MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 10/1998

# *Elementare und Analytische Zahlentheorie*

**08.03.1998 – 14.03.1998**

This conference on *"Elementary and Analytic Number Theory"* was organized by

Jörg Brüdern, Stuttgart
Hugh L. Montgomery, Ann Arbor
Hans Peter Schlickewei, Marburg
Eduard Wirsing, Ulm

About fifty mathematicians from sixteen different countries accepted the invitation of the Institute. All lectures presented during the week gave a stimulating survey of current progress in Analytic Number Theory. Approximately forty of the participants considered a wide variety of topics in Analytic and Elementary Number Theory, such as

> Artin's Conjecture, Diophantine approximation, distribution of prime numbers, exponential sums, lattice points, linear recurrence sequences, moments of the Riemann zeta-function and $L$-functions, partitions, primes in arithmetic progressions, the Selberg Class, transcendence, set addition, Waring's Problem,

while in parallel sessions a smaller group of ten focussed on a very special, but important Diophantine topic, namely the Schmidt Subspace Theorem.

In the beautiful and relaxed atmosphere of the Institute, the participants enjoyed sharing their questions and ideas. The organizers and participants of this conference express their thanks to the Land Baden-Württemberg, the Director of the Institute, Prof. Kreck, and his staff for providing this productive experience.

**Dank einer Unterstützung im Rahmen des EU-Programmes TMR (Training and Mobility of Researchers) konnten zusätzlich einige jüngere Mathematiker zu der Tagung eingeladen werden. Dies ist einerseits eine hervorragende Förderung des wissenschaftlichen Nachwuchses und gibt andererseits den etablierten Kollegen die Gelegenheit, besonders begabte junge Mathematiker kennenzulernen.**

# *Conference Program*

## *Monday, March 9*

| | | |
|---|---|---|
| 9:15 – 10:15 | Wolfgang M. Schmidt | The Zero Multiplicity of Linear Recurrence Sequences |
| 10:25 – 11:15 | Etienne Fouvry | Exponential Sums and Divisibility of Class Numbers |
| 11:25 – 11:55 | Aleksandar Ivić | The Mellin Transform and the Riemann Zeta-Function |
| 12:00 – 12:30 | Matti Jutila | The Mellin Transform of the Fourth Power of Riemann's Zeta-Function |
| Hall 2 | | |
| 11:25 – 12:10 | Patrice Philippon | Some Remarks on Methods of Diophantine Approximation |
| | LUNCH | |
| 16:00 – 16:30 | Adolf J. Hildebrand | Partitions into Primes |
| 16:40 – 17:10 | Jerzy Kaczorowski | On the Structure of the Selberg Class |
| 17:20 – 17:50 | Alberto Perelli | Linear Independence in the Selberg Class |
| 18:00 – 18:30 | Jean-Marc Deshouillers | A Step Beyond Kneser's Addition Theorem |
| | DINNER | |

## *Tuesday, March 10*

| | | |
|---|---|---|
| 9:00 – 9:50 | Roger Heath-Brown | Solutions of Diagonal Cubic Equations |
| 10:00 – 10:30 | Helmut Maier | The Distribution of the Values of the Riemann Zeta-Function in Short Intervals of the Critical Line |
| 10:40 – 11:20 | Yoichi Motohashi | The Complex Binary Additive Divisor Problem and the Spectral Theory of the Three-Dimensional Hyperbolic Upper Half-Space |
| 11:30 – 11:55 | Dieter Wolke | A Prime Number Theorem with Weights |
| 12:00 – 12:25 | Cècile Dartyge | Almost Prime Numbers with Missing Digits |
| Hall 2 | | |
| 10:00 – 10:50 | Damien Roy | Heights and Siegel's Lemma |
| 11:30 – 12:20 | Jeff L. Thunder | An Old Idea of Hermite Receives New Life |
| | LUNCH | |

| | | |
|---|---|---|
| 16:00 – 16:50 | Trevor D. Wooley | Exponential Sums and Diophantine Equations in Many Variables |
| 17:00 – 17:30 | Koichi Kawada | Sums of Fourth Powers and Related Topics |
| 17:40 – 18:00 | Morley Davidson | Local Solubility in the Waring-Siegel Problem |
| 18:10 – 18:30 | Jörg Brüdern | On Artin's Conjecture, Local Case |
| | DINNER | |
| 20:00 | PROBLEM SESSION | |

## *Wednesday, March 11*

| | | |
|---|---|---|
| 9:00 – 9:45 | Philippe Michel | Non-Vanishing of Critical Values of $L$-Functions |
| 10:00 – 10:50 | Jan-Hendrik Evertse | On the Norm Form Inequality $\lvert F(\underline{x})\rvert \leq M$ |
| 10:55 – 11:25 | Kai-Man Tsang | Lattice Points in Spheres |
| 11:30 – 12:00 | Imre Z. Ruzsa | Additive Completion |
| 12:05 – 12:30 | David W. Farmer | Non-Vanishing of $L$-Functions and the Irreducibility of Hecke Polynomials |
| Hall 2 | | |
| 11:30 – 12:20 | Gisbert Wüstholz | Modular Varieties, Hypergeometric Series and Transcendence |
| | LUNCH | |
| | EXCURSION | |

## *Thursday, March 12*

| | | |
|---|---|---|
| 9:00 – 9:40 | Robert C. Vaughan | Primes in Arithmetic Progressions |
| 9:50 – 10:20 | Stephan Daniel | Lattice Point Methods and Divisor Sum Problems |
| 10:30 – 11:10 | Gérald Tenenbaum | On the Gutman-Ivić-Matula Function and Related Topics |
| 11:20 – 11:50 | Steve G. Gonek | The Variance of Small Powers of Primitive Roots |
| 12:00 – 12:25 | Manfred Peter | The Almost Periodicity of the Normalized Sequence of Class Numbers |
| Hall 2 | | |
| 10:00 – 10:45 | Robert Tijdeman | On the Number of Digit Changes |
| 11:30 – 12:15 | Roberto G. Ferretti | Mumford's Degree of Contact and Diophantine Approximations |

LUNCH

| | | |
|---|---|---|
| 15:45 – 16:30 | Peter D. T. A. Elliott | Primes and Products |
| 16:40 – 17:10 | Jeffrey D. Vaaler | On the Number of Polynomials over $\mathbb{Z}$ having Bounded Height and Bounded Mahler Measure |
| 17:20 – 17:50 | Jürgen W. Sander | Rational Points on a Class of Superelliptic Curves |
| 18:00 – 18:30 | Lutz G. Lucht | Arithmetical Results on Certain Functional Equations |

Hall 2

| | | |
|---|---|---|
| 15:45: – 16:30 | Hans Peter Schlickewei | The Subspace Theorem and Geometry of Numbers |

DINNER

## *Friday, March 13*

| | | |
|---|---|---|
| 9:00 – 9:30 | Hugh L. Montgomery | Beyond Pair Correlation |
| 9:40 – 10:10 | András Biro | On an Extremal Problem Related to Gaussian Sums |
| 10:30 – 11:10 | Daniel A. Goldston | Primes in Short Segments of Arithmetic Progressions |
| 11:20 – 11:50 | Régis de la Bretèche | A Summation Process |
| 12:00 – 12:25 | Alla Lavrik-Männlin | On the Zeros of the Hardy $Z$-function and its Derivatives |

Hall 2

| | | |
|---|---|---|
| 9:45 – 10:30 | Helmut Locher | On the Number of Good Approximations of Algebraic Numbers by Algebraic Numbers of Bounded Degree |
| 11:00 – 11:45 | Yuri V. Nesterenko | On an Equation of ( oormaghtigh |

LUNCH

| | | |
|---|---|---|
| 16:00 – 16:30 | Martin N. Huxley | Integer Points Close to Curves and Exponential Sums |
| 16:40 – 17:15 | Andrew Pollington | Haar Wavelets and Irregularities of Distribution |
| 17:30 – 18:15 | Ulrike M. A. Vorhauer | Three Two-Dimensional Weyl Steps in the Circle Problem |

DINNER

# *Abstracts of the Lectures*

## *A Summation Process*

*Régis de la Bretèche, University of Orsay*

We define $P$-convergence and $P$-regularity, a notion which was introduced by Fouvry and Tenenbaum in 1991. Let $P(n) = \max_{p|n} p \;\; (n > 1)$, $P(1) = 1$. We say that a series $\sum_{n\geq 1} \alpha_n$ is $P$-convergent if $\sum_{P(n)\leq y} \alpha_n$ converges for each $y \geq 2$ and if

$$\lim_{y\to\infty} \Big( \sum_{P(n)\leq y} \alpha_n \Big) = \alpha .$$

We say that a series $\sum_{n=1}^{\infty} \alpha_n$ is $P$-regular if it is $P$-convergent and if $\alpha = \sum_{n=1}^{\infty} \alpha_n$.

For multiplicative functions $f$ with $|f| \leq 1$ we study the series

$$\sum_{n=1}^{\infty} f(n) \left(\log n\right)^k \frac{e(\theta n)}{n}$$

with respect to $P$-regularity.

