; \forall \Phi \text{ convex} \nearrow .$$

R. W. BARNARD:

Approximation theory and the confluent hypergeometric function

We discuss the verification of a 10-year old conjecture on the Kummer function and show how it determines a sharp norm estimate for approximating functions with Gram polynomials.

W. BERGWEILER:

Applications of quasiconformal maps in iteration and value distribution

The Fatou set $F(f)$ of a meromorphic function f is the set where the family $\{f^n\}$ of iterates of f is normal. For a component $U_\circ$ of $F(f)$ we denote by U_n the component of $F(f)$ that contains $f^n(U)$. A component $U_\circ$ is called wandering if $U_n \neq U_m$ for $n \neq m$. A famous theorem of Sullivan says that rational functions do not have wandering domains. This has been extended to certain classes of transcendental functions by Eremenko, Lyubich, Goldberg, Keen, Devaney, Baker, Stallard, and others. We use Sullivan's technique to obtain a result about Baker domains. By definition, a component $U_\circ$ of $F(f)$ is called a Baker domain if $U_p = U_\circ$ for some p, $f^{pn}|_{U_\circ} \to z_\circ \in \partial U_\circ$, and $f^p(z_\circ)$ is not defined. We show that if the family of all meromorphic functions $f_\Phi = \Phi \circ f \circ \Phi^{-1}$, where Φ is K-quasiconformal, depends on only finitely many parameters, then $\cup_{j=\circ}^{p-1} U_j$ contains a singularity of f^{-1}, if $U_\circ$ is a Baker domain of period p.

The result has applications to linear differential equations. Suppose that $u'' + Pu = 0$, where P is a polynomial of degree m. Classical results of Hille give the asymptotics of u in sectors. Now sectors where $u \to 0$ correspond to Baker domains of $f(z) = z - \frac{u(z)}{u'(z)}$.

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Using the above mentioned result on Baker domains we deduce that if P has only k zeros where $k < \frac{m+2}{2}$, then the Nevanlinna deficiency $\delta(0,u)$ satisfies $\delta(0,u) \leq \frac{k}{m+2-k} < 1$.

[joint work with N. Terglane]

M. BONK:

Weights and the Gehring - Hayman theorem

A well-known theorem by Gehring and Hayman says that among all curves in the unit disc $\mathbb{D}$ with given endpoints the hyperbolic geodesic has shortest image length under a conformal map f up to a multiplicative constant only depending on f.

This theorem can be generalized as follows. Suppose w is a positive continuous weight on $\mathbb{D}$. Using w a notion of length and area can be defined corresponding to length and area in the image domain of f if $w(z) = |f'(z)|^2$, f conformal on $\mathbb{D}$.

If w satisfies a condition similar to the Koebe distortion theorem and a growth estimate for area, then an analogue of the Gehring - Hayman theorem is true.

M. CHUAQUI:

Homeomorphic extensions of conformal mappings and the Nehari class

According to a theorem of Gehring and Pommerenke (1985) an analytic function defined in the unit disc and satisfying Nehari's univalence criterion (1949)

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2} \qquad (1)$$

admits a spherically continuous extension to the closed disc. Furthermore, the image of the disc under such a map is a Jordan domain with the single exception of Möbius images of a parallel strip.

Under the stronger assumption

$$|Sf(z)| \leq \frac{2t}{(1-|z|^2)^2}$$

for some $0 \leq t < 1$, Ahlfors and Weill showed in 1962 that f admits a $\frac{1+t}{1-t}$ - quasiconformal extension to the sphere. They gave an explicit formula for the extension in case the mapping was regular up to the boundary.

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In our work with B. Osgood we show that the Ahlfors-Weill extension remains a homeomorphism of the sphere to itself when f satisfies (1) and is not of the exceptional type. No analyticity up to the boundary is required.

P. DUREN:

Robin capacity and conformal mapping

Let Ω be a finitely connected domain containing ∞. Divide the boundary into two disjoint parts : $\partial\Omega = A \cup B$. The Robin function $R(z,\infty)$ is harmonic in $\Omega\setminus\{\infty\}$ with $R(z,\infty) - \log|z|$ harmonic near ∞, $R(z,\infty) = 0$ on A, and $\frac{\partial R}{\partial n}(z,\infty) = 0$ on B. The Robin capacity of A with respect to Ω is $\delta(A) = e^{-\gamma(A)}$, where $\gamma(A) = \lim_{z\to\infty}\{R(z,\infty) - \log|z|\}$. It is known that $\delta(A) \leq d(A)$, the ordinary capacity of A. Thus by conformal invariance $\inf_f d(f(A)) \geq \delta(A)$, where the infimum extends over all conformal mappings with the form $f(z) = z + b_o + \frac{b_1}{z} + \cdots$ near ∞. Equality occurs at least if A is composed of entire boundary components. For such sets $A \subset \partial\Omega$, let $\hat{\Omega} \supset \Omega$ be the domain with boundary $\partial\hat{\Omega} = B = \partial\Omega\setminus A$; and define the Neumann function $N(z,\zeta)$ to be harmonic in $\hat{\Omega}\setminus(\{\zeta\}\cup\{\infty\})$ with $N(z,\zeta) = \frac{1}{|z-\zeta|} + 0(1)$ as $z \to \zeta$, $N(z,\zeta) = -\log|z| + o(1)$ as $z \to \infty$, and $\frac{\partial N}{\partial n}(z,\zeta) = 0$ for $z \in B$. Let $I(\mu) = \int_E\int_E N(z,\zeta)d\mu(z)d\mu(\zeta)$ be the energy integral of a positive unit measure μ on E, the closed interior of A. If $V = \inf_\mu I(\mu)$ is the minimum energy, then the Robin capacity is $\delta(A) = e^{-V}$. There is a unique minimizing measure μ, supported on $A = \partial E$ and given by $d\mu(z) = \frac{1}{2\pi}\frac{\partial R}{\partial n}(z,\infty)|dz|$.

[Joint work with J. Pfaltzgraff and R. Thurman]

C. J. EARLE:

An application of the frame mapping criterion

Let X be a Riemann surface whose universal covering surface is the unit disk. Suppose $f : X \to X$ is a qc map that is in the Teichmüller class of the identity map. Our first result uses a minor extension of a method of Reich and Strebel to generalize a theorem of Teichmüller. It has been observed independently by Fred Gardiner and probably by others.

