MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Mathematical Continuum Mechanics

22.06. — 28.06.1997

The meeting was organized by J.M. Ball (Oxford), R.D. James (Minneapolis), and A. Mielke (Hannover). The guiding topic was the mathematical analysis of problems in continuum mechanics, mainly in the field of elastic deformations of solids. However, the area was wide enough to include new developments from material science on the one side and from abstract partial differential equations on the other side. A focus was set on the development of microstructure in the deformations of crystals and their analysis using, for instance, Young measures or convex integration.

Each morning session combined two to four talks under one special topic and included an extended discussion in the plenum. The topics were 'Fracture and cavitation', 'Microstructure', 'Time-dependent problems', 'New mechanical theories', and 'Lower-dimensional theories'.

Abstracts

Stuart Antman

Quasilinear problems of nonlinear viscoelasticity

We study the quasilinear initial-boundary-value problems governing the motion in space of nonlinearly viscoelastic rods (of strain-rate type). We desribe reasonable constitutive restrictions that ensure that these problems have globally defined classical solutions. (Our methods are applicable to the study of the motion of any nonlinearly viscoelastic bodies of strain-rate type that are governed by quasilinear initial-boundary-value problems with one independent spatial variable.) We devote special attention to the characteristic technical difficulties of rod theories that follow from the underlying geometrical significance of the governing equations, and from the consequent dependence on space and time of the natural basis for all geometrical and mechanical vector-valued functions. These difficulties prevent our analysis from being a routine application of available techniques.

We employ very general models for rods that can suffer flexure, extension, torsion and shear. In these models the contact forces and couples depend on strains measuring these effects and on the time derivatives of these strains. We ensure that there is a strong mechanism of internal friction by requiring that the mapping taking the strain rates to the contact forces and couples be uniformly monotone. The governing equations form an eighteenth-order quasilinear system of parabolic-hyperbolic partial differential equations in one space variable. This system is singular in the sense that certain constitutive functions appearing in the principal part of the differential operator blow up if the strain variables lie on a surface corresponding to a "total compression". The existence theory

for this system or even for its restricted version governing planar motions has never been studied. The heart of our paper is the introduction of reasonable constitutive hypotheses that enable us to obtain an a priori bound preventing a total compression and a priori bounds on the strains and strain rates. These bounds on the arguments of our constitutive functions allow us to replace the original singular problem with an equivalent regular problem, which we analyze by using the Faedo-Galerkin method, with a key convergence proof relying on the monotonicity of the contact forces and couples with respect to the strain rates. Most of our effort is devoted exploiting the structure of the governing equations, which reflects the geometrical, mechanical and constitutive behavior of the rods.

Andrea Braides

Non-monotone stress-strain relations and surface energies

(This is a joint work with G. Dal Maso and A. Garroni)

In a discrete model let an elastic bar be identified with a system of n equally spaced material points interacting through an array of non-linear springs connecting neighbouring points. We suppose that the force due to each spring depends on its relative elongation ε following a law $\sigma = \psi_n(\varepsilon)$. We assume that there exist $0 < \varepsilon_n < \varepsilon_n^*$ such that ψ_n is increasing on $(-\infty, \varepsilon_n]$, ψ_n is decreasing on $[\varepsilon_n, \varepsilon_n^*]$, and vanishes on $[\varepsilon_n^*, +\infty)$. This means that each spring has a (nonlinear) elastic behaviour up to the critical value ε_n of the relative elongation, and that a softening phenomenon occurs between this value and the fracture threshold ε_n^*. In addition we assume as usual that $\psi_n(0) = 0$, and $\psi_n(\varepsilon) \to -\infty$ as $\varepsilon \to -\infty$.

Let x_i^n, $i = 0, \dots, n$, denote the locations of the material points in the reference configuration, and let u_i^n denote the corresponding displacements. If $\lambda_n = 1/n$ is the distance of two neighbouring points, the energy of the system is

$$\mathcal{E}_n(u^n) = \sum_{i=1}^{n} \lambda_n \Psi_n \left(\frac{u_i^n - u_{i-1}^n}{\lambda_n} \right), \tag{1}$$

where Ψ_n is the primitive of ψ_n vanishing at 0. The displacement $u^n = (u_i^n)$ corresponding to an equilibrium configuration in this discrete model is a stationary point for $\mathcal{E}_n$ with appropriate boundary conditions.