## *On Artin's Conjecture, Local Case*

*Jörg Brüdern, University of Stuttgart*

For a fixed $k \geq 3$, consider the statement: Any system of equations

$$\sum_{i=1}^{N} a_{ij}\, x_i^k = 0 \qquad (a_{ij} \in \mathbb{Z},\ 1 \leq j \leq R)$$

admits a non-trivial solution $x_i \in \mathbb{Z}$ whenever $N \geq N_0(k, R)$. According to a well-known conjecture of Artin, this should be true with

(1) $$N_0 = Rk^2 + 1,$$

but this has been confirmed only when $R = 1$ or when $R = 2$ and $k$ is odd (by Davenport and Lewis, middle 60ies). It is known that

$$N_0 = 3R^2 k \log(3Rk) \quad (k \text{ odd}), \qquad N_0 = 48Rk^3 \log(3Rk^2) \quad (\text{else})$$

are admissible choices. For odd $k$, this is very satisfactory in the $k$-aspect, but for even $k$, the $k$-aspect is $k^3 \log k$ which falls considerably short of the expected $k^2$ in (1). In joint work with H. Godinko (Brasilia) we showed

THEOREM 1. *Let* $R \geq 3$. *Then* $N_0 = R^3 k^2$ *is admissible unless* $R = 3$, $k = 2^\tau$ *in which case one may take* $N_0 = 36k^2$.

Refinements are possible for small $R$ or $k$. We discuss in detail pairs $(R = 2)$. Here Davenport and Lewis showed that $N_0 = 7k^3$ is enough when $k$ is even.

THEOREM 2.

(i) *If* $k = 2 \cdot 5^{\tau}$ *or* $k = p^{\tau}(p-1)$ *with* $p > 2$ *prime, then* $N_0(k,2) = 6k(k-1)$ *is admissible.*

(ii) *If* $k$ *is not of the form considered in* (i) *but* $k = 2^{\tau}k_0$ *with* $k_0 \in \{1,3,5\}$, *then* $N_0(k,2) = 16k^2k_0^{-1}$ *is admissible.*

(iii) *For all other* $k$, *the choice* $N_0(k,2) = 3k(k-1)$ *is admissible.*

## *Lattice Point Methods and Divisor Sum Problems*

*Stephan Daniel, University of Stuttgart*

For some residue class $a(\bmod q)$, $q \in \mathbb{N}$, we define

$$E(\underline{M},\underline{N},q,a) = \#\{(x_1,x_2)\in\mathbb{Z}^2 : M_i < x_i \le M_i + N_i,\ x_1 \equiv a x_2 (\bmod q)\} - \frac{N_1N_2}{q}.$$

Let $f$ denote an irreducible polynomial with integer coefficients. We show that for $Q \ge 1$

$$\sum_{q\le Q} \sum_{\substack{a(\bmod q)\\ f(a)\equiv 0(\bmod q)}} \max_{\substack{0<N_1,N_2\le N\\ \underline{M}\in\mathbb{R}^2}} |E(\underline{M},\underline{N},q,a)| \ll \sqrt{Q}\log Q\, N + Q.$$

We deduce

$$\#\{(x_1,x_2,a,q):\ q\le Q,\ x_1\equiv a x_2(\bmod q),\ f(a)\equiv 0(\bmod q),\ \alpha_i < \tfrac{x_i}{q} \le \alpha_i + \eta_i\} \sim \eta_1\eta_2\, cQ^2$$

holds for some constant $c = c(f)$. By the same method we can show the mean valve evaluation

$$\sum_{x_1,x_2\le N} d(|g(x_1,x_2)|) = cN^2\log N + O(N^2\sqrt{\log N})$$

and similar estimates, where $g$ is an irreducible binary form of degree 4.

## *Almost Prime Numbers with Missing Digits*

*Cècile Dartyge, University of Nancy I*

*Joint work with Christian Mauduit, University of Marseille II*

Let $r \in \mathbb{N}$, $r \ge 3$, $\mathfrak{D} = \{0, d_2, \ldots, d_t\} \subset \{0, \ldots, r-1\}$ with $2 \le t \le r-1$ and so that $\gcd(d_2,\ldots,d_t) = 1$. Define $\mathfrak{W}_{\mathfrak{D}} = \{n \in \mathbb{N}:\ n = \sum_{j=0}^{N} \varepsilon_j r^j \text{ with } \varepsilon_j \in \mathfrak{D}\}$. Then, there exists $k = k(r,\mathfrak{D})$ such that $\mathfrak{W}_{\mathfrak{D}}$ contains infinitely many integers with at most $k$ prime factors.

## *A Step beyond Kneser's Addition Theorem*

*Jean-Marc Deshouillers, University of Bordeaux*

*Joint work with Gregory A. Freiman, Tel Aviv*

A general philosophy is that if you consider in a monoid a set $\mathcal{A}$ such that $\mathcal{A} \cdot \mathcal{A} = \{a \cdot b : a \in \mathcal{A},\ b \in \mathcal{A}\}$ is small compared with $\mathcal{A}$, in the sense of cardinality, measure or density, then $\mathcal{A}$ has a special structure.

THEOREM. *Let $\mathcal{A} \in \mathbb{Z}/n\mathbb{Z}$ which is not included in a coset modulo a proper subgroup of $\mathbb{Z}/n\mathbb{Z}$ with $|\mathcal{A}| \le 10^{-9}n$ and*

$$|\mathcal{A} + \mathcal{A}| \le 2.04\,|\mathcal{A}|. \tag{1}$$

*Then there exists a proper subgroup $\mathcal{H}$ of $\mathbb{Z}/n\mathbb{Z}$ such that*
*($i$) either $\mathcal{A}$ is included in an arithmetic progression of $\ell$ cosets modulo $\mathcal{H}$ with*

$$(\ell - 1)\,|\mathcal{H}| \le |\mathcal{A} + \mathcal{A}| - |\mathcal{A}|, \tag{2}$$

*or*
*($ii$) $\mathcal{A}$ is included in three cosets modulo $\mathcal{H}$ and (2) holds with $\ell - 1$ replaced by 3.*

COMMENTS

- This is the first result of this type in $\mathbb{Z}/n\mathbb{Z}$ for general $n$ with a constant larger than 2 in (1).
- When the constant in (1) is less than 2, then Kneser's Theorem permits to study the structure of $\mathcal{A}$.
- A similar result with a larger constant in (1) has been obtained by Freiman in the early 60's when $n$ is prime.
- The values of the constants are by no mean best possible.
- The general case of the Theorem is ($i$), which means that $\mathcal{A}$ belongs to an arithmetic progression of cosets modulo $\mathcal{H}$ which is well-filled by $\mathcal{A}$.
- As soon as the constant in (1) is at least 2, there is no way to dispense with ($ii$). Indeed, if $\mathcal{A}$ consist of three cosets modulo $\mathcal{H}$ in general position, we have $|\mathcal{A} + \mathcal{A}| = 2|\mathcal{A}|$.
- The proof combines analytic and combinatorial ideas.

## *Primes and Products*

*Peter D. T. A. Elliott, University of Boulder, Colorado*

THEOREM 1. *There are infinitely many representations*

$$2 = \frac{p_1 + 1}{q_1^2 + 5} \cdot \frac{p_2 + 1}{q_2^2 + 5} \cdot \frac{q_3^2 + 5}{p_3 + 1}$$

*with $p_i, q_j$ prime.*

THEOREM 2. *There are infinitely many representations*

$$2 = \frac{p_1+1}{q_1^3+5} \cdot \frac{p_2+1}{q_2^3+5} \cdot \frac{q_3^3+5}{p_3+1}$$

*with* $p_i$, $q_j$ *prime.*

Let $f$ be a polynomial with integer coefficients and leading coefficient positive. Call a prime $\ell$ singular (w.r.to $f$) if for some $m \in \mathbb{Z}$, the congruence $mf(n) \equiv 1 (\bmod \ell)$ has $\ell - 1$ reduced residue class solutions $n (\bmod \ell)$. Then $\ell \le 1 + \deg f$. Let $\Delta$ be the product of the primes singular w.r.to $f$. There are classes $n_j (\bmod \Delta)$, $1 \le j \le J$, such that $m \equiv n_j (\bmod \Delta)$ for some $j$ iff no congruence $mf(n) \equiv 1 (\bmod \ell)$ has $\ell - 1$ reduced solutions for any prime $\ell$.
Define

$$d(\mathcal{E}) = \liminf_{x \to \infty} \left( \frac{Jx}{\Delta} \right)^{-1} \sum_{j=1}^{J} \sum_{\substack{n \le x, n \in \mathcal{E} \\ n \equiv n_j (\bmod \Delta)}} 1$$

for sets of rational integers $\mathcal{E}$.