Theorem 1 *Let p be any point of X and let $X' = X\setminus\{p\}$. If f is as above, then the Teichmüller class of the restricted map $f|X'$ contains a unique extremal map, and it is a Teichmüller mapping with a quadratic differential of finite norm.*

When X is the punctured unit disk Theorem 1 provides a negative answer to a question I

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was asked last year by Genadi Levin. With the help of Theorem 1, some previous results, and some new ideas we obtain

Theorem 2 *On every fiber of the forgetful map from Teich(X') to Teich(X), the infinitesimal Teichmüller distance along the fiber is (strictly) less than the infinitesimal Poincaré distance at every point.*

J. GARNETT:

Interpolating Blaschke products

H^∞ is the ring of bounded analytic functions on the unit disc. An *interpolating* Blaschke product is a Blaschke product

$$B(z) = \prod_n \frac{-\overline{z_n}}{|z_n|} \frac{z - z_n}{1 - \overline{z_n} z}$$

such that

$$\inf_n (1 - |z_n|^2)|B'(z_n)| > 0$$

In 1976 D.E. Marshall proved H^∞ is the uniform closure of polynomials in Blaschke products. In 1994, Artur Nicolau and I proved H^∞ is the uniform closure of polynomials in *interpolating* Blaschke products.

A. GRINSHPAN:

The Grunsky operator and coefficients of univalent functions

We shall consider some coefficient properties of univalent functions related to their Grunsky operators and quasiconformal extendibility. Some bounds of the sharp growth order in n and numerical estimates for coefficients and coefficient differences of analytic and univalent functions in the unit disk will be given. These results depend on a restriction of the norm of the Grunsky operator.

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J. HEINONEN:

Quasiconformal maps in $\mathbb{R}^n$ and beyond

There is an increasing motivation, much of it coming from geometry and combinatorial group theory, to understand quasiconformality in circumstances more general than Euclidean/Riemannian. In this talk, I discuss how to obtain global properties of quasiconformal maps in metric spaces that satisfy certain bounds on their mass and geometry. The traditional smoothness assumption is replaced by the assumption that our metric space admits a suitable Poincaré inequality. It turns out that much of the classical quasiconformal analysis of $\mathbb{R}^n$ can be done in spaces where a Poincaré type inequality holds, and that there are many nontrivial examples of such spaces including non-Riemannian '*Carnot spaces*'. Furthermore, the Poincaré inequality that we need is substantially weaker than what is true in $\mathbb{R}^n$ while many analytic results remain equally strong. Finally, I discuss what quasiconformal invariants metric spaces can harbour. Geometric topology offers deep examples of spaces such as finite polyhedra that are known to be homeomorphic to a round sphere, but very little is known about the geometry of these maps.

[The results are joint with P. Koskela.]

W. HENGARTNER:

Non-parametric minimal surfaces with a univalent Gauss map onto a half-ball

Let S be a non-parametric surface over a domain Ω of the complex plane, i.e.

$$S = \{(u, v, s(u, v)) : (u, v) \in \Omega\}.$$

Then S is a (regular) minimal surface if and only if there is a plane domain D and a univalent orientation-preserving harmonic map $f(z) = u(z) + iv(z)$ from D onto Ω such that $s_z^2 = -af_z^2$, where $\overline{f_{\bar{z}}(z)} = a(z)f_z(z)$. Observe that $\sqrt{a(z)} \in H(D)$ and that $|a(z)| < 1$ on D. Therefore, f is a locally quasiconformal mapping on D. If Ω is a proper simply connected domain of the complex plane, then one may choose for D the unit disk U. The normal vector $\vec{N}, N_3 > 0$, of S, called the Gauss map,

$$\vec{N} = \frac{(2\,\mathrm{Im}\sqrt{a}, 2\,\mathrm{Re}\sqrt{a}, 1 - |a|)}{1 + |a|}$$

depends only on the second dilatation function $a(z)$ of f. In particular, the Gauss map is univalent and maps U onto the upper half-ball if and only if there is a univalent solution $f = u + iv$ of $\overline{f_{\bar{z}}(z)} = z^2 f_z(z)$ which maps D onto Ω. In the following theorem we characterize such domains Ω.

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Theorem *(a) Let Ω be a simply connected concave Jordan domain with four points of convexity. Then there exists a non parametric minimal surface over Ω with a univalent Gauss map onto the upper half-ball if and only if $L_1 + L_3 = L_2 + L_4$. There is essentially only one such minimal surface.*

(b) Let Ω be a simply connected concave Jordan domain with three points of convexity. Then there exists always a non parametric minimal surface over Ω with a univalent Gauss map onto the upper half-ball. There are at most three essentially different minimal surfaces.

(c) If Ω is a simply connected Jordan domain which does not belong to one of the two cases mentioned above, then there is no non parametric minimal surface over Ω with a univalent Gauss map onto the upper half-ball.

E. HOY:

Eine a-priori-Abschätzung für die Gaußsche Krümmung einer Quasiminimalfläche

Es sei $x = x(u, v) : \{(u, v) : u^2 + v^2 < 1\} \to \mathbb{R}^3$ eine Quasiminimalfläche, deren sphärisches Bild eine Umgebung auf der Einheitssphäre ausläßt. O. B. d. A. enthalte diese Umgebung den Nordpol der Einheitssphäre. Für die Gaußsche Krümmung K an einer festen Stelle wird eine Abschätzung hergeleitet, in die für eine gewisse Klasse von Quasiminimalflächen nur eine Zahl $Q > 1$ als (bekanntes) Maß für die Quasikonformität der sphärischen (Gaußschen) Abbildung, eine Schranke für die Projektion des sphärischen Bildes und der Abstand der festen Stelle zum Rand der Fläche eingehen. Insbesondere ergibt sich hieraus für vollständige Quasiminimalflächen ein Satz vom Bernstein-Typ.