In order to derive a continuum model consider the variational limit of the functionals (1) as n tends to $+\infty$. The critical strains ε_n and the maximum stresses $\sigma_n = \psi_n(\varepsilon_n)$ are assumed to be equibounded. Let f_n be any increasing function coinciding with ψ_n on $(-\infty, \varepsilon_n]$, and let g_n be the rescaled function

$$g_n(t) = \psi_n \left(\frac{t}{\lambda_n} + \varepsilon_n \right), \qquad t \geq 0.$$

As these functions are monotonic it is not restrictive to assume that they converge, up to a subsequence, pointwise to two functions f and g, respectively. Let F and G be their primitives vanishing at 0, and define $\mathcal{E}$ on $SBV(0,l)$ as

$$\mathcal{E}(u) = \int_0^1 F\big(\dot{u}(x)\big)\, dx + \sum_{x \in S_u} G\left([u](x)\right), \qquad ([u] \geq 0) \tag{2}$$

where $\mathcal{S}_u$ is the set of discontinuity points of u. The functionals $\mathcal{E}_n$ converge in a variational sense to the functional $\mathcal{E}$; moreover the local minima and stationary points of $\mathcal{E}$ can be completely characterized. Under natural equi-coerciveness conditions, we also obtain that the minimum values of $\mathcal{E}_n$ for a given total elongation δl converge to the minimum value of $\mathcal{E}$ with the same elongation, and (a suitable interpolation of) the corresponding displacements converge (up to a subsequence) to a displacement which minimizes $\mathcal{E}$.

This approximation property provides a first justification of the choice of an energy $\mathcal{E}$ of the form (2) for a continuum model allowing for damage and fracture, and a compactness argument shows that such functionals are the only ones that can be obtained by a limiting procedure starting from a disrete model with non-linear springs with qualitative properties described above.

Oscar Bruno
Energetics in Martensites

The behavior of shape-memory alloys is governed by a shape-deforming phase transition between a high-temperature, low-strain phase (austenite) and a low-temperature, high-strain phase (martensite). The conserved and dissipated energies in martensitic transformations are macroscopic material properties which can be determined from experiment and which, together with the properties of heat release and exchange, determine the form in which these phase transitions take place.

In this talk I will discuss this principle in connection with pure crystals as well as polycrystalline martensites, and I will present a number of associated mathematical results. Finally, I will compare our predictions with corresponding experimental results.

Carsten Carstensen
Numerical analysis of the relaxed double well problem

Motivated by applications in microstructures of material science, in micromagnetics, or in optimal design problems, the double well potential is the most prominent example of a non-convex energy density. The scalar variant

$$W(F) = |F - F_1|^2 \cdot |F - F_2|^2 \quad (F \in \mathbb{R}^n)$$

(for distinct given F_1, F_2) is under question where F denotes the gradient of the displacement $u : \Omega \to \mathbb{R}$.

A direct minimization of related variational problems is a hard task since there may be no (classical) solution in the continuous case and a cluster of local minimizers in a discrete case. If the main interest is on the macroscopic displacement field (i.e. the weak limit of a minimizing sequence), on the Young measure (generated by minimizing sequences), then a simple minimization of the relaxed problem (replacing W by its lower convex envelope W^{**}) is shown to be sufficient and can be performed utilizing standard software. A priori and a posteriori error estimates for the stress field and parts of the strain and Young measure are shown. This justifies the numerical experiments and allows the design of efficient self-adaptive mesh-refinement algorithms. Partial regularity of solutions as absolute error control (involving explicit estimates for all arising constants) for a reliable computation are indicated.

Sophia Demoulini

Young measure solutions for systems of mixed type and the recovery of distributional solutions

The method of implicit time discretisation (i.t.d.) can be used to construct measure-valued solutions of the evolutionary problems which are associated with a non-convex potential and admit a variational formulation. I focus the analysis partly on such constructions and partly on the validity of the i.t.d. as a method to construct regularisations for conservation laws. As a model problem I construct solutions to the anti-plane shear problem $w_t = \nabla v$, $v_t = \nabla.\sigma(w)$ on an open bounded set in n dimensions, with $\sigma = \nabla\phi$ and ϕ non-convex. Non-convexity renders the problem hyperbolic-elliptic and to solve it I discretise in time the equivalent quasilinear wave equation $u_{tt} = \nabla.\sigma(\nabla u)$. The key observation here is that the energy non-increase must be used to obtain estimates for the approximate solutions. By exactly the same observation the i.t.d. yields weak solutions for the equation of viscoelasticity $u_{tt} = \nabla.\sigma(\nabla u)+\Delta u_t$. It is useful to compare with the Yo[illegible] measure solution constructed by Kinderlehrer and Pedregal for the forward-backward heat equation $u_t = \nabla.\sigma(\nabla u)$. I show that this solution is unique within the class of measure-valued solutions which satisfy an *independence* property, namely that σ and the identity are independent variables with respect to the measure. Regarding the admissibility of the method of i.t.d. I show that in the parabolic case one can recover classical weak solutions in the case of a strictly convex potential. I also discuss results in joint studies with Stuart and Tzavaras currently underway: i) the i.t.d. gives regularisations for conservation laws which satisfy generalised entropy-entropy flux inequalities (Lax entropies averaged by the Young measure) as introduced by Tartar and used by DiPerna ii) the method can be used to construct entropy satisfying distributional solutions for the equations of 1-d elastodynamics.