THEOREM 3. *The density $d$ of the set of integers representable in the form $(p+1)f(q)^{-1}$ with $p$, $q$ prime, is at least $1/4$.*

## *On the Norm Form Inequality $|F(\underline{x})| \le M$*

*Jan-Hendrik Evertse, University of Leiden*

A major tool in estimating the number of solutions is the quantitative Subspace Theorem. The first such result was obtained by W. M. Schmidt in 1989. Thanks to many improvements, due to the replacement of Roth's Lemma by Faltings' Product Theorem and the replacement of the adelic version of Minkowski's Theorem on successive minima of convex bodies by McFeat and Bombieri–Vaaler by the absolute Minkowski Theorem of Roy and Thunder, Schlickewei and Evertse succeeded in deriving an absolute quantitative Subspace Theorem, a special case of which is as follows:

THEOREM 1. *Let $L_1(\underline{x}), \dots, L_n(\underline{x})$ be linearly independent linear forms in $x_1, \dots, x_n$ with coefficients in a number field of degree $D$, and with absolute Weil heights $H(L_i) \le H$. Suppose that $|L_i| := \max|$coeff. of $L_i| = 1$. Let $0 < \delta < 1$, and let $\overline{\mathbb{Z}}$ be the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. Then the set of solutions of*

$$(1) \qquad \prod_{i=1}^{n} \max_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} |L_i(\sigma(\underline{x}))| \le H(\underline{x})^{-\delta} \qquad \text{in } \underline{x} \in \overline{\mathbb{Z}}^n$$

*with $H(\underline{x}) \ge (2nH)^{2nD/\delta}$ is contained in the union of at most $4^{(n+6)^2}\delta^{-2n-4}\log_4 D \log\left(\frac{\log_4 D}{\delta}\right)$ proper linear subspaces of $\overline{\mathbb{Q}}^n$ defined over $\mathbb{Q}$.*

Now let $F(\underline{x}) = cN(\alpha_1 x_1 + \dots + \alpha_n x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ be a norm form of degree $r$. In 1971 Schmidt showed that if $F$ is non-degenerate then for every $M \ge 1$, the number $Z_F(M)$ of solutions of $|F(\underline{x})| \le M$ in $\underline{x} \in \mathbb{Z}^n$ is finite. Using his quantitative Subspace

Theorem, he gave in 1989 an upper bound for the number of solutions $Z_F(1)$ of $|F(\underline{x})| = 1$ depending only on $r = \deg F$ and $n$. He conjectured that $Z_F(M) \le c(n,r)M^{n/r}$. I proved the following weaker result, using Theorem 1:

THEOREM 2. *Suppose $F$ is non-degenerate. Then*

$$Z_F(M) \le (\delta r)^{(n+7)^3/3} M^{(n+\sum_{m=2}^{n-1}\frac{1}{m})\frac{1}{r}} (1+\log M)^{\frac{1}{2}n(n+1)-1}.$$

## *Non-Vanishing of L-Functions and the Irreducibility of Hecke Polynomials*

*David W. Farmer, Bucknell University, Lewisburg*

Kohnen and Zagier asserted that the $L$-functions associated to Hecke eigenforms $f \in S_k(1)$ do not vanish at the critical point if the Hecke algebra of $S_k(1)$ is simple (i.e. at least one Hecke generator $T_n$ has an irreducible $/\mathbb{Q}$ characteristic polynomial). An outline of a proof of that result was described. Several results and calculations on the irreducibility and factorization (mod $\ell$) of these polynomials were described.

## *Mumford's Degree of Contact and Diophantine Approximations*

*Roberto G. Ferretti, Inst. des Hautes Études Scientifiques, Bures-sur-Yvette*

Given linear forms $L_0, \dots, L_n$ with coefficients in a number field $L$. Then, under some more conditions, the Schmidt Subspace Theorem implies that the solutions $\underline{x} \in \mathbb{P}^n(K)$ for a subfield $K \subset L$ of the inequalities

$$\frac{|L_i(\underline{x})|}{|\underline{x}|} < H(\underline{x})^{-r_i}$$

l e in finitely many subspaces of $\mathbb{P}^n$ if

(1) $$\sum_{i=0}^{n} r_i > n+1.$$

If we consider solutions $\underline{x} \in X(K)$ for some algebraic subvariety $X \subset \mathbb{P}^n$, can we weaken the condition (1)? The answer is positive in several cases. We consider some examples given by ruled surfaces, Weierstrass fibrations and blow-ups.

## *Exponential Sums and Divisibility of Class Numbers*

*Etienne Fouvry, University of Orsay*

If $\Delta$ is a fundamental discriminant, we denote by $h(\Delta)$ the class number of $\mathbb{Q}(\sqrt{\Delta})$. We sketched the proofs of

THEOREM 1 *(joint with S. Daniel). There exist infinitely many positive fundamental discriminants such that* $\Delta + 4$ *is also a fundamental discriminant and such that* $h(\Delta)$ *and* $h(\Delta + 4)$ *are both odd.*

THEOREM 2 *(joint with Belabas). There exist infinitely many primes* $p \equiv 1 (\bmod 4)$ *such that* 3 *does not divide* $h(p)$.

THEOREM 3. *There exist infinitely many positive fundamental discriminants such that* $\Delta+4$ *is also a fundamental discriminant,* $h(\Delta)$ *is odd and such that* 3 *does not divide* $h(\Delta+4)$.

Some tools which we use are the Gauß criterion for the 2-rank of quadratic fields, Davenport -Heilbronn results on the average behavior of the 3-rank of quadratic fields, the average behavior of primes in arithmetic progressions (Bombieri–Friedlander–Iwaniec result) and how to bound the exponential sums

$$\sum_{\Delta(a,b,c,d)\equiv 0 (\bmod p)} e\left(\frac{ah_1 + bh_2 + ch_3 + dh_4}{p}\right)$$

and

$$\sum_{\Delta(a,b,c,d)+4\equiv 0 (\bmod p)} e\left(\frac{ah_1 + bh_2 + ch_3 + dh_4}{p}\right)$$

with $\Delta(a,b,c,d) = b^2c^2 + 18abcd - 27a^2d^2 - 4b^3d - 4c^3a$ by using either the algebraic properties of the function $\Delta$ or a result of Katz–Laumon about exponential sums.

## *Primes in Short Segments of Arithmetic Progressions*

*Daniel A. Goldston, San Jose State University*

*Joint work with C. Y. Yildirim.*

Let

$$I(x,h,q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_x^{2x} \left(\psi(y+h,q,a) - \psi(y,q,a) - \frac{h}{\varphi(q)}\right)^2 dy\,,$$

where

$$\psi(x,q,a) = \sum_{\substack{n\le x \\ n\equiv a (\bmod q)}} \Lambda(n)\,.$$

Assuming a Twin Prime Conjecture we prove

$$I(x,h,q) \sim hx \log\left(\frac{xq}{h}\right) \qquad \text{for } 1 \le \frac{h}{q} \le x^{1/2-\varepsilon}.$$

If we replace the Twin Prime Conjecture with the Generalized Riemann Hypothesis, then we can still prove

$$I(x,h,q) \sim hx \log\left(\frac{qx}{h}\right) \qquad \text{for almost all } q \text{ with } h^{3/4+\varepsilon} \le q \le h^{1-\varepsilon};$$

$$\sum_{q \le Q} I(x,h,q) \sim Qhx \log\left(\frac{Qx}{h}\right) \qquad \text{for } h^{1/2}(\log x)^6 \le Q \le h \le x;$$

$$I(x,h,q) \ge (1-\varepsilon)\frac{hx}{2} \log\left(\left(\frac{q}{h}\right)^3 x\right) \qquad \text{for } 1 \le \frac{h}{q} \ll x^{1/3-\varepsilon}.$$

These results have applications to pair correlation of $L$-functions.

## *The Variance of Small Powers of Primitive Roots*

*Steve G. Gonek, University of Rochester*

For $g$ a primitive root $\pmod p$, let

$$\mathcal{N} = \{g^{\nu} \pmod p : 1 \le \nu \le N\}$$

and let $f(m,H)$ be the number of elements of $\mathcal{N}$ that are also in the interval $(m, m+H]$, where $1 \le H, N \le p$ and $m = 1, \ldots, p$. H. Montgomery established an asymptotic formula for the variance of $f(m,H)$ when $p^{5/7+\varepsilon} \le N \le p^{1-\varepsilon}$ and asked to what extent the range of $N$ would be increased if one were to average over all the primitive roots $\pmod p$. We show that in this case we can take $p^{2/3+\varepsilon} \le N \le p^{1-\varepsilon}$ and prove an analogous result when the primitive root $\pmod p$ is fixed, but we average over primes. In this case we can take $p^{19/27+\varepsilon} \le N \le p^{1-\varepsilon}$.

## *Solutions of Diagonal Cubic Equations*

*Roger Heath-Brown, Magdalen College, Oxford*

THEOREM 1. *Let* $p_1 \equiv p_2 \equiv p_3 \equiv p_4 \equiv p_5 \equiv 8 \pmod 9$ *be primes. Then, under the General Riemann Hypothesis* $\sum_{i=1}^{5} p_i x_i^3 = 0$ *has a non-zero integral solution.*

THEOREM 2. *Let* $p_1 \equiv p_2 \equiv p_3 \equiv p_4 \equiv 2 \pmod 3$ *be primes. Then, assuming the Parity Conjecture for elliptic curves,* $\sum_{i=1}^{4} p_i x_i^3 = 0$ *has a non-zero integral solution.*

In Theorem 1 we assume the General Riemann Hypothesis for $L$-functions with Größencharacters over $\mathbb{Q}(\frac{1+\sqrt{-3}}{2})$. In Theorem 2 the Parity Conjective is needed for curves $x^3+y^3 = A$ only. The key idea is as follows: Suppose the 3-Selmer rank of $x^3+y^3 = A$ is 1 and that the arithmetic rank is also 1, either because in Theorem 2 we assume the Parity Conjecture, or because in Theorem 1 we arrange the analytic rank to be 1. Then any $\alpha x^3+\alpha^{-1}y^3 = A$, which is everywhere locally solvable, over $\mathbb{Q}(\frac{1+\sqrt{-3}}{2})$, has rational points there. This enables us to get points on $p_1x^3+p_2y^3 = p$ for suitable primes $p$. In Theorem 2 we solve two equations $p_1x^3+p_2y^3 = p = p_3u^3+p_4v^3$ in this way. For Theorem 1 we consider all fifteen possible pairs of equations. By showing (under GRH) that the average analytic rank is at most 2, we can find one pair where both equations correspond to analytic rank 1, which suffices.