S. KRUSHKAL:

Quasireflections over analytic arcs and a quasiconformal dynamic property of the disk

The problem of evaluation of quasireflection coefficients for curves and arcs arises in various questions. The case of arcs has specific features and is less studied. We consider the quasireflections over analytic arcs. These turn out to be intrinsically connected with the geometry of Teichmüller spaces and holomorphic extension by the classical Bernstein-Walsh-Siciak approximation theorem. One of the basic points here is the r^2-property of the disk: its conformal embeddings into the complex plane possesses r^2-quasiconformal extensions over r-level lines of the Green function. We show also that this is a characteristic property of the disk: any simply connected domain which is not a disk does not admit an

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r^2-property.

[joint work with R. Kühnau]

J. G. KRZYŻ:

Conformally natural extensions of quasisymmetric maps

Let M denote the class of all Möbius automorphisms of the unit disk Δ and let QS stand for the class of all quasisymmetric automorphisms of $T = \partial\Delta$. An operator H which assigns to each $\gamma \in QS$ a quasiconformal automorphism $H[\gamma]$ of Δ and satisfies $H[\mu \circ \gamma \circ \nu] = \mu \circ H[\gamma] \circ \nu$ for any $\mu, \nu \in M$ is said to be conformally natural. Douady and Earle constructed an operator $E[\gamma]$ with this property.

An innate problem arises to find some other conformally natural operators. A method leading to a relevant construction is proposed.

R. S. LAUGESEN:

Extremals of zeta functions of Laplacians

Over forty years ago, G. Pólya and G. Szegö showed that for simply connected plane domains, the first eigenvalue of the Laplacian is maximal for a disk, under a conformal mapping normalization. That is, if $f(z)$ is a conformal map of a disk D onto a bounded, simply connected plane domain Ω and if $|f'(0)| = 1$ then

$$\lambda_1(\Omega) \leq \lambda_1(D).$$

We extend this result from the first eigenvalue to the zeta function of the Laplacian, which we show is minimal for the disk:

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j(\Omega)^s} \geq \sum_{j=1}^{\infty} \frac{1}{\lambda_j(D)^s}$$

for all $s > 1$, where the λ_j are the eigenvalues of the Laplacian under Dirichlet boundary conditions.

We further prove similar theorems for simply and doubly connected surfaces having curvature bounded above, and we prove that for a compact two-dimensional Riemannian manifold M_g with smooth boundary, the zeta function of the Laplace–Beltrami operator

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Δ_{wg} is a convex functional of the conformal factor w. (Here w should be positive and smooth on the surface.)

These extremal results for the zeta function all have analogues for the finite sum of reciprocal eigenvalues $\sum_{j=1}^{m} \lambda_j^{-1}$, for each m.

[joint work with C. Morpurgo]

I. NEZHMETDINOV:

Classes of uniformly convex and starlike functions as dual sets

Let $\mathcal{A}$ be the class of functions $f(z) = z + a_2(f)z^2 + \ldots$, regular in the unit disk E. A. Goodman (1991) introduced the subclass UST of uniformly starlike functions such that Re $\{(z-\zeta)f'(z)/[f(z)-f(\zeta)]\} > 0$ for any $(z,\zeta) \in E \times E$, and proved, in particular, that a condition of the form $(*) \sum_{n=2}^{\infty} n|a_n(f)| \leq \delta$ is sufficient for $f \in \mathcal{A}$ to be in UST, whatever $\delta \leq \sqrt{2}/2 = 0.70\ldots$ may be, but for no $\delta > \sqrt{3}/2 = 0.86\ldots$. We represent the class UST as the dual set (in St. Ruscheweyh's sense) of a certain two-parametric family of functions from $\mathcal{A}$. Then the best value of the constant δ for which the underlined condition $(*)$ still implies $f \in UST$ is proved to equal $0.7963\ldots$. A similar representation is constructed for the subclass UCV of uniformly convex functions. This allows us to deduce necessary and sufficient conditions for the Hadamard convolution of functions from T-δ-neighborhoods (defined by T. Sheil-Small and E. Silvia) of certain subclasses to lie in UST (or UCV) and determine the best radii of such neighborhoods.

K. ØYMA:

Interpolating sequences

A sequence $\{z_n\}$ in the upper half plane is called an interpolating sequence if for every bounded sequence $\{w_n\}$ the interpolating problem $f(z_n) = w_n$ has a solution in H^∞. For $E \subset \mathbb{R}$ let $\omega(z, E)$ be the harmonic measure of E w.r.t. the upper half plane.

Theorem $\{z_n\}$ *is an interpolating sequence if and only if there exist disjoint subsets* $E_n \subset \mathbb{R}, \delta > 0, \quad \varepsilon > 0$, *such that*

$$\begin{cases} \left|\frac{z_n - z_m}{z_n - \bar{z}_m}\right| \geq \delta & \text{for all} \quad n \neq m, \\ \omega(z_n, E_n) \geq \varepsilon. & \end{cases}$$

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D. PARTYKA:

The generalized harmonic conjugation operator and the Hilbert transformation

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and let $\mathbf{Q}_{\mathbb{T}}$ be the class of all homeomorphic self-mappings of $\mathbb{T}$ which admit a quasiconformal extension to $\mathbb{D}$. For every integrable function $f \in L^1(\mathbb{T})$ define

$$f_{\mathbb{D}}(z) := \frac{1}{2\pi}\int_{\mathbb{T}} f(u)\mathrm{Re}\frac{u+z}{u-z}|du|, \quad z \in \mathbb{D},$$

and set

$$\boldsymbol{H} := \{f \in \mathrm{Re}\, L^1(\mathbb{T}) : \int_{\mathbb{D}} |(f_{\mathbb{D}})'|^2 dS < \infty\},$$

where $dS := dxdy\,(z = x+iy)$. For $f, g \in \boldsymbol{H}$ we identify f and g and write $f = g$ whenever $f - g$ is a constant almost everywhere on $\mathbb{T}$. Then $\boldsymbol{H}$ is a real Hilbert space with the inner product

$$(f,g)_{\boldsymbol{H}} := \frac{1}{2}\mathrm{Re}\int_{\mathbb{D}} (f_{\mathbb{D}})'\overline{(g_{\mathbb{D}})'}dS, \quad f,g \in \boldsymbol{H}.$$