Georg Dolzmann

Existence of minimizers for a variational problem in magneto-elasticity

We prove existence of energy minimizing configurations for a two-dimensional, variational model of magnetoelastic materials capable of large deformations. The model is based on an energy functional which is the sum of the nonlocal self-energy and the local anisotropic energy. Since the functional fails to be weakly lower semicontinuous the direct method in the calculus of variation cannot be applied. Existence is obtained by rewriting the minimization problem as a partial differential relation and using recent existence theorems for Lipschitz solutions.

Irene Fonseca

Optimal design results for elastic membranes

The first part of this talk concerns joint works with Gilles Francfort. Here we search to characterize the effective, or relaxed, energy of a mixture of two materials when the thickness of the sample approaches zero. The underlying mathematical model is also relevant to the study of brittle damage, and for this case we are able to fully identify the damage evolution, induced by a *min min* principle, while for the former optimal design problem the question remains open, as we are unable to tackle the corresponding *max min* problem associated to the minimization of the compliance.

Precisely, with $\Omega_\varepsilon := \omega \times (-\varepsilon, \varepsilon)$, $\omega \subset \mathbb{R}^2$ open, bounded domain, and with $u : \Omega_\varepsilon \to \mathbb{R}^3$ a deformation of the sample, following earlier work by Ciarlet, Le Dret and Raoult, and others. Let $\Omega := \omega \times (-1, 1)$, $y_1 = x_1$, $y_2 = x_2$, $y_3 = \varepsilon^{-1} x_3$, $v_\varepsilon := u_\varepsilon(y_1, y_2, \varepsilon y_3)$, so that an energy

$$\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W\big(D_\alpha u | D_3 u\big)\, dx \qquad (\alpha = 1, 2)$$

now becomes

$$\int_\Omega W\big(D_\alpha v | \varepsilon^{-1} D_3 v\big)\, dy.$$

Here, $W(\bar{F}, \xi_3) := W(F)$ where F is a 3×3 matrix whose first 2 columns are the columnes of $\bar{F}$ and the third column is ξ_3. Assuming that W_1, W_2 grow at infinity as $|\xi|^p$, $p > 1$, and that $W_i(\bar{\xi} | \xi_3) = W_i(\bar{\xi} | -\xi_3)$ for $i = 1, 2$ and for all 3×3 matrix $(\bar{\xi}, \xi_3)$, we can show that

$$J(\theta_0, v) = \inf\Big\{ J_\omega(\theta, v) \,:\, \fint_\omega \theta = \theta_0 \Big\} \qquad \text{with} \quad J_\omega(\theta, v) = 2 \int_\omega \overline{W}(\theta, Dv)\, dx_\alpha$$

for all $\theta_0 \in [0, 1]$, $v \in W^{1,p}(\omega, \mathbb{R}^3)$ and $\theta \in L^\infty\big(\omega, [0, 1]\big)$. We have the following characterizations

$$\begin{aligned}
J(\theta_0, v) &:= \inf\Big\{ \underline{\lim}\, I_\varepsilon(\chi_\varepsilon, v_\varepsilon) \,:\, \fint_\omega \chi_\varepsilon = \theta_0,\ v_\varepsilon \to v \text{ in } L^p \Big\}, \\
J_\omega(\theta_0, v) &:= \inf\Big\{ \underline{\lim}\, I_\varepsilon(\chi_\varepsilon, v_\varepsilon) \,:\, \chi_\varepsilon \overset{*}{\rightharpoonup} \theta_0,\ v_\varepsilon \to v \text{ in } L^p \Big\}, \\
I_\varepsilon(\chi, v) &:= \int_\Omega \Big[\chi W_1\big(D_\alpha | \varepsilon^{-1} D_3 v\big) + (1 - \chi) W_2\big(D_\alpha v | \varepsilon^{-1} D_3 v\big) \Big]\, dx \\
\overline{W}(\theta_0, \bar{\xi}) &:= \inf\Big\{ \int_{(0,1)^2} \Big[\chi \widetilde{W}_1\big(\bar{\xi} + D_\alpha \phi\big) + (1 - \chi) \widetilde{W}_2\big(\bar{\xi} + D_\alpha \phi\big) \Big]\, dx_\alpha \,:\, \fint_{(0,1)^2} \chi = \theta_0 \\
&\qquad \phi \in W_0^{1,p}\big((0, 1)^2, \mathbb{R}^3\big) \Big\}
\end{aligned}$$

with $\widetilde{W}_i(\bar{\xi}) := \inf\big\{ W_i(\bar{\xi} | \xi_3) \,:\, \xi_3 \in \mathbb{R}^3 \big\}$, and where we only consider cylindrical mixtures, i.e. χ_ε are characteristic functions independent of x_3.