## *Partitions into Primes*

*Adolf J. Hildebrand, University of Illinois, Urbana*

The ordinary partition function $p(n)$ denotes the number of representations of $n$ as a sum of non-increasing positive integers and has a generating function $\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-1}$. We consider the function $p_\Lambda(n)$ defined by

$$\sum_{n=0}^{\infty} p_\Lambda(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-\Lambda(n)},$$

where $\Lambda(n)$ is the von Mangoldt function. This function represents a weighted count of the number of partitions into prime powers. Since $\Lambda(n)$ is 1 on average, one may expect that the behavior of $p_\Lambda(n)$ is similar to that of the ordinary partition function $p(n)$. This expectation is confirmed, to some extent, by the following res lt of B. Richmond (1975): Let $\Delta(n) = \log p_\Lambda(n) - \log p(n)$. Then

$$\Delta(n) \ll \sqrt{n}\exp\left(-(\log n)^{4/7-\varepsilon}\right). \tag{1}$$

Moreover, under the Riemann Hypothesis, (1) can be sharpened to

$$\Delta(n) \ll n^{1/4}. \tag{2}$$

For comparison, $\log p(n)$ has order of magnitude $\sqrt{n}$. Recently, my student Yi-Fan Yang improved these results as follows:

THEOREM.

(*i*) *The unconditional estimate* (1) *can be sharpened to*

$$\Delta(n) \ll \sqrt{n}\exp\left(-C\frac{\log n}{(\log\log n)^{2/3}(\log\log\log n)^{1/3}}\right).$$

(*ii*) *The estimate* (2) *is best-possible in the sense that* $\Delta(n) = \Omega_\pm(n^{1/4})$.

(*iii*) *If* (2) *holds in the weaker form* $\Delta(n) = O_\varepsilon(n^{1/4})$ *then the Riemann Hypothesis is true. Thus* (2) *is equivalent to the Riemann Hypothesis.*

## *Integer Points Close to Curves and Exponential Sums*

*Martin N. Huxley, University of Cardiff*

The methods for bounding an exponential sum

(1)
$$\sum_m e\left(TF\left(\frac{m}{M}\right)\right)$$

and estimating $R$, the number of solutions of

(2)
$$\left|n - NF\left(\frac{m}{M}\right)\right| \le \delta\,,$$

are compared with analogues for rational points

(3)
$$\left|\frac{r}{q} - \lambda F\left(\frac{m}{n}\right)\right| \le \frac{\delta}{Q^2}$$

or projective rational points

(4)
$$\left|\frac{n}{q} - \lambda F\left(\frac{mQ}{qM}\right)\right| \le \frac{\delta}{Q}\,.$$

New results include

$$R \ll \delta^{1/3}M + M^{(9-2\alpha)/10}, \qquad \text{where } \alpha = \frac{\log N}{\log M}\,,\ \ \frac{3}{2} \le \alpha < 2\,,$$

in (2) under standard conditions and a bound for (3). There are applications to means of differences between square-free numbers

$$\sum_{s_{i+1} \le N} (s_{i+1} - s_i)^\eta \sim \beta(\eta)\,N$$

and to the exponential sums given in (1).

## *The Mellin Transform and the Riemann Zeta-Function*

*Aleksandar Ivić, University of Belgrade*

Let

$$Z_2(s) = \int_1^\infty \left|\zeta(\tfrac{1}{2} + ix)\right|^4 x^{-s}\,dx \qquad (\operatorname{Re} s > 1)\,.$$

By using analytic properties of $Z_2(s)$ several results have been obtained. These include two-sided omega results for $\int_0^T E_2(t)dt$ and $L(T)$, where $E_2(T)$ is the error term in the asymptotic formula for $\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt$ and $L(T)$ is the error term in the asymptotic formula for $\int_0^T |\zeta(\frac{1}{2} + it)|^4 e^{-t/T} dt$. It is proved that $\int_0^T E_2^2(t)dt \gg T^2$, which complements the estimate $\int_0^T E_2^2(t)dt \ll T^2(\log T)^C$, obtained jointly with Y. Motohashi in 1994. Mean square estimates for $Z_2(s)$ $(\frac{1}{2} < \operatorname{Re}(s) < 1)$ and possibilities to use $Z_2(s)$ to bound $\int_0^T |\zeta(\frac{1}{2} + it)|^6 dt$ and $\int_0^T |\zeta(\frac{1}{2} + it)|^8 dt$ are discussed. The latter is joint work with M. Jutila and Y. Motohashi.

## *The Mellin Transform of the Fourth Power of Riemann's Zeta-Function*

*Matti Jutila, University of Turku*

The function

$$Z_2(s) := \int_1^\infty \left|\zeta(\tfrac{1}{2}+ix)\right|^4 x^{-s}\,dx$$

has been introduced and studied by Y. Motohashi, who showed its meromorphic continuation and spectral decomposition. In the half-plane $\operatorname{Re} s > -1/2$, this function has poles at $1$, $\frac{1}{2} \pm i\kappa_j$ and $\varrho/2$, where $\kappa_j = \sqrt{\lambda_j - 1/4}$ with $\lambda_j$ standing for eigenvalues of the hyperbolic Laplacian and $\varrho$ summing over non-trivial zeros of the Riemann zeta-function. It is interesting that

$$\zeta_f(s) = \sum_{n=1}^\infty d(n)\,d(n+f)\,n^{-s} \qquad (f \neq 0),$$

related to the additive divisor problem, has the same poles. Moreover, the latter function is of polynomial growth on vertical lines, if neighborhoods of the poles are excluded. By analogy, one would expect the same to be true for $Z_2(s)$ as well, and a proof of this is in fact outlined in the lecture. More precisely,

$$Z_2(u+iv) \ll (v+1)^{(7-6u)/4+\varepsilon}$$

for $u > -1/2$, $v \geq 0$. This should be compared with

$$\zeta_f(u+iv) \ll (v+1)^{1-u+\varepsilon},$$

which may be viewed to represent the conjectured order of $Z_2(s)$, again by the same analogy. The basic idea of the proof of the estimate of $Z_2(s)$ is to replace $\left|\zeta(\frac{1}{2}+ix)\right|^4$ in its definition by a local weighted average, for which a spectral decomposition due to Motohashi is available. Then a correction term has to be added and in the resulting decomposition of $Z_2(s)$ into a sum of two functions, both of them can be shown to be of polynomial order. This method works also, at least to some extent, for automorphic $L$-functions.

## *On the Structure of the Selberg Class*

*Jerzy Kaczorowski, University of Poznan*

This is a report on a work in progress. Let $\mathcal{S}_d$ denote the set of $L$-functions from the Selberg class $\mathcal{S}$ having degree $d$. Functions from $\mathcal{S}_d$ are fully characterized for $d \leq 1$ only (Bochner, Richert, Conrey–Ghosh, Kaczorowski–Perelli). The basic conjecture in this context is the so called Degree Conjecture saying that $\mathcal{S}_d = \emptyset$ unless $d \in \mathbb{Z}$. We prove the following

THEOREM *(joint with A. Perelli). There are no $F \in \mathcal{S}_d$ with poles if $1 < d < 2$.*

The proof depends on the study of the suitable twist of $F(s) = \sum_n a_n n^{-s} \in \mathcal{S}_d$:

$$F^*(s) = \sum_{n=1}^\infty a_n n^{-s} \exp\left(-2\pi i A_F n^{1/(d-1)}\right),$$

where $A_F = (d-1)\,q_F^{1/(d-1)}$ and $q_F$ is the modulus of $F$.

## *Sums of Fourth Powers and Related Topics*

*Koichi Kawada, Iwate University, Morioka*

*Joint work with Trevor D. Wooley, University of Michigan*

We first prove a good lower bound for $N(X)$, the number of natural numbers $\leq X$, which are the sum of 5 fourth powers. Instead of 5 genuine fourth powers, let $r(n)$ be the number of representations of $n$ in the form $n = 2m^2 + u^4 + v^4$, where $u, v \in \mathbb{N}$ and $m$ is an integer written as $m = x^2 + xy + y^2$ with $x, y \in \mathbb{N}$. Then, by a well-known argument, one can easily show that $\#\{n \leq X : r(n) > 0\} \gg X^{1-\epsilon}$. On the other hand, $r(n) > 0$ means that $n$ is a sum of 5 fourth powers, since

$$2(x^2 + xy + y^2)^2 = x^4 + y^4 + (x+y)^4. \tag{1}$$

Therefore we have $N(X) \gg X^{1-\epsilon}$. Using Tenenbaum's method to estimate $\sum_{n \leq x} r(n)^2$, we further obtain

THEOREM 1. *$N(X) \gg X(\log X)^{-1-\epsilon}$ for any fixed $\varepsilon > 0$.*

The identity (1) is attributed to F. Roth in Dickson's book "History of the Theory of Numbers". We can apply this idea to various additive problems involving fourth powers. On occasion, however, we must admit that some residue classes modulo 16 are definitely out of grasp of our method, because the three integers $x$, $y$ and $x+y$ cannot be odd simultaneously. More precisely, one sees that $x^4 + y^4 + (x+y)^4 \equiv 0$ or $2 \pmod{16}$, while sums of 3 genuine fourth powers represent 0, 1, 2 and $3 \pmod{16}$. Anyway, some of our results are:

THEOREM 2. *When $4 \nmid k$, every sufficiently large integer is the sum of 10 fourth powers and a $k$-th power. When $4 \mid k$, every sufficiently large integer $\equiv r \pmod{16}$ with $1 \leq r \leq 9$ is the sum of 10 fourth powers and a $k$-th power.*

THEOREM 3. *Every sufficiently large integer $\equiv r \pmod{16}$ with $1 \leq r \leq 10$ is the sum of 11 fourth powers.*

## *On the Zeros of the Hardy Z-function and its Derivatives*

*Alla Lavrik-Männlin, ETH Zürich*

Hardy's $Z$-function is a real-valued function, whose zeros coincide with those of the Riemann zeta-function on the critical line. We discuss the problem of mutual localization of zeros of $Z(t)$ and its derivatives, as well as its connection with a problem of gaps between consecutive zeros of the Riemann zeta-function on the critical line.