With any $\gamma \in \mathbf{Q}_{\mathbb{T}}$ we can associate a linear hoeomorphism $A_\gamma : \boldsymbol{H} \to \boldsymbol{H}$ called the generalized harmonic conjugation operator assigned to γ. Given a domain $\Omega \subset \hat{\mathbb{C}}$ let $A^2(\Omega) := \{F \in A(\Omega) : \int_\Omega |F|^2 dS < \infty\}$. The class $A^2(\Omega)$ with the usual inner product

$$(F,G)_2 := \int_\Omega F\overline{G}dS, \quad F,G \in A^2(\Omega),$$

is a complex Hilbert space. We recall that the two-dimensional Hilbert transformation T_Ω maps every integrable analytic function F on Ω into an analytic function $T_\Omega(F) \in A(\Omega)$ by means of the singular integral

$$T_\Omega(F)(z) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi}\int_{\Omega \setminus \mathbb{D}(z,\varepsilon)} \frac{\overline{F(u)}}{(u-z)^2}dS_u, \; z \in \Omega,$$

where $\mathbb{D}(z,\varepsilon) := \{u \in \mathbb{C} : |u-z| < \varepsilon\}$. The Hilbert transformation T_Ω is an antilinear bounded operator of $A^2(\Omega)$ into itself and its supremum norm satisfies $||T_\Omega|| \leq 1$.

If Ω is bounded by a quasicircle $\Gamma \subset \mathbb{C}$ then a welding homeomorphism γ of Γ belongs to $\mathbf{Q}_{\mathbb{T}}$ and the operators T_Ω and A_γ are strictly related. We discuss this topic and present examples of applications.

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CH. POMMERENKE:

The integral means spectrum of univalent functions

Let f be a conformal map of $\mathbb{D}$ onto a bounded domain $\subset \mathbb{C}$ and define

$$\beta_f(p) = \limsup_{r\to 1} \left(\log \int_0^{2\pi} |f'(re^{it}|^p dt\right) / \log \frac{1}{1-r} \quad (-\infty < p < +\infty).$$

The integral means spectrum of (bounded) univalent functions is defined by

$$B(p) = \sup\{\beta_f(p) : f \text{ bounded univalent}\}.$$

It is only partially known, in particular by recent work of Carleson, Jones and Makarov. For instance $B(1)$ determines the growth behaviour of the coefficients. Brennan, Carleson, Jones and Kraetzer have conjectured that

$$B(p) = \frac{1}{4}p^2 \text{ for } -2 \le p \le 2 \qquad \text{(BCJK conjecture)}$$

which would imply that

$$B(p) = |p| - 1 \text{ for } |p| \ge 2.$$

D. V. PROKHOROV:

Starlikeness of solutions of the Hele-Shaw equation

The Hele-Shaw equation

$$\operatorname{Re} \dot{w}/zw' = 1/|w'|^2, \quad |z| = 1, \quad w|_{t=0} = f_0(z),$$

is known in mechanics and applications. This is a kind of the Loewner-Kufarev differential equation with boundary feedback. There was posed a problem to describe those geometric properties of the initial mapping $f_0(z)$ which are inherited by all the solutions $w(z,t)$, $t > 0$. We prove the

Theorem *If $f_0(z)$ is starlike and the Hele-Shaw equation has its solutions $w(z,t)$ on $(0,t_0)$, then $w(z,t)$ are also starlike.*

A similar statement is true for $w(z,t)$ mapping the exterior of the unit disc onto domains with convex complement.

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E. REICH:

Some examples of least non-analytic extensions (with applications to quasiconformal mappings)

Let $D = \{|z| < 1\}$. Suppose $f \in C^1(\partial D)$. There exists a unique extension $F_0(z)$ of f to $D \cup \partial D$ with the property that

$$m_0[f] = \inf\{||\bar\partial F||_\infty : F \text{ is an extension of } f \text{ to } D \cup \partial D\} = ||\bar\partial F_0||_\infty.$$

In the simplest situation

$$(*) \qquad \bar\partial F_0 = m_0[f]\frac{\overline{\Phi_0'(z)}}{\Phi_0'(z)},$$

with Φ_0 holomorphic in D and $Area[\Phi_0(D)] < \infty$. We investigate the question of whether F_0 indeed is of the type (*) when f is a trigonometric polynomial.

A related question for quasiconformal self-mappings of D is whether or not knowledge of the supremum of moduli ratios $M(f(Q))/M(Q)$ for quadrilaterals Q with domain D and vertices on ∂D is sufficient to determine the maximal dilatation of the extremal extension of a quasisymmetric homeomorphism f.

S. RICKMAN:

Quasiregular maps with branching on wild Cantor sets

We have shown that a K-quasiregular map can have arbitrary high local index on a sufficiently regular Antoine's necklace with K an absolute bound, more precisely:

Theorem *Let $A \subset \mathbb{R}^3$ be a necklace of Antoine with a selfsimilar construction. There exist finite positive numbers $K, M,$ and N such that for every $\lambda > N$ there is a K-quasiregular map $f : \mathbb{R}^3 \to \mathbb{R}^3$ of polynomial type with local index $i(x, f) = \lambda$ for $x \in A$, and $i(x, f) \leq M$ for $x \in \mathbb{R}^3 \setminus A$.*

Earlier Rickman used his method in the Picard example construction to prove that high local index can occur on a tame Cantor set. A modified method applies also here for the wild Cantor set A. S. Semmes showed that the function $w(x) = \min\{1, \text{dist}(x, A)^s\}$ is a strong A_∞ weight in his terminology when $s > 0$ and that there cannot be any bi-Lipschitz map from the metric space $(\mathbb{R}^3, D_w)$ to $\mathbb{R}^3$ when $s > 3$, where D_w is the metric associated to the weight w. The map in the theorem can be used to answer negatively a stronger version of a question raised by Semmes:

Corollary *There is a strong A_∞ continuous weight w in $\mathbb{R}^3$ such that the associated metric space $(\mathbb{R}^3, D_w)$ is not bi-Lipschitz equivalent to $\mathbb{R}^3$ although $\mathbb{R}^3$ receives regular maps (in the sense of G. David and S. Semmes) from $(\mathbb{R}^3, D_w)$.*

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The method gives also a proof for the existence of a quasiconformal group, which is not topologically equivalent to any Moebius group but which is semiconjugate to one by a quasiregular map.