On the second part of the talk, and in joint work with Emilio Acerbi and Nicola Fusco, we consider particular "bulk" energy densities for membranes

$$\overline{W}(\bar{\xi}) := \frac{1}{2} |\bar{\xi}|^2 + f\big(\nu(\bar{\xi})\big),$$

where $\bar{\xi} = (\xi_1, \xi_2)$ is a $d \times 2$ matrix, and $\nu(\bar{\xi}) := \xi_1 \wedge \xi_2 \in \mathbb{R}^d$. Here $0 \le f(t) \le c(1 + |t|)$ is a C^1 function, not necessarily convex. We show that local minimizers $u \in W^{1,2}(\omega, \mathbb{R}^d)$ of

$$v \mapsto \int_\omega \frac{1}{2} |Dv|^2 + f\big(\nu(v)\big)\, dx,$$

with $\nu(v) := D_1 v \wedge D_2 v$, are $C^{0,\gamma}$ for all $0 < \gamma < 1$. This is obtained using the $L^{p,\gamma}$ regularity theory, as well as, following similar ideas by Baumann, Phillips, and Dougherty, higher integrability of $A := \frac{|D_1 u| - |D_2 u|^2}{2}$, $B := D_1 u - D_2 u$.

This regularity result, in turn, provides existence of "classical" solutions for discontinuity problems that appear in fracture mechanics.

Gilles Francfort
Fracture mechanics as seen from an energy minimization standpoint
(This is a joint work with J. J. Marigo)

Griffith's theory of brittle fracture is somewhat at a loss when trying to predict crack initiation in an uncracked elastic sample; it also finds it difficult to determine the crack path; finally, it is restricted to smooth evolutions of the crack length, even along predefined crack paths.

We propose a model which does away with the previously mentioned obstacles. Of course it does so at a price, at least for the time being: the only permissible "loads" are boundary displacements.

A surface energy of Griffith type is introduced for any crack Γ (closed subset) of $\overline{\Omega}$ (Ω being the N-dimensional domain occupied by the sample), namely,

$$E_s(\Gamma) = k\mathcal{H}^{N-1}(\Gamma),$$

where k is the fracture toughness of the material. The bulk energy is characterized by an elastic energy density W. Then, for any crack-state Γ, and any displacement "load" U, the bulk energy is

$$E_d(\Gamma, U) = \min_u \Big\{ \int_\Omega W\big(\nabla u(x)\, dx\big) \,:\, u = U \text{ on } \partial\Omega\backslash\Gamma \Big\}.$$

In a time-discretized evolution corresponding to a sequence of loads $U_1, \ldots, U_n, \ldots$ the crack-states Γ_i are then such that

$$E_d(\Gamma_i, U_i) + E_s(\Gamma_i) \leq E_d(\Gamma, U_i) + E_s(\Gamma)$$

for all Γ with $\Gamma_{i-1} \subset \Gamma$.

A corresponding time-continuous evolution is presented in the case of monotonically increasing one-parameter family of "loads".

We show that the formulation produces crack initiation in finite time, as well as a complete determination of the crack growth along a predefined path. Also embedded in our model is the determination of a (or the) optimal crack path.

The formulation lends itself to numerical computations in complex loading and geometric situations. A few numerical resuts are presented.

Jens Frehse
Some regularity for nonlinear mixed boundary problems

Elliptic equations or systems

$$D_i F_i(x, \nabla u) = D_i f_i, \tag{1}$$

with linear growth of F_i, are considered in a domain of $\mathbb{R}^n$ with convex corners. Dirichlet and Neumann boundary may touch in a smooth line or in corners. (1) is supposed to be an

Euler equation of a variational problem. A simple trick yields weighted $H^{3/2,2}$-estimates for the solution.

Further considerations yield, for $n = 3$, $\nabla u \in L^{3+\delta}$ in corners, $\nabla u \in L^{10/3+\delta}$ if the separating line between Dirichlet and Neumann is smooth. For $n = 3$ the second derivatives are in $L^{6/5-\delta}$ (which is probably not optimal). The reader may think of the equation $\Delta u = f$, however, the theory is very general and applies in many nonlinear situations, also for Navier-Stokes or $\Delta\Delta$.