## On the Number of Good Approximations of Algebraic Numbers by Algebraic Numbers of Bounded Degree

*Helmut Locher, University of Marburg*

Let $\alpha \in \overline{\mathbb{Q}}$, $d \in \mathbb{N}$, $\delta > 0$. Consider the inequality

$$|\alpha - \beta| < h(\beta)^{-2d^2-\delta}, \qquad \beta \in \overline{\mathbb{Q}}, \ \deg \beta \le d.$$

An explicit lower bound in terms of $\deg \alpha$, $h(\alpha)$ and $\delta$ is given, where $h(\cdot)$ denotes the absolute multiplicative height. Also a $p$-adic version of this result was presented.

## Arithmetical Results on Certain Functional Equations

*Lutz G. Lucht, University of Clausthal*

The classical system of functional equations

$$\frac{1}{n} \sum_{\nu=0}^{n-1} F\Big(\frac{x+\nu}{n}\Big) = n^{-s} F(x) \qquad (n \in \mathbb{N})$$

with $s \in \mathbb{C}$ is extended to

$$\frac{1}{n} \sum_{\nu=0}^{n-1} F\Big(\frac{x+\nu}{n}\Big) = \sum_{d=1}^{\infty} \lambda_n(d)\, F(dx) \qquad (n \in \mathbb{N})$$

with sequences $\lambda_n : \mathbb{N} \to \mathbb{C}$. We determine the periodic integrable solutions $F : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ and show that, under suitable assumptions concerning the sequence $(\lambda_n(1))$, aperiodic continuous solutions $F : \mathbb{R}_+ \to \mathbb{C}$ can only occur in the classical case. This solves an open problem in the theory of functional equations via arithmetical methods.

## The Distribution of the Values of the Riemann Zeta-Function in Short Intervals of the Critical Line

*Helmut Maier, University of Ulm*

We study the behavior of $\zeta(\frac{1}{2} + i\tau)$ for $\tau \in [t, t + (\log t)^{-\alpha}]$, $0 < \alpha < 1$, and show that $\log \zeta(\frac{1}{2} + it)$ is normally distributed with expectations depending on $t$ for most $t$-values. For the proof we use an approximation formula of Selberg to show that

$$\log \zeta(\tfrac{1}{2} + it) \sim \sum_{p \le x} p^{-\frac{1}{2} - it}$$

for most $t$-values and then replace $p^{-it}$ by independent random variables $X_p$.

# *Non-Vanishing of Critical Values of L-Functions*

*Philippe Michel, University Paris -Sud, Orsay*

*Joint work with Emmanuel Kowalski, Rutgers University*

Let $q$ be a prime, and let $S_2^p(q)$ be the set of primitive cusp forms over $\Gamma_0(q)$. We study the average order of vanishing of $L(f,s)$ for $f \in S_2^p(q)$ at the critical point $s=1$. This can be interpreted in terms of the rank of $J_0(q) = \mathrm{Jac}\, X_0(q)$ by the Birch, Swinnerton-Dyer Conjecture (BSD). We prove

THEOREM 1. *There is an absolute constant $c > 0$ such that*

$$\sum_{f \in S_2^p(q)} \operatorname*{ord}_{s=1} L(f,s) \leq \big(c + o(1)\big) \left|S_2^p(q)\right| \qquad \text{for } q \to +\infty .$$

For the first time such a bound is given unconditionally, without GRH like in former works of Brunner or Ram Murty. Moreover, one can take $c < 10$. On the other hand we also prove non-vanishing results:

THEOREM 2. *We have*

$$(1) \qquad \left|\{f \in S_2^p(q) : \operatorname*{ord}_{s=1} L(f,s) = 0\}\right| \geq \big(\tfrac{1}{6} + o(1)\big)\, \tfrac{1}{2} \left|S_2^p(q)\right| ,$$

$$(2) \qquad \left|\{f \in S_2^p(q) : \operatorname*{ord}_{s=1} L(f,s) = 1\}\right| \geq \big(\tfrac{19}{54} + o(1)\big)\, \tfrac{1}{2} \left|S_2^p(q)\right| .$$

There are similar results of Balasubramanian and Murty for the case of Dirichlet $L$-functions except that much better constants are obtained by our method. By works of Gross, Gross–Zagier, Kolyvagin–Logachev, these imply an arithmetic statement about the existence of large quotients of $J_0(q)$ satisfying the BSD Conjecture.
In particular (1) provides a lower bound for the dimension of the winding quotient of Merel $J_e$ which has rank 0, satisfies BSD and $\dim J_e \geq (\frac{1}{3} + o(1)) \dim J_0(q)$. By Gross–Zagier, (2) provides the lower bound

$$\operatorname{rank} J_0(q) \geq \big(\tfrac{19}{54} + o(1)\big) \dim J_0(q) .$$

These results have also applications to forms of weight $3/2$. Our methods are based on estimates for modified mean squares

$$\sum_f \left|L(f,s)\, M(f,s)\right|^2 , \qquad \text{where } M(f,s) = \sum_{m \leq M} \lambda_f(m)\, \kappa_m\, m^{-s} ,$$

and $s$ is either $1$ or $1 + 1/\log q + it$. By the estimates in Theorem 1, we prove an analogue of an old density theorem of Selberg to derive our upper bound. These methods generalize to other families of automorphic $L$-functions.

## *Beyond Pair Correlation*

*Hugh L. Montgomery, University of Michigan*

Assuming the Riemann Hypothesis, it is known that the Pair Correlation Conjecture is equivalent to the assertion that

$$\int_1^x \Big(\psi(x+h) - \psi(x) - h\Big)^2 dx \sim \tfrac{1}{2}x^2 h \log\left(\tfrac{x}{h}\right),$$

for $x^\varepsilon < h < x^{1-\varepsilon}$. Since the Cramér model would predict $\log x$ on the right-hand side in place of $\log(x/h)$, the distribution of $\psi(x+h) - \psi(x) - h$ is unclear. In joint work with Soundararajan, we give reasons to believe that this quantity is normally distributed with mean 0 and variance $\log(x/h)$. Equivalently, in terms of zeros, if $x^\varepsilon < T < x^{1-\varepsilon}$, then

$$\sum_{0<\gamma<T} x^{i\gamma}$$

is distributed, for $X \le x \le 2X$, like a sum of $N(T)$ unimodular independent random variables.

## *The Complex Binary Additive Divisor Problem and the Spectral Theory of the Three-Dimensional Hyperbolic Upper Half-Space*

*Yoichi Motohashi, University of Tokyo*

My original motivation was to find something lying inbetween the fourth power and the eighth power moments of the Riemann zeta-function. One of many possibilities is the fourth power moment of the Dedekind zeta-function of a given imaginary quadratic field. Naturally one may consider the same problem for any real quadratic number field; such a theory is now under construction. The problem is essentially equivalent to the complex binary additive divisor problem:

$$\sum_n \sigma_\alpha(n)\,\sigma_\beta(n+f)\,w\left(\tfrac{n}{f}\right) \qquad (f \ne 0).$$

Here $n$ runs over integers of a given imaginary quadratic field; $\sigma_\alpha$ is the sum-of-powers-of-divisors function of the field, and $w$ is a smooth weight. The use of Ramanujan's Fourier expansion of $\sigma_*$ leads us to an expression that is a sum of Kloosterman sums

$$S(m,n;\ell) = \sum_{\substack{h \,(\mathrm{mod}\, \ell) \\ (h,\ell)=1 \\ hh^* \equiv 1 \,(\mathrm{mod}\, \ell)}} e\left(\mathrm{Re}\left(\tfrac{h}{\ell}\,\overline{m}\right) + \mathrm{Re}\left(\tfrac{h^*}{\ell}\,\overline{n}\right)\right),$$

where as usual $e(x) = e^{2\pi i x}$, if the situation is simplified with the assumption that the field is $\mathbb{Q}(\sqrt{-1})$, though the generic case is very similar.

Then, following the example in the case of a rational number field due to Kuznetsov, we are led to the spectral theory of the upper half-space. We have already proved the corresponding trace formula. The formula contains an integral transform involving a product of two Bessel functions. Now, the problem has essentially been reduced to the "inversion" of this integral transform. Here, still a lot of work has to be done.