[Joint work with Juha Heinonen]

S. ROHDE:

Porosity of Collet-Eckmann Julia sets

Consider a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ together with its Julia set J. A weak notion of hyperbolicity is as follows: f is said to satisfy the Collet-Eckmann condition, if there are $C > 0, \lambda > 1$, such that

$$|{f^n}'(f(c))| \geq C\lambda^n \qquad (CE)$$

for all critical points c of f in J whose forward orbit does not meet other critical points, and for all n.

For instance, it is known that the set of real $a \in [-2, 0]$ for which $z \to z^2 + a$ satisfies (CE) (and $0 \in J$) has positive measure. Based on pioneering work of Przytycki, we show that the Julia set of f has a certain porosity property if f has (CE), $J \neq \hat{\mathbb{C}}$ and no parabolic periodic point. Roughly this porosity property says that around every $x \in J$, the scales on which the complement of J contains a disk of radius comparable to the scale have density uniformly bounded away from 0. This implies $\dim J < 2$ for the Minkowski - (thus Hausdorff -) dimension, answering a question of Graczyk and Smirnov.

[joint work with F. Przytycki]

G. SCHMIEDER:

Lokalisierung der Ableitungsnullstellen eines Polynoms

Es werden Einschließungssätze für die Ableitungsnullstellen eines Polynoms in Termen seiner Nullstellen behandelt, z. B.:

Satz *Es sei* $p(z) = \prod_{j=1}^{n}(z - z_j)$ *ein komplexes Polynom mit* $n > 1$ *sowie* $a \in \mathbb{C}$ *und* $r > 0$. *Für* $j = 1, \ldots, n$ *gelte* $|z_j - a| < r$ *und auf* $|z - a| = r$ *liege weder eine Nullstelle noch eine Ableitungsnullstelle von* p. *Gilt dann für jedes* $k \in \{m+1, \ldots, n\}$ *eine der Ungleichungen*

$$(i)\quad |a - z_k| \leq r(1 - 2^{\frac{n-m}{m}}) \quad oder \quad (ii)\quad |a - z_k| \geq r(1 + 2^{\frac{n-m}{m}}),$$

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so enthält $K_r(a) = \{|z-a| \leq r\}$ *genau* $M-1$ *Nullstellen von* P', *wenn* M *die Anzahl der* $j \in \{1, \ldots, n\}$ *mit* $|a - z_j| \leq r$ *bezeichnet.*

O. SCHRAMM:

The conformal invariance conjecture for critical percolation

Percolation can be heuristically described as the study of connected components of random subsets of R^d (or other manifolds/metric spaces). The conformal invariance conjecture states that important properties of percolation in a domain D in the plane are invariant under conformal homeomorphisms of D. In the talk, a new model for percolation will be described, which is better suited for the study of this conjecture. The new model is based on Voronoi tessellations. We prove a conformal invariance theorem for Voronoi percolation (which is neither stronger nor weaker than the conjecture).

[joint work with I. Benjamini]

A. YU. SOLYNIN:

Estimates and extremal problems for harmonic measures

Two methods of estimating harmonic measures are presented.

Using the first method we obtain numerous new estimates for the harmonic measures of continua, concerning several problems which have been considered for a long time by T. Hall, D. Gaier, J. A. Jenkins, W. K. Hayman and W. Wu, and others. We usually give a complete solution of a problem, including a description of the extremal configurations. Our approach is based on some new results in the Modul theory developed by J. A. Jenkins and K. Strebel.

The second method is based on the polarization transformation. It leads to complete solutions in problems connected with the integral means of harmonic measures, Green functions and the Poincaré metrics.

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K. STEPHENSON:

Classical analysis & circle packing

Two survey lectures were presented on circle packings and their connections with classical analytic functions and conformal geometry.

I. A circle packing is a configuration P of circles with a specified abstract pattern K of tangencies. Each packing has dual nature as combinatoric and geometric object.

Theorem (Koebe - Andreev - Thurston) *Given a triangulation K of a topological surface S, there exists a unique conformal structure on S and an essentially unique circle packing P_K in S in the corresponding intrinsic metric so that P_K satisfies: (1) the pattern of tangencies specified by K, (2) circles with mutually disjoint interiors, and (3) the carrier fills S.*

Removing requirements for extremality, univalence, and completeness and introducing natural branch points, there may be a huge variety of packings sharing combinatorics, and this suggests the following.

Definition A discrete analytic function $f : Q \to P$ is a map between circle packings which preserves tangency and orientation.

We obtain analogues of many classical maps: discrete entire, rational, polynomial, Blaschke product, and so forth. Likewise, analogues of classical theorems: discrete Schwarz-Pick, uniformization, Liouville, and so on. Moreover, these discrete objects converge to their classical analogues as the underlying packings become finer. A pleasant surprise is the faithfulness of the discrete objects to the classical modes of behavior, even for very coarse packings.

II. The second lecture highlighted emerging links with specific classical topics:
The '*type*' problem, discrete and circle packing extremal length, harmonic measure and discrete potential theory, tailored random walks on circle packings, '*packable*' surfaces in Teichmüller space, work (with Phil Bowers) on discrete Belyi functions and '*dessins d'Enfants*' of Grothendieck, and applications in graph embedding. The lecture concluded by describing methods of circle packing which motivated recent progress by Z - X. He and O. Schramm on Koebe's well-known Kreisnormierungsproblem.

K. STREBEL:

Point shift differentials and extremal quasiconformal mappings

We consider quasiconformal selfmappings f of the disk with given boundary values. For fixed $z_\circ$ the set of image points $\{f_\circ(z_\circ)\}$ for all extremal mappings $f_\circ$ is called the variability set $V[z_\circ]$ of $z_\circ$. Extremal mappings f_m which take $z_\circ$ into points $w_m \notin V[z_\circ]$ are

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called point shift mappings. They are Teichmüller mappings associated with quadratic differentials φ_m of norm $||\varphi_m|| = 1$, called point shift differentials. It is shown that the φ_m form Hamilton sequences for the extremal mappings which take $z_\circ$ into boundary points of $V[z_\circ]$. From that it follows, using the frame mapping criterion, that the φ_m depend continuously in norm of the point w_m. Their constant dilatation $K[w_m]$ is called the dilatation function. Based on a variational method for point shift mappings it is shown that the level lines of $K[w_m]$ are Jordan curves separating $V[z_\circ]$ from $\partial \mathbf{D}_w$. Thus, $V[z_\circ]$ is a compact, connected set without holes.