Bernd Kirchheim
Microstructures with finite surface energy: The 3-well problem

Let $K \subset \mathbb{M}^{3\times 3}$ be the 3 cubic-to-tetragonal wells, i.e. $K = \bigcup_{j=1}^{3} SO(3)D^j$ where

$$D^1 = \operatorname{diag}(\lambda^{-2}, \lambda, \lambda), D^2 = \operatorname{diag}(\lambda, \lambda^{-2}, \lambda) \text{ and } D^3 = \operatorname{diag}(\lambda, \lambda, \lambda^{-2}).$$

We prove that any Lipschitz $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies

i) $\nabla u(x) \in K$ for almost every $x \in \Omega$.

ii) each set $E^j = \{x\ ;\ \nabla u(x) \in SO(3)D^j\}$ is of finite perimeter.

is a local laminate. This means for each $x \in \Omega$ there is a radius $R > 0$ such that $u_{|B(x,R)}$ is affine either on the whole ball or on both halfballs determined by a plane through x. The proof is based on a careful analysis of the geometry of the phase boundaries $\bigcup_j \partial_* E^j$. This together with the Liouville theorem for sets of finite perimeters from work by Dolzmann and Müller about the 2-well problem enables us to overcome difficulties due to the noncommutativity of $SO(3)$. We derive sufficiently easy to handle algebraic constraints leading to the nonexistence of nonlaminatial microstructure whenever $\lambda \neq 1$. Although we essentially use special symmetries of K, our approach should also exclude the existence of nonlaminatial zero-energy states for more general wells (and generic choices of the parameters of the wells).

Jan Kristensen
Nonlocality of quasiconvexity

In this talk I present an example of a smooth function defined on $n \times m$ matrices ($n \geq 3, m \geq 2$), which equals a quasiconvex function on any ball of radius 117, but which is not itself quasiconvex. As a consequence we deduce that in dimensions $n \geq 3, m \geq 2$ there is no "local condition" which for smooth functions is equivalent to quasiconvexity. In particular it follows that there can be no condition involving only the function and a finite number of its derivatives, which is both necessary and sufficient for quasiconvexity.

Hervé Le Dret
Nonlinear lower dimensional theories: asymptotics and the projection method
(This is a joint work with Annie Raoult)

The presentation is in two parts. First, we recall our derivation of nonlinear membrane theory from three-dimensional nonlinear elasticity via Γ-convergence arguments. Starting from a three-dimensional cylinder of thickness 2ε, made of a nonlinearly elastic material

with stored energy function $W : M_3 \to \mathbb{R}$, we show that when $\varepsilon \to 0$ energy minimizing deformations converge (in an appropriate sense) toward deformations $\bar{\phi}$ from $\omega \subset \mathbb{R}^2$ into $\mathbb{R}^3$ that minimize an energy of the form $\int_\omega QW_0(\nabla\bar{\phi})\,dx +$ force terms, where QW_0 is constructed from W as follows: Set $W_0(\bar{F}) = \inf_{z\in\mathbb{R}^3} W(\bar{F}|z)$, i.e., minimize W with respect to the third column vector, and let QW_0 be the quasi-convex envelope of W_0. Various properties of the membrane energy are discussed: frame indifference, isotropy and degeneracy under compression. The case of the Saint Venant-Kirchhoff material is entirely computed.

In the second part of the talk, we consider the same kind of questions when kinematic assumptions of Cosserat type are made at the onset. Typically, 3D deformations are assumed to be of the form $\phi(x) = \bar{\phi}(x_1, x_2) + x_3\vec{d}(x_1, x_2)$, and $\vec{d}$ is a director field. It is thus a mix of the projection method and the asymptotic method in the nonlinear case. We show that the unconstrained Cosserat hypothesis leads to the right limit nonlinear membrane behavior described above, while such constrained director assumptions as $|\vec{d}| = 1$ do not and are thus inappropriate in the membrane regime. In order to describe the asymptotic behavior of the director field, we are led to introduce necessary and sufficient conditions for weak lower semi-continuity and relaxation results for functionals of the calculus of variations on $W^{1,p} \times L^p$. These conditions generalize the classical notions of convexity and quasi-convexity.

Ingo Müller:
Rational Extended Theromdynamics

Extended Thermodynamics uses balance equations and local constitutive equations as field equations for the field u

$$F^A(u)_{|A} = \Pi(u).$$

Every solution is a thermodynamic process. The entropy inequality $h^A(u)_{|u} = \Sigma(u) \geq 0$ is used to find restrictions on the constitutive functions which appear in the form of integrability conditions for the entropy density and the entropy flux h^α $(\alpha = 1, 2, 3)$.

The instructive application of extended thermodynamics lies in the field on monatomic gases where the moments of the particle distribution function may be chosen as variables.

At high pressures the light scattering properties of such gases are governed by the phenomological relations of Navier-Stokes-Fourier, the constitutive relations of ordinary thermodynamics. But at low pressures the list of variables has to be extended (sic!) to include hundreds, or even thousands of moments.

The interesting field of application of extended thermodynamics occurs with shock waves. The shock structure problem, i.e. the dependence of the width of the shock on the Mach number, is a promising field. It seems now that the proper description will again require the use of very many moments.