## *On an Equation of Goormaghtigh*

*Yuri V. Nesterenko, University of Moscow*

*Joint work with T. N. Shorey, Tata Institute, India*

The equation of Goormaghtigh asks for integers that can be written with all digits 1 with respect to two distinct bases. It has been conjectured that this problem has only finitely many solutions. For fixed positive integers $m > 2$ and $n > 2$ in the equation

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1} \tag{1}$$

H. Davenport, D. J. Lewis and A. Schinzel proved in 1961 that indeed only finitely many solutions in integers $x > 1$ and $y > 1$ with $x \neq y$ exist. This result is extended in the following quantitative sense:

THEOREM 1. *Let $m - 1 = dr$, $n - 1 = ds$, where $d, r, s$ are positive integers, $d \geq 2$, $\gcd(r, s) = 1$. Then (1) with $x < y$ implies that*

$$x < \max\left(9, \frac{gD_r}{2} + 1\right),$$

*where*

$$g = \frac{d+1}{d^2} \quad \text{and} \quad D_r = d^m \prod_{p \mid d} p^{\operatorname{ord}_p(r!)}.$$

The theorem yields all solutions of (1) for small values of $d$, $r$ and $s$. For example

THEOREM 2. *Equation (1) with $x < y$, $m \equiv 1 \pmod 2$ and $n = 3$ implies that*

$$m \geq 25 \quad \text{unless} \quad (x, y, m) = (2, 5, 5) \text{ or } (2, 90, 13).$$

## *Linear Independence in the Selberg Class*

*Alberto Perelli, University of Genova*

Motivated by the Countability Problem for the Selberg class $\mathcal{S}$, i.e. the class of Dirichlet series admitting meromorphic continuation, functional equation and Euler product, we prove the following theorem, which is essentially a result on multiplicative functions.

THEOREM 1 *(joint with J. Kaczorowski). Distinct functions in $\mathcal{S}$ are linearly independent over $\mathbb{C}$.*

Calling two functions $f$, $g$ equivalent if $f(p^m) = g(p^m)$ for all $m$ and all but finitely many primes $p$, we also have

THEOREM 2 *(joint with J. Kaczorowski). Pairwise non-equivalent multiplicative functions are linearly independent over* $\mathbb{C}$.

In fact, Theorem 1 follows immediately from Theorem 2 by a result of Murty–Murty, asserting that coefficients of functions in $S$ are non-equivalent.

## *The Almost Periodicity of the Normalized Sequence of Class Numbers*

*Manfred Peter, University of Freiburg*

Let $h(d)$ be the number of equivalence classes of binary primitive quadratic forms of discriminant $d$. It is shown that the sequence $d \mapsto h(-d)d^{-1/2}$, $d \in \mathbb{N}$, $\equiv 0, 1 (\text{mod } 4)$, no square, is almost periodic. This can be generalized to sequences $d \mapsto L(s, \chi_d)$ with $\text{Re}\, s > 1/2$ and $\chi_d$ the Jacobi character associated to $d$. Other possible generalizations are related to Hurwitz' class numbers and the numbers of representations of natural numbers by a positive integral ternary quadratic form. As a consequence the existence of limit distributions and mean values of these sequences over certain subsets of $\mathbb{N}$ can be shown.

## *Some Remarks on Methods of Diophantine Approximation*

*Patrice Philippon, Paris*

Hoping for a shake-hand between methods from Diophantine Approximation Theory and Transcendance Theory, we show how zero estimates from Transcendence Theory imply Roth's type lemmas (including the Product Theorem), we also recall how the Subspace Theorem can deal with forms of higher degrees and finally we formulate some strong conjecture on lower bounds for linear forms in logarithms of rational numbers with rational coefficients, inspired by the Subspace Theorem and which would imply, for example, the *abc*-Conjecture.

## *Haar Wavelets and Irregularities of Distribution*

*Andrew Pollington, Brigham Young University, Provo*

We study the discrepancy function of $N$ points in the unit $d$-dimensional cube and obtain lower bounds for the discrepancy with respect to rectangles with sides parallel to the coordinate axes. The method adopted is to use the Haar system, where the fundamental building blocks are squares. Using this method we obtain lower bounds for $||D||_1$.

## *Heights and Siegel's Lemma*

*Damien Roy, University of Ottawa*

E. Bombieri and J. Vaaler showed that, if $V$ is a subspace of $\overline{\mathbb{Q}}^n$ of dimension $m$ defined over a number field $K$, then there is a basis $\{\underline{x}_1, \dots, \underline{x}_m\}$ of $V$ contained in $K^n$ which satisfies

$$H(\underline{x}_1)\cdots H(\underline{x}_m) \leq m^{m/2}|\mathrm{Disc}(K)|^{\frac{m}{2d}} H(V),$$

where $H$ denotes the absolute Weil's height on $\overline{\mathbb{Q}}^n$, $d$ the degree of $K$ and $\mathrm{Disc}(K)$ its discriminant. In a joint work with J. Thunder, we prove that a dependence on the field $K$ is needed if looking for a basis of $V$ in $K^n$ but not for a basis of $V$ in $\overline{\mathbb{Q}}^n$. In the latter case, we prove that, for any constant $c > c(m)^m$, where $c(m) = \sqrt{2}^{m-1}$, there exists a basis $\{\underline{x}_1, \dots, \underline{x}_m\}$ of $V$ with

$$H(\underline{x}_1)\cdots H(\underline{x}_m) \leq cH(V).$$

We call this an *absolute Siegel's Lemma.* Let $K$ be a number field and let $K_{\mathbf{A}}$ denote its ring of adeles. To each element $A$ of $\mathrm{GL}_n(K_{\mathbf{A}})$, we associate a height function $H_A$ on $\overline{\mathbb{Q}}^n$. When $A$ is the identity, this is the usual absolute height on $\overline{\mathbb{Q}}^n$ denoted $H$ above. We also define $H_A(V)$ for a subspace $V$ of $\overline{\mathbb{Q}}^n$ and $H_A(P)$ for a polynomial $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$, and we indicate properties of these. The main one is that, given an injective linear map $\varphi: \overline{\mathbb{Q}}^m \to \overline{\mathbb{Q}}^n$ defined over $K$ and an element $A$ of $\mathrm{GL}_n(K_{\mathbf{A}})$, there exists an element $B$ of $\mathrm{GL}_m(K_{\mathbf{A}})$ such that $H_A(\varphi(V)) = H_B(V)$ for any subspace $V$ of $\overline{\mathbb{Q}}^m$. These twisted heights introduced by J. Thunder are an essential ingredient in the proof of our absolute Siegel's Lemma.

## *Additive Completion*

*Imre Z. Ruzsa, Math. Inst. of the Hungarian Academy of Sciences, Budapest*

We say that two sets $\mathcal{A}, \mathcal{B}$ are additive complements if all except finitely many positive integers are of the form $a + b$, $a \in \mathcal{A}, b \in \mathcal{B}$. We say that a complement $\mathcal{B}$ of $\mathcal{A}$ is economical, if $A(x)B(x) \ll x$, with $A(x)$, $B(x)$ denoting the respective counting functions, and it is exact, if $A(x)B(x)/x \to 1$.

By a result of Narkiewicz for a pair of exact complements we have $A(2x)/A(x) \to 1$ and consequently $A(x) = O(x^\varepsilon)$, or the analogous statements for $\mathcal{B}$. Hence in order to have an exact complement, $\mathcal{A}$ must be either very thin or very dense.

We found that very thin sets automatically have an exact complement. If $\mathcal{A} = \{a_1, a_2, \dots\}$, such that $a_{n+1}/(na_n) \to \infty$, then $\mathcal{A}$ has an exact complement. We show with a modification of the method that the same is true for $\mathcal{A} = \{2^n : n \in \mathbb{N}\}$.

We also prove that the primes do not have an exact complement. Every completion $\mathcal{B}$ of the set of primes must satisfy $\liminf B(x)/\log x \geq e^\gamma$, where $\gamma$ denotes the Euler constant. We conjecture that it does not even have an economic complement.