T. SUFFRIDGE:

Harmonic mappings in the complex plane

Let S_H denote the complex valued mappings f that are harmonic and sense preserving in the unit disk and that satisfy $f(0) = 0$ and $f_z(0) = 1$. Also, let $S_H^0 = \{f \in S_H : f_{\bar{z}}(0) = 0\}$. We introduce some slit mappings and show that the inner mapping radius of the image of the unit disk can be arbitrarily near to 4 for f in S_H and arbitrarily near 2 for f in S_H^0. This gives a counterexample to the conjecture of Sheil-Small that the inner mapping radius of the image of the disk for functions in S_H is bounded by $\pi/2$. In addition, we give some results concerning harmonic sense preserving polynomials.

A. VOLBERG:

Green's boundary condition and an estimate of Hausdorff dimension of Julia sets

We consider a condition similar to a quasihyperbolic boundary condition but in which quasihyperbolic distance is replaced by Green's function. Then one can prove that exactly as in the case of Hölder domains the Hausdorff dimension of the boundary is strictly less than d (domains are in $\mathbb{R}^d$). The result is similar to those of Smith-Stegenga and Koskela-Rohde and follows the idea of a Jones-Makarov theorem $B(p) = p-1+O((p-1)^2)$, $p \sim 2$. One can combine the result with the recent estimates of Collet-Eckmann maps.

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A. VOLBERG:

Hunt-Muckenhoupt-Wheeden theorem with and without the inverse Hölder inequality

One can give a very simple proof of the classical Hunt-Muckenhoupt-Wheeden theorem about A_2 weights using only the formula of Green. We can give 2 such proofs. The second of them indicates an unexpected connection of this subject with Bellman's price function. These new proofs can be applied to obtain matrix A_2 condition.

Berichterstatter: Prof. Dr. Stephan Ruscheweyh, Würzburg

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Problems

This is a collection of problems presented at the Oberwolfach meeting on Function Theory, March 1996. They have not been edited, except for some typographical unifications.

K. ASTALA:

Problem *Let $B = \{|z| < 1\}$; recall the Beurling transform*

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2} dm(\zeta).$$

If $E \subset B$ and $k > 0$ find optimal bounds for the quantity

$$\int_{B\setminus E} |S\chi_E| dm + k \int_E |S\chi_E| dm,$$

when χ_E is the characteristic function of E.

The bound should depend only on k and $|E|$; recall that when $k = 0$

$$\int_{B\setminus E} |S\chi_E| dm \leq |E| \log \frac{\pi}{|E|}$$

and equality holds for $E = B(r) = \{z : |z| < r\}$.

The problem is connected to finding optimal bounds for conductivity properties of compound materials.

A. BAERNSTEIN II:

Conjecture *Let $f : \mathbb{C} \to \mathbb{C}$ be a Lipschitz function with compact support, and $2 < p < \infty$.*

Then

$$\int_{\mathbb{C}} (|\partial f| - (p-1)|\bar{\partial} f|)(|\partial f| + |\bar{\partial} f|)^{p-1} dm \leq 0. \qquad (1)$$

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The conjecture is suggested by work of Burkholder on sharp inequalities for Martin-Gall transforms. If true, the conjecture will imply a conjecture by T. Iwaniec that the L^p norm of the Beurling transform S satisfies

$$||S||_P = \max\,(p-1, \frac{1}{p-1}),\ 1 < p < \infty. \tag{2}$$

From work of Iwaniec, it is known that the validity of (2) in the limit $p \to \infty$ implies the sharp area distortion inequality for QC mappings, which was proved by Astala in 1994.

We note that (1), for $p = 2$, holds with equality for every Lipschitz f with compact support. For f of the form

$$f(re^{i\Theta}) = g(r)e^{i\Theta},$$

with $g \in \mathrm{Lip}_c(\mathbb{R}^+)$ satisfying $r|g'(r)| \leq g(r)$, (1) holds with equality for every $p \in [2, \infty)$.

P. DUREN:

The Robin capacity $\delta(a)$ of a set $A \subset \partial\Omega$ with respect to a domain Ω containing $\{\infty\}$ is no larger than the ordinary capacity: $0 \leq \delta(A) \leq d(A)$. Thus $d(A) = 0 \Rightarrow \delta(A) = 0$. However, the converse is false even for Jordan domains. There exist a Jordan curve $\partial\Omega$ and a subset $A \subset \partial\Omega$ with $\delta(A) = 0$ and $d(A) > 0$.

On the other hand, if $\partial\Omega$ is a quasicircle this cannot happen; then for $A \subset \partial\Omega$, $\delta(A) = 0$ if and only if $d(A) = 0$. [Results of P. Duren, J. Pfaltzgraff, and R. Thurman, to appear]

Problem *Describe the Jordan curves $\partial\Omega$ with property that $\delta(A) = 0 \Leftrightarrow d(A) = 0$ for all $A \subset \partial\Omega$.*

C. J. EARLE and F. P. GARDINER:

Let φ be a nontrivial quadratic differential on a compact Riemann surface X of genus $g \geq 2$. Let $G(\varphi)$ be the group of Teichmüller mappings of X onto itself whose initial and terminal quadratic differentials are scalar multiples of φ (we permit complex scalars). By definition the Teichmüller disk $D(\varphi)$ is the subset of the Teichmüller space of X determined by all the Teichmüller maps whose domain is X and whose initial quadratic differential is a scalar multiple of φ. If f is such a map then so is $f \circ g$ for every g in $G(\varphi)$. So the group $G(\varphi)$ acts on the disk $D(\varphi)$. This action is like the action of a Fuchsian group on the unit disk.

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Problem *Given $g \geq 2$ find φ and X to minimize the Poincaré area of the quotient space $D(\varphi)/G(\varphi)$.*

Comment. Examples of Veech show that the minimum is always finite.