Michael Ortiz
Nonconvex energy minimization and dislocation structures in ductile single crystals
(This is a joint work with E. A. Repetto)

Plastically deformed crystals are often observed to develop intricate dislocation patterns such as labyrinth, mosaic, fence and carpet structures. In this paper, such dislocation

structures are given an energetic interpretation with the aid of direct methods of the calculus of variations. We formulate the theory in terms of deformation fields and regard the dislocations as manifestations of the incompatibility of the plastic deformation gradient field. Within this framework, we show that the incremental displacements of inelastic solids follow as minimizers of a suitably defined pseudoelastic energy function. In crystals exhibiting latent hardening, the energy function is nonconvex and has wells corresponding to single-slip deformations. This favors microstructures consisting locally of single slip. Deformation microstructures constructed in accordance with this prescription are shown to be in correspondence with several commonly observerd dislocation structures. Finally, we show that a characteristic length scale can be built into the theory by taking into account the self energy of the dislocations. The extended theory leads to scaling laws which appear to be in good qualitative agreement with observation.

Felix Otto

Domain branching in uniaxial ferromagnets

(This is a joint work with R. Choksi and B. Kohn)

Consider a ferromagnet with a single preffered axis ("easy axis") for the magnetization (mathematically speacking a three–dimensional vector field of unit length) in absence of an applied magnetic field. It is observed that there are domains where the magnetization varies smoothly ("Bloch domains"), separated by discontinuity surfaces ("Bloch walls"). It is further observed that the size of these domains strongly decreases when approaching a boundary plane of the sample which is perpendicular to the easy axis ("basal plane"), a phenomenon called domain branching.

We show that this phenomenon can be understood as coming from the minimization of the micromagnetic energy. The micromagnetic energy, a functional of all vector fields of unit length, is the sum of three contributions: The exchange energy (which penalizes the Bloch walls), the anisotropy energy (which penalizes magnetizations not parallel to the easy axis) and the magnetostatic energy (which can be seen as penalizing the divergence of the vector field, including its singular contribution at the boundary of the sample).

This non convex variational problem is not explicitely solvable; it even seems difficult to make use of the corresponding Euler–Lagrange equations. Instead, we turn to estimating the minimial energy directly. More precisely, our analysis is based on rigorously deriving the scaling of the minimal energy in the physical parameters. This approach has been introduced by Bob Kohn and Stefan Müller.

We compare the minimal energy among all vector fields which do not refine towards the basal plane (vector fields which are constant in the direction of the easy axis) with the minimal energy among all vector fields. We find that the latter one is smaller in the regime of interest because it scales differently in the physical parameters.

The analysis consists of two parts: Establishing an upper bound by a special construction and proving a lower bound by some interpolation arguments. The construction yielding the upper bound is a modification of a construction by Privorotskii. The lower bound can be reduced to the fact that a certain interpolation of the three spaces BV, L^∞ and H^{-1} imbeds into L^2. This imbedding has enough flexibility to allow us to derive the scaling (in the physical parameters) of the typical size of Bloch domains away from the basal plane.

Robert Rogers
Phase transition in quartz

At about 547 °C, quartz undergoes a phase transition from a high temperature/high symmetry (point group 622) hexagonal phase to a lower temperature/lower symmetry (point group 32) trigonal phase. The transition, usually called β-α transition, is accomponied by a reduction in volume.

The kinematics of quartz in a neighborhood of the transition is described by changes in its crystallographic dimensions, but also by atomic "shuffle" within the unit cell. These shuffles can be described by a scalar order parameter, that can be identified physically and measured experimentally. One of the most interesting features of the transitions is that in a narrow range of temperatures near the critical point, the order parameter exhibits a fine triangular microstructure.

In this talk, I discuss how one can use local bifurcation theory to analyze the triangular microstructure. To do this I apply results from singularity theory (developed by Golubitsky and several collaborators) describing solution branches for problems with hexagonal symmetry. I use these results to analyze a discrete model problem in detail and give some indications as to how one can apply the results to existing continuum models for quartz.

Piotr Rybka
Asymtotics for equations related to martensitic phase transitions
(This is a joint work with Karl-Heinz Hoffmann)

We study asymptotic behavior of

$$\rho u_{tt} = \operatorname{div} \sigma(\nabla u) + \Delta u_t - \delta^2 \Delta^2 u, \tag{1}$$

where $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, $\sigma(\xi) = DW(\xi)$ and W has several local minima, and $\rho = 0$ or $\rho = 1$. For $\rho = 1$ equation (1) is the equation of the viscoelasticity with capillarity. The case of $\rho = 0$ corresponds to neglecting the inertial effects. Both equations arise in the studies of van der Waals fluids and phase transitions in solids.