## *Rational Points on a Class of Superelliptic Curves*

*Jürgen W. Sander, University of Hannover*

A famous Diophantine equation is given by

$$y^k = (x+1)(x+2)\cdots(x+m). \tag{1}$$

For $k \geq 2$ and $m \geq 2$, all integer solutions of (1) are $x = -j$ $(j = 1, \ldots, m)$, $y = 0$, by a remarkable result of Erdős and Selfridge in 1975. From the viewpoint of Algebraic Geometry, equation (1) represents a plane curve for fixed $k$ and $m$. Therefore it is natural to ask for rational solutions. For $k \geq 2$, $m \geq 2$ and $k + m > 6$, we know from Faltings' proof of Mordell's Conjecture that (1) has at most finitely many rational solutions. In this talk we shall use Wiles' recent method and results, which led to the celebrated proof of Fermat's Last Theorem, in order to deduce the following

THEOREM. *For $k \geq 2$ and $2 \leq m \leq 4$, all rational points $(x; y)$ on the superelliptic curve (1) are the trivial ones with $x = -j$ $(j = 1, \ldots, m)$, $y = 0$, except for the case $k = m = 2$, where we have exactly those satisfying*

$$x = \frac{2c_1^2 - c_2^2}{c_2^2 - c_1^2}, \qquad y = \frac{c_1 c_2}{c_2^2 - c_1^2}$$

*with coprime integers $c_1 \neq \pm c_2$.*

## *The Subspace Theorem and Geometry of Numbers*

*Hans Peter Schlickewei, University Marburg*

*Joint work with Jan-Hendrik Evertse, Leiden*

The classical Subspace Theorem of W. M. Schmidt (1972) says the following:

Let $L_1, \ldots, L_n$ be linearly independent linear forms in $X_1, \ldots, X_n$ with algebraic coefficients. Suppose $\delta > 0$. Then there exist finitely many proper linear subspaces $T_1, \ldots, T_t$ of $\mathbb{Q}^n$ such that the set of solutions $\underline{x} \in \mathbb{Z}^n$ of the inequality $|L_1(\underline{x}) \cdots L_n(\underline{x})| < |\underline{x}|^{-\delta}$ is contained in the union $T_1 \cup \ldots \cup T_t$. Here we give a quantitative, parametric version of this theorem. A very special version of our result is the following:

Let $K$ be a number field of degree $d$. Write $\mathfrak{M}(K)$ for the set of places of $K$. Suppose that for each $v \in \mathfrak{M}(K)$ we are given linearly independent linear forms $L_1^{(v)}, \ldots, L_n^{(v)}$ with coefficients in $K$. Assume that we have $L_1^{(v)} = X_1, \ldots, L_n^{(v)} = X_n$ for almost all $v \in \mathfrak{M}(K)$. Let $\underline{c} = (c_{iv};\ v \in \mathfrak{M}(K),\ i = 1, \ldots, n)$ be a tuple of real numbers with

$$\sum_{v \in \mathfrak{M}(K)} \sum_{i=1}^{n} c_{iv} = 0, \qquad \sum_{v \in \mathfrak{M}(K)} \max_i c_{iv} \leq 1$$

$$c_{1v} = \ldots = c_{nv} = 0 \qquad \text{for almost all } v \in \mathfrak{M}(K).$$

For $v \in \mathfrak{M}(K)$ write $\|\ \|_v$ for the absolute value corresponding to $v$, normalized such that the product formula holds. For a finite extension $F$ of $K$ and for $w \in \mathfrak{M}(F)$ lying above $v \in \mathfrak{M}(K)$ write:

$$L_i^{(w)} = L_i^{(v)}, \quad d(w,v) = [F_w : K_v]/[F : K], \quad c_{iw} = d(w/v)\, c_{iv}\,.$$

Finally, for $v \in \mathfrak{M}(K)$ put $s(v) = 1$ if $v \mid \infty$ and $s(v) = 0$ if $v$ is finite. Now for given $Q > 1$ consider the inequalities

$$(1) \qquad \left\| L_i^{(w)}(\underline{x}) \right\|_w < \Delta_w^{1/n}\, Q^{c_{iw} - \delta \frac{d_v}{d} d(w/v) s(v)} \qquad w \in \mathfrak{M}(F),\ v \in \mathfrak{M}(K),\ w \mid v,\ i = 1, \ldots, n,$$

$0 < \delta < \frac{1}{n}$ and where $\Delta_w = \left\| \det(L_1^{(w)}, \ldots, L_n^{(w)}) \right\|_w$. Let $C$ be defined by

$$(2) \qquad C = \max\left\{ H(L_i^{(v)}),\, n^{n/\delta} \right\}.$$

THEOREM. *Suppose that we have $R$ systems of forms $\{L_1^{(\varrho)}, \ldots, L_n^{(\varrho)}\}$ such that for any $v \in \mathfrak{M}(K)$ the system $\{L_1^{(v)}, \ldots, L_n^{(v)}\}$ is a permutation of $\{L_1^{(\varrho)}, \ldots, L_n^{(\varrho)}\}$ for a suitable $\varrho$ with $1 \le \varrho \le R$. Then there exist proper linear subspaces $T_1, \ldots, T_t$ of $\overline{\mathbb{Q}}^n$, defined over $K$,*

$$t \le 2^{2(n+4)^2} \delta^{-n-4} \log 4R \, \log\log 4R$$

*with the following property: For every finite extension $F$ of $K$ and for every $Q$ with $Q > C$ and $C$ as in (2) the set of solutions $\underline{x} \in F^n$ of (1) is contained in the union $T_1 \cup \ldots \cup T_t$.*

The theorem already has led to applications estimating the number of solutions of Diophantine equations. It is a main ingredient in W. M. Schmidt's proof that the multiplicity of a non-degenerate linear recurrence sequence of order $k$ is bounded in terms of $k$ only.
At a crucial point in our proof we use a recent result by Roy and Thunder, an absolute version of Minkowski's Theorem.

## *The Zero Multiplicity of Linear Recurrence Sequences*

*Wolfgang M. Schmidt, University of Colorado, Boulder*

Consider a linear recurrence sequence $\{u_n\}_{n \in \mathbb{Z}}$ of order $t$, so that $u_n \in \mathbb{C}$ and $u_n = c_1 u_{n-1} + \ldots + c_t u_{n-t}$ $(n \in \mathbb{Z})$ with fixed coefficients $c_1, \ldots, c_t$. Such a sequence is of the form $u_n = \sum_{i=1}^k P_i(n) \alpha_i^n$, where $\alpha_i \in \mathbb{C}^\times$ and $P_i \in \mathbb{C}[X]$ with $\sum_{i=1}^k (1 + \deg P_i) = t$. The sequence is non-degenerate if no quotient $\alpha_i/\alpha_j$ $(i \ne j)$ is a root of 1. The zero-multiplicity is the number of $n$ with $u_n = 0$. Clearly this is the number of solutions $x \in \mathbb{Z}$ of the equation

$$\sum_{i=1}^k P_i(x)\, \alpha_i^x = 0$$

of mixed polynomial-exponential type. According to a classical theorem of Skolem–Mahler–Lech, a non-degenerate linear recurrence sequence has finite zero-multiplicity. Much progress has been made during the last decade on estimating this multiplicity, with contributions by Bombieri, Evertse, Faltings, van der Poorten, Roy, Schlickewei, Thunder, Zagier, Zannier,

S. Zhang, and the author. I now can prove that a non-degenerate linear recurrence of order $t$ has zero-multiplicity below some bound $c(t)$ depending on $t$ only.

## *On the Gutman-Ivić-Matula Function and Related Topics*

*Gérald Tenenbaum, University de Nancy I*

*Joint work with Régis de la Bretèche, Orsay*

The function referred to in the title has first been defined in 1968 by Matula for purposes in theoretical chemistry. It is the only completely additive arithmetical function such that $f(p_k) = 1 + f(k)$ $(k \geq 1)$, where $p_k$ denotes the $k$-th prime. We define a vector space $\mathcal{E}$ which contains both, the above function and the logarithm. By means of a general result which links the average of an arbitrary function $g(n)$ to the asymptotic behavior of

$$R(x;g) := \frac{1}{x}\sum_{n\leq x} g(n) - \frac{1}{x}\sum_{p_k\leq x} g(k)\Big[\frac{x}{p_k}\Big],$$

we obtain remainder asymptotic formulae for all functions of $\mathcal{E}$. A quantitative mean value theorem for multiplicative functions $h$ with certain links between $h(k)$ and $h(p_k)$ enables us to obtain convergence to the Gaussian law of elements $f$ in $\mathcal{E}$ for $\big(f(n) - C_1\log n\big)/D_1\sqrt{\log n}$ for suitable $C_1 = C_1(f)$ and $D_1 = D_1(f) > 0$. An estimate of the rate of convergence is given.

## *An Old Idea of Hermite Receives New Life*

*Jeff L. Thunder, Northern Illinois University, De Kalb*

*Joint work with Damien Roy, Ottawa*

Let $K$ denote a number field and let $n$ be a positive integer. For $A \in \mathrm{GL}_n(K_{\mathbf{A}})$ let $H_A$ be the twisted height as defined in the abstract of D. Roy. Define minima $\mu_1(A) \leq \mu_2(A) \leq \ldots \leq \mu_n(A)$ as follows:

$$\mu_i(A) = \inf\{\mu > 0\colon \exists\, \underline{x}_1,\ldots,\underline{x}_i \in \overline{\mathbb{Q}}^n,\ \text{lin. indep.},\ H_A(\underline{x}_j) \leq \mu \text{ for all } j \leq i\}.$$

We prove the following absolute version of Minkowski's second Convex Bodies Theorem:

THEOREM. *Let $k, n$ and $A$ as above. Then*

$$H_A(\overline{\mathbb{Q}}^n) = |\det(A)|_{\mathbf{A}} \leq \prod_{i=1}^{n}\mu_i(A) \leq c(n)^n|\det(A)|_{\mathbf{A}},$$

*where* $c(n) = \sqrt{2}^{\,n-1}$.

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This theorem implies our absolute Siegel's Lemma stated by Roy in his abstract. It can be shown that this theorem is implied by the inequality

$$\mu_1(A) \leq c(n)\,|\det(A)|_{\mathbb{A}}^{1/n}.$$

We prove this inequality in the case $n = 2$ and then show that

$$c(n) \leq c(n-1)^{(n-1)/(n-2)}$$

for $n > 2$, giving $c(n) \leq c(2)^{n-1}$. The line of argument is similar to Hermite's method of bounding the Hermite constant $\gamma(n)$ from above by first showing that $\gamma(2) = 2/\sqrt{3}$ and then $\gamma(n) \leq \gamma(n-1)^{(n-1)/(n-2)}$ for $n > 2$.