D. GAIER:

In journal publications, and also at conferences, one can observe that the nature of mathematical problems becomes more and more specialized. It often happens that even the formulation of a problem is full of technical details that only the specialist understands who has worked in that area himself, and the same is true also for the result. In contrast, the ideal problem permits a simple formulation, in clear language and without technicalities; the result is simple to express and understandable to "everybody"; while the proof of the result is non-trivial, sophisticated, and contains new ideas sometimes from other mathematical areas. A typical problem of this sort is: Does $f \in S$ imply $|a_n| \leq n$?

I present four related problems on the analytic dependence of domain functionals.

Problem 1 *Let $G(r)$ be the doubly connected domain bounded by the unit square Q (centered at 0) and by the circle $\{z : |z| = r\}$, with $0 < r < 1$. Let $M(r)$ be the modulus of $G(r)$. Is $M(r)$ an analytic function of r? (Power series expansion of $M(r)$ at each r_0 with $0 < r_0 < 1$.)*

Problem 2 *Let $G(x)$ be the doubly connected domain bounded on the exterior by the upper half of the unit square Q and the lower half of $\{z : |z| = 1\}$, and on the interior by the slit $\{z : 0 \leq z \leq x\}$, with $0 < x < 1$. Is the modulus $M(x)$ of $G(x)$ an analytic function of x?*

Problem 3 *Let γ_1, γ_2 be two Jordan arcs lying in $\{z : 0 \leq \text{Im } z \leq 1\}$, each connecting $\{z : \text{Im } z = 0\}$ to $\{z : \text{Im } z = 1\}$. Let h be the horizontal distance between γ_1 and γ_2, and let $m(h)$ be the modulus of the quadrilateral with γ_1 and γ_2 as opposite sides. Consider this configuration with fixed γ_1, γ_2 but varying h. Is $m(h)$ an analytic function of h?*

Note: The asymptotic behavior of $m(h) - h$, as $h \to \infty$, was studied in two recent papers by Gaier and Hayman.

Problem 4 *Let $\Gamma : z = z(t)$ with $0 \leq t < \infty$ be an analytic Jordan arc (t is arc length) such that $0 \notin \Gamma$ and $z(t) \to \infty$ as $t \to +\infty$. Let $\Gamma_s : z = z(t)\,(t \geq s)$ be a subarc of Γ, and let $R(s)$ be the conformal radius of $\mathbb{C} \backslash \Gamma_s$ at zero. Is $R(s)$ an analytic function of s?*

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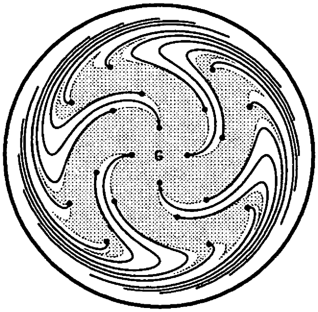
D. GAIER, W. LAUF, G. SCHMIEDER:



Let k be a conformal mapping of the unit disk $\mathbb{D} \subset \mathbb{C}$ onto a simply-connected and bounded domain $G \subset \mathbb{C}$. Then $\Sigma(G)$ denotes the group of all conformal self-mappings of G endowed with the topology of *uniform* convergence in G, i.e.,

$$d(f,g) := \sup_{w \in G} |f(w) - g(w)|, \quad f, g \in \Sigma(G).$$

Problem *$\Sigma(G)$ connected $\Rightarrow$ k continuously extensible to $\partial\mathbb{D}$?*

A. GRINSHPAN:

Let $S(\kappa)$ be the class of functions $f(z) = z + c_2 z^2 + \ldots, f(0) = f'(0) - 1 = 0$, regular and univalent in the open unit disk E with $||G_f|| \leq \kappa$, where G_f is the Grunsky operator.

Problem 1 *Let $f(z) = z + c_2 z^2 + \ldots \in S^{1/2}(\kappa) = \{f \in S : \sqrt{f(z^2)} \in S(\kappa)\}$, $\kappa < 1$. Estimate the coefficients $c_n, n \geq 2$.*

Problem 2 *Let $f(z) = z + c_3 z^3 + c_4 z^4 + \ldots \in S(\kappa)$, $\kappa < 1$. Estimate $|f(z)|$, $z \in E$.*

A. GRINSHPAN:

Let S be the class of regular and univalent functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disk.

It is known that $\max |a_n|, n \geq 2$, is realized by $(n-1)$-symmetric functions in the subclass S^M of functions $f \in S, |f(z)| < M$ $(M = M_n$ is close to 1), and in the subclass $S_\kappa(\infty)$

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of functions $f \in S$ which have a κ-quasiconformal extension $\hat{f}$ onto the whole complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \hat{f}(\infty) = \infty$, $\kappa = \kappa_n$ is small).

Problem *Describe other sets Q of close to the identity functions $f \in S$ such that $\max_Q |a_n|, n \geq 2$, is realized by an $(n-1)$-symmetric function.*

J. HEINONEN:

Let f be a quasiconformal mapping of the open unit ball $\mathbf{B}^n$ of $\mathbf{R}^n, n \geq 3$, onto a Jordan domain D in $\mathbf{R}^n$. Then f extends homeomorphically to the boundary ∂D. Assume that ∂D is $(n-1)$- rectifiable (in the sense of Federer) and that the Hausdorff $(n-1)$-measure H_{n-1} of ∂D is finite.

Problem 1 *Is it true that the image at every set $E \subset \partial\mathbf{B}^n$ of positive H_{n-1} measure has positive H_{n-1} measure?*

Problem 2 (Gehring many years ago). *As above but assume that f extends quasiconformally to all of $\mathbf{R}^n$?*

Problem 3 (Heinonen - Koskela). *As above, but remove all the metric assumptions on ∂D. What is the smallest Hausdorff dimension $f(E)$ can have if $E \subset \partial\mathbf{B}^n$ has positive H_{n-1} measure? Is this number independent of the quasiconformal dilatation of f?*

Comments

Problem 1. It follows from recent works of Heinonen, Semmes and Väisälä that the answer to the dual problem $(H_{n-1}(E) = 0 \overset{?}{\Rightarrow} H_{n-1}(fE) = 0)$ is YES if and only if H_{n-1} almost every point of ∂D is a point of inner tangency of D. Moreover, this condition fails only if the set of n-density $(=1)$ points of $\mathbf{R}^n \backslash D$ on ∂D has positive H_{n-1} measure.
(See Heinonen, Semmes, Revista Matem. Iberoamericana, to appear)

Problem 3. Shown by Heinonen - Koskela (Pacific J. Math. 94) that there is a bound $(2K(f))^{\frac{1}{1-n}} > 0$.