We adopt special boundary conditions, such that the influence of the boundary on the overall behavior is little. We set $\Omega = [0, \omega] \times [0, L]$ and

$$u_{x_2} = \Delta u_{x_2} = 0 \quad \text{at} \quad x_2 = 0, L, \qquad u \text{ is anti-periodic in } x_1.$$

We define the energy functional to be

$$E(u) = \int_\Omega \left[W(\nabla u(x)) + \frac{\delta^2}{2} |\Delta u(x)|^2 \right] dx,$$

where $W(F_1, F_2) = \Phi(F_1) + \frac{1}{2} F_2^2$, and $\Phi \in C^\infty(\mathbb{R})$ is such that, $\Phi(t) \geq 0$, $\Phi(t) = 0$ iff $|t| = 1$ and $\Phi(t) = \Phi(-t)$.

We note that because of the anti-periodic condition we have circles of equilibria. Moreover, global minimizers have particularly simple structure, i.e. they do not depend on x_2. We prove that if u is a unique solution of (1) for $\rho = 0$ then it converges to a unique point on the manifold of global minimizers of E provided that $E(u(\cdot, 0))$ is sufficiently small. We conjecture a similar result holds in the case $\rho = 1$.

We point out that the right-hand-side of (1) linearized at any global minimizer has a two-dimensional kernel. Thus, standard tools of the theory of dynamical systems are not applicable here. So, we resort to methods based on analyticity of W.

Ekhard Salje
Strain as the dominant interaction mechanism in ferroelastic and co-elastic materials

The concepts of ferroelastic ($\sim$ ferroelastic hysteresis) and co-elastic ($\sim$ elastic degrees of freedom relevant for effective Hamiltonion) material behavior involve longranging correlations which have direct implication on the formation of microstructures. The correlation between the atomic interaction models (e.g. Φ_ψ model) and the macroscopic Gibbs free energy including dispersion effects $(\nabla\nabla u)^2$, $(\Delta\Delta u)^2$ were discussed. Results from computer simulation on systems with more than 10^6 interacting particles were used to classify microstructures and their (slow)time evolution (walls, junctions, needles, etc.). Pattern formation of the type $Q_t = -Q + Q^3 + Q_{xx} - \gamma Q_{xxxx}(+\delta Q_{xxx\,xxx})$ are discussed in the γ-δ phase diagramme.

Friedemann Schuricht
Obstacle problems in elasticity and finer nonsmooth methods

Obstacle problems in elasticity are usually described by variational inequalities. This approach works for simplified models where the set of admissible deformations is convex in some function space. In general nonlinear elasticity this is, however, not the case. Furthermore variational inequalities, which can be understood as one of the roughest nonsmooth tool, do not have enough structure to provide a detailed description of the forces exerted by obstacles. Thus finer nonsmooth methods are necessary to handle such problems efficiently. On the basis of the very general Cosserat theory describing planar deformations of shearable nonlinearly elastic rods it is demonstrated how this can be done. Using Clarke's calculus of generalized gradients, instead of a variational inequality the Euler-Lagrange equations can be derived for very general obstacle problems. This way we obtain a very natural representation of the contact reaction which finally serves as basis for further regularity results. In particular an interesting qualitative difference in regularity between shearable and unshearable material can be observed.

Jey Sivaloganathan
On the location of singularities arising in Nonlinear Elasticity
(This is a joint work with Scott Spector)

Consider an isotropic, hyperelastic material occupying the unit ball in $\mathbb{R}^3$ in its reference state. Next subject the boundary of the material to a radial displacement and minimize the total energy stored in the ball in the class if radially symmetric deformations. It is known from the work of J. M. Ball that for some materials a discontinuity may form in the minimizer for this variational problem (the phenomenon of cavitation).

We suggest an approach to predicting the most energetically favorable point at which to form a hole (as the boundary of the ball is displaced radially outwards) with assumptions of symmetry on the admissible deformations.

Our results yield some interesting and elegant formulae and suggest intimate links between cavitation and classical engineering approaches to modelling defects resp. fracture using singular solutions of linear elasticity.

Scott Spector:
Critical Cavitation Loads in Elastic Solids
(This is a joint work with Stefan Müller and Jey Sivaloganathan)

Consider the model stored energy density

$$W(F) = |F|^p + h(\det F),$$

where h is convex, nonnegative, and becomes infinite as its argument goes to zero or infinity. Here $\det F$ is the determinant of the 3×3 matrix F. Let θ^3 denote the unique minimizer of h. If $1 \leq p < 3$ then John Ball has shown that ther is a $\lambda_{radial} > \theta$ such that for all $\lambda > \lambda_{radial}$ the linear map λx is not a global minimizer of total energy

$$E(u) := \int_B W(\nabla u(x))\, dx,$$

among those functions u in the Sobolev space $W^{1,p}(B, \mathbb{R}^3)$ that are one-to-one a.e. and satisfy $u(x) = \lambda x$ on ∂B, where B is the unit ball centered at the origin in $\mathbb{R}^3$. Moreover, for such values of λ there exists a singular deformation that minimizes E in the class of radial maps. This radial minimizer creates a new cavity at the center of B.