## *On the Number of Digit Changes*

*Robert Tijdeman, University of Leiden*

It follows from work of Senge and Straus (1973) and Stewart (1980) that the number of non-zero digits of a large positive integer can only be small with respect to two bases $b_1$ and $b_2$ if $\log b_1 / \log b_2 \in \mathbb{Q}$. Stewart proved a corresponding result for terms of a linear recurrence expressed in base $b$. In a similar way, Blecksmith, Filaseta and Nicol (1993) proved that the number of digit changes of $a^n$ in base $b$ tends to infinity with $n$ unless $\log a / \log b \in \mathbb{Q}$. In joint work with Barat and Tichy such results have been generalized to linear number system expansions. It turns out that the ineffective Thue-Siegel-Roth-Schmidt method and the effective Gelfond-Baker method yield results of different types.

## *Lattice Points in Spheres*

*Kai-Man Tsang, University Hong Kong*

We consider $P_3(R)$, the remainder term in the asymptotic formula for the number of lattice points inside the three-dimensional sphere of radius $R$, centered at the origin. The upper bound $P_3(R) \ll R^{21/16+\varepsilon}$ was obtained recently by D. R. Heath-Brown. For $\Omega$-results, it is known that

$$P_3(R) = \Omega_-(R\sqrt{\log R}) \quad \text{and} \quad P_3(R) = \Omega_+(R\log\log R).$$

We introduce a different approach to prove that

$$P_3(R) = \Omega_\pm(R\sqrt{\log R})$$

holds.

# On the Number of Polynomials over $\mathbb{Z}$ having Bounded Height and Bounded Mahler Measure

*Jeffrey D. Vaaler, University of Texas, Austin*

Let $M : \mathbb{R}^N \to [0, \infty)$ denote the Mahler measure of the polynomial having $\underline{x}$ in $\mathbb{R}^N$ as its vector of coefficients. So

$$M(\underline{x}) = \exp\left\{ \int_0^1 \log\left| \sum_{n=1}^{N} x_n\, e\big((N-n)\theta\big) \right| d\theta \right\}$$

for $\underline{x} \in \mathbb{R}^N$. From this point of view, $M$ is a symmetric distance function in the sense of the geometry of numbers and

$$\mathcal{S}_N = \{ \underline{x} \in \mathbb{R}^N : M(\underline{x}) < 1 \}$$

is an open, bounded starbody. It follows, moreover, that

$$\sum_{\substack{\underline{\ell} \in \mathbb{Z}^N \\ M(\underline{\ell}) < T}} 1 = \mathrm{Vol}_N(\mathcal{S}_N)\, T^N + O_N(T^{N-1}) \qquad \text{as } T \to \infty .$$

Note that $\mathcal{S}_N$ is not convex if $N \geq 3$. We show that

$$\mathrm{Vol}_N(\mathcal{S}_N) = \frac{2^N N^{[N/2]}}{N!} \prod_{n=1}^{N-1} n^{(N-n)(-1)^n}$$

for each $N \geq 1$. The proof uses the analytic function

$$F_N(s) = \int_{\mathbb{R}^N} M\left(\binom{1}{\underline{x}}\right)^{-s} d\underline{x} \qquad (\mathrm{Re}(s) > N)$$

and the discovery that

$$F_N(s) = A_N\, s^{[\frac{N-1}{2}]+1} \prod_{0 \leq m \leq [\frac{N-1}{2}]} (s - N + 2m)^{-1} ,$$

with $A_n \in \mathbb{Q}_\times$. Similar – but easier – results hold when $\mathbb{R}$ is replaced by $\mathbb{C}$ or by a non-archimedean local field.

## *Primes in Arithmetic Progressions*

*Robert C. Vaughan, University of Michigan*

Let

$$\psi(x,q,a) = \sum_{\substack{n\le x \\ n\equiv a (\mathrm{mod}\, q)}} \Lambda(n)\,,$$

$$V(x,q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left|\psi(x,q,a) - \frac{x}{\phi(q)}\right|^2,$$

$$U(x,q) = x\log q - x\Big(\gamma + \log 2\pi + \sum_{p|q}\frac{\log p}{p-1}\Big),$$

$$M_k(x,Q) = \sum_{Q/2<q\le Q} \left|V(x,q) - U(x,q)\right|^k.$$

Then the following theorem was obtained.

THEOREM. *Suppose that $A$ is a positive number and that $k$ is a positive integer. Then for every $Q$ and $x$ with $x(\log x)^{-A} \le Q \le x$ we have*

$$M_k(x,Q) \ll Q\,x^k F\Big(\frac{x}{Q}\Big)^k + \frac{Qx^k}{(\log x)^A}$$

*where, for $y \ge 1$,*

$$F(y) \ll y^{-1/2}\exp\Big(-\frac{c(\log 2y)^{3/5}}{(\log\log 3y)^{1/5}}\Big)$$

*with $c$ a positive constant.*

## *Three Two-Dimensional Weyl Steps in the Circle Problem*

*Ulrike M. A. Vorhauer, University of Ulm*

*Joint work with Eduard Wirsing, Ulm*

We study the circle problem and its generalization involving the logarithmic mean. Most non-trivial results depend on estimates of exponential sums. Chen has carried out such estimates using three two-dimensional Weyl steps in complicated techniques. Our approach is simpler and clearer. Crucial is a good understanding of the Hessian determinant in question and a simple estimate for certain exponential integrals. We determine the order of magnitude of the Hessian as well as that of the maximum of the second derivatives for the third order differences of the two-dimensional Euclidean vector norm.
The classical tool for estimating two-dimensional exponential integrals is a theorem of Titchmarsh that was refined by Min among others. Apart from its difficult proof and somewhat doubtful formulation it has the disadvantage that it requires a system of complex side conditions that are hard to check or to satisfy. We propose for the same purpose a similar

theorem which is somewhat weaker but which, on the other hand, needs few and simple assumptions and is considerably easier to prove:

THEOREM. *Let $\mathcal{G} \subset \mathbb{R}^2$ be a convex, compact region of diameter $\ell$ with boundedly many algebraic arcs for its boundary $\partial\mathcal{G}$. Let $\mathcal{U}$ be an open neighborhood of $\mathcal{G}$ and $f : \mathcal{U} \to \mathbb{R}$ be a real algebraic function such that on $\mathcal{G}$*

$$|f_{xx}|,\ |f_{xy}|,\ |f_{yy}| \le \lambda_2 ,$$
$$|f_{xx}f_{yy} - f_{xy}^2| \ge H > 0 .$$

*Then*

$$J = \iint_{\mathcal{G}} e(f)\,dx\,dy \ll \frac{\lambda_2}{H} \log\left(2 + \sqrt{\lambda_2}\,\ell\right),$$

*where, as usual, $e(x) = e^{2\pi i x}$, and the $O$-constant depends only on the total degree of the minimal polynomial $F(x,y,f)$ of $f$ and on the number and degrees of the boundary arcs.*

The convexity condition can easily be relaxed, but it is convenient to assume and suffices for our applications. This theorem is best possible apart, possibly, from the log-factor. Any improvement in the direction of the Titchmarsh-Min lemma must use stronger assumptions. This can be seen from an instructive example that is given by the function $f(x,y) = \frac{1}{2}(r - R)^2$, $r = \sqrt{x^2 + y^2}$, on the circular ring $R/2 \le r \le R - R^\alpha$ with a parameter $\alpha \in (0,1)$. Here $\lambda_2 \asymp 1$, $H(x,y) \gg R^{\alpha-1}$ and $J \asymp R^{1-\alpha}$. The same holds for the convex hull of, say, one quarter of the above ring.

## *A Prime Number Theorem with Weights*

*Dieter Wolke, University of Freiburg*

The following weighted version of the Prime Number Theorem is discussed. There is a function $g\colon \mathbb{P} \to \mathbb{R}$ such that, with numerical constants $c_1, c_2 > 0$

$$g(p) = 1 + O\left(\exp\left(-c_1 \frac{(\log p)^{1/3}}{(\log\log p)^{1/3}}\right)\right), \qquad \sum_{p \le x} g(p) = \operatorname{li} x + O(x^{1-c_2}).$$

As I. Ruzsa and E. Wirsing remark, this can be derived very easily from a Hoheisel-Ingham type Prime Number Theorem. We get it from an analytic process which may be of interest in itself. Consider the partial fraction expansion

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_{\varrho}\left(\frac{1}{s-\varrho} + \frac{1}{\varrho}\right) + B,$$

where $\varrho$ runs over the trivial and non-trivial zeros of $\zeta(s)$. The principal idea is to erase the poles at $\varrho$ by adding $-(\zeta'/\zeta)(s + 1 - \varrho)$. As this produces new poles there are severe convergence problems. However, it can be done by using a generalized form of an approximate formula for $-\zeta'/\zeta$ due to Selberg. By this we produce a function $H(s)$ such that $H(s) - (s-1)^{-1}$ is regular for $\operatorname{Re}(s) > 0$. $H(s) = \sum \Lambda^*(n)\,n^{-s}$ in $\operatorname{Re}(s) > 1$, where $\Lambda^*$ is very close to $\Lambda$, and is of not too large order of magnitude for $\operatorname{Re}(s) > 1/2$.

# *Exponential Sums and Diophantine Equations in