P. KOSKELA:

Assume that $\Omega \subset \mathbf{R}^n$ satisfies

$$(*) \qquad k_\Omega(x, x_0) \leq C_1 \log \frac{1}{d(x, \partial\Omega)} + C_2$$

for a fixed $x_0 \in \Omega$ and all $x \in \Omega$. Here

$$k_\Omega(x, x_0) = \inf_{\gamma_x} \int_{\gamma_x} \frac{ds}{d(y, \partial\Omega)}$$

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where the infimum is taken over all rectifiable curves that join x and x_0 in Ω. If $n = 2$ and Ω is simply connected, $(*)$ implies that there exist C_3, C_4 s.t.

$$C_3 \bmod (B_0, F; \Omega) \geq \left(\log \frac{C_4}{\operatorname{diam}(F)}\right)^{-1}$$

for each continuum $F \subset \Omega \backslash B_0$, where $B_0 = B(x_0, d(x_0, \partial\Omega)/2)$. Here $\bmod (B_0, F; \Omega)$ is the modulus of the curve family that joins B_0 and F in Ω, or equivalently, the conformal capacity of B_0 and F relative to Ω.

Problem *Does $(*)$ always guarantee that*

$$C_3 \bmod (B_0, F; \Omega) \geq \left(\log \frac{C_4}{\operatorname{diam}(F)}\right)^{1-n}$$

for fixed constants C_3, C_4 and all continua $F \subset \Omega \backslash B_0$?

R. LAUGESEN:

Let $f(z)$ be a conformal map of the unit disk D onto the bounded, simply connected plane domain Ω. (That is, $f \in S$ is bounded and $\Omega = f(D)$.) Write $\lambda_j(\Omega)$ for the j-th eigenvalue of the Laplacian on Ω with Dirichlet boundary conditions, so that $-\Delta\Psi_j = \lambda_j(\Omega)\Psi_j$ in Ω for the j-th eigenfunction Ψ_j, and $\Psi_j = 0$ on $\partial\Omega$.

Conjecture *The trace of the heat kernel is minimized for D; that is,*

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega)} \geq \sum_{j=1}^{\infty} e^{-t\lambda_j(D)}, \qquad t > 0.$$

Remarks

1. The proposer [1] can prove this only for $t \geq 2/\lambda_1(\Omega)$.
2. Letting $t \to \infty$ recovers the old Pólya-Szegö inequality $\lambda_1(\Omega) \leq \lambda_1(D)$.
3. The conjecture is true asymptotically as $t \to 0$ since $\sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega)} \sim \dfrac{\operatorname{area}(\Omega)}{4\pi t}$ as $t \to 0$ and $\operatorname{area}(\Omega) \geq \operatorname{area}(D)$.

Reference:

[1] R.S. Laugesen and C. Morpurgo, '*Extremals for eigenvalues of Laplacians under conformal mapping*', preprint.

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CH. POMMERENKE:

Let f be a conformal map of the unit disk $\mathbb{D}$ into $\mathbb{C}$. For $-\infty < p < \infty$, let

$$\beta_f(p) = \limsup_{r\to 1} \left[(\log \int_0^{2\pi} |f'(re^{it})|^p dt)/(-\log(1-r)) \right].$$

Makarov has defined the universal integral means spectrum (of bounded univalent functions) by

$$B(p) = \sup\{\beta_f(p) : f \text{ bounded univalent in } \mathbb{D}\}.$$

Its determination would be of great interest for various problems about the boundary behaviour of conformal maps. Starting from the known results and the conjectures of Brennan and Carleson - Jones and furthermore from numerical results using the Julia sets of quadratic polynomials, Philipp Kraetzer made the following

Conjecture

$$B(p) = \begin{cases} \frac{1}{4}p^2 & for \quad |p| \le 2 \\ |p|-1 & for \quad |p| \ge 2 \end{cases} \qquad (BCJK conjecture).$$

E. REICH:

This problem deals with the question of whether a limit of analytic functions, subject to a further condition that describes how good the convergence is, is necessarily analytic. (The question is of interest in connection with a uniqueness condition for extremality of quasiconformal mappings.)

Suppose

(i) $f_n(z)$ is analytic for $|z| < 1$, $\iint_{|z|<1} |f_n(z)| dxdy < \infty$, $\qquad n = 1, 2, 3, \ldots,$

(ii) $\lim_{n\to\infty} f_n(z) = F(z)$ a.e. for $|z| < 1$, and $0 < \iint_{|z|<R} |F(z)| dxdy < \infty$ whenever $0 < R < 1$.

(iii) $\lim_{n\to\infty} \iint_{|z|<1} \left\{ |f_n(z)| - \text{ Re} \left[\frac{\overline{F(z)}}{|F(z)|} f_n(z) \right] \right\} dxdy = 0.$

Problem *Do conditions (i), (ii), (iii) imply that $F(z)$ is analytic for $|z| < 1$?*

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J. VÄISÄLÄ:

Problem *Let $n \geq 2$ be an integer. What is the best possible constant $a_n > 0$ such that the following is true: Let I^n be the unit cube of side 1, and let $A \subset I^n$ be a Borel set with Hausdorff $(n-1)$-measure $m_{n-1}(A) < a_n$. Then there is $y \in I^n$ such that $L \cap A = \emptyset$ for each line L through y parallel to a coordinate axis.*

In particular, is $\inf\{a_n : n \geq 2\} > 0$?

Results.

(1) $a_n \leq 1$. Proof. Take $A =$ face of I^n.

(2) $a_n \geq 1/n$. Proof. Let $p_1, \ldots, p_n$ be the projections onto the coordinate planes and set $E_i = I^n \cap p_i^{-1} p_i A$. Then $m(E_i) < 1/n$, and hence there is $y \in I^n \setminus (E_1 \cup \ldots \cup E_n)$.

(3) $a_2 = 1$

(4) $a_3 \leq \sqrt{2} - 1/2 = 0.9142\ldots$

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