In this talk I will present some joint work with S. Müller and J. Sivaloganathan, which shows that for $2 < p < 3$ there is a $\lambda_0 \in (0, \lambda_{radial})$ such that for all $0 < \lambda < \lambda_0$ the linear map λx is indeed a global minimizer of E among those maps $u \in W^{1,p}(B, \mathbb{R})$ that satisfy $u(x) = \lambda x$ on ∂B, and whose extension to all of $\mathbb{R}^3$ (as the linear deformation λx) satisfies condition (INV). Thus, in particular, no holes (radial or nonradial) can form in the material at such values of λ.

Roughly speaking, condition (INV) is the requirement that the deformation u be monotone in the sense of Lebesgue and that holes in one part of B are not filled by material from another part of B. The condition $\det \nabla u > 0$ a.e. together with condition (INV) prohibits interpenetration of matter, that is, these conditions together imply that u is one-to-one almost everywhere.

The heart of our proof is an estimate on the difference of two Jacobians; for every $p > 2$ there is a constant $\alpha = \alpha(p) > 0$ such that for every $\lambda > 0$ and every bounded open region $\Omega \subset \mathbb{R}^3$

$$0 \leq \int_\Omega \left[\det \lambda I - \det \nabla u(x)\right] dx \leq \alpha \lambda^{3-p} \int_\Omega \left|\nabla u(x) - \lambda I\right|^p dx$$

for all deformations $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ that satisfy $u(x) = \lambda x$ on $\partial\Omega$, $\det \nabla u > 0$ a.e., and whose extension to all of $\mathbb{R}^3$ (as the linear deformation λx) satisfies condition (INV). If $p \geq 3$ this estimate is clear since the first integral is zero due to the fact that the Jacobian is a null Lagrangian. For $2 < p < 3$ one can express this integral in terms of the singular part of the distributional Jacobian of u, which is a Radon measure under our Hypotheses. This singular measure is then estimated locally via isopermetric inequality and a standard covering argument finishes the proof.

Pius Sprenger
Stability, quasiconvexity at the boundary and null Lagrangians

A basic problem in elastostatics is to minimize energy functionals of the type

$$I(u) = \int_\Omega W(\nabla u(x))\,dx + \int_\Gamma g(x)\cdot u(x)\,da,$$

where W is the stored energy function of a hyperelastic material and g denotes the surface force on the free boundary part Γ. Ball and Marsden proved that for a deformation u_0 to be a strong local minimizer of the functional I at every free boundary point $x_0 \in \Gamma$ the stored energy function has to be quasiconvex at the boundary in $(\nabla u_0(x_0), \nu(x_0))$, where $\nu(x_0)$ denotes the outer normal at Γ in x_0. This condition is not local and therefore it may be rather difficult or impossible to verify it.

With the aid of null Lagrangians we give a new pointwise condition, called polyconvexity at the boundary, which is an extension of convexity and is stronger than quasiconvexity at the boundary. It is shown that in nonlinear elasticity theory the assumption of polyconvexity at the boundary is compatible with physical properties of the stored energy function.

Moreover, an equilibrium satisfying polyconvexity at the boundary in $(\nabla u(x_0), \nu(x_0))$, with $x_0 \in \Gamma$, is a spatially localized minimum of I at x_0.

Florian Theil
Long-time dynamics for Young measure solutions of nonlinear PDEs

We study two different model equations for phase transitions:

$$\rho\ddot{u} = \sigma(u_x)_x + \beta\dot{u}_{xx} - \alpha u \qquad \text{(V)}$$
$$\rho\ddot{u} = \sigma(u_x)_x + \beta\ddot{u}_{xx} \qquad \text{(M)}$$

with a nonmonotonous stress-strain relation σ. Equation (V) is refered to as "viscoelastically regularized wave equation" and has been studied by several authors; equation (M) has been proposed by V. Levitas as a model for microkinetic fluctuations during phase transitions. We will explain that both equations admit Young-measure solutions which we construct with a method inspired by transport theory.

It is interesting that their long-time behaviours differ strongly. No solution of model (V) can develop additional microstructure even if Young-measure solutions are admitted, instead they decay to a stationary state. This contrasts with the long-time behaviour of solutions of (M). Here, solutions can indeed develop fine sturcture and converge to a genuine Young measure even if only classical initial conditions are admitted.

Matthias Winter
Microstructure and surface energy, analytical results

We present an example of microstructure arising from a potential of the gradient with incompatible minima. We show that the Young measure is homogeneous and unique. Including surface energy we show that for the infinite sequentially laminated microstructure the energy scales like $\exp\left(-\sigma\sqrt{\log\varepsilon^{-1}}\right)$, $0 < \sigma\sqrt{2\log 2}$ in the limit of $\varepsilon \to 0$ (ε being the factor of the surface energy part) at least in the upper bound.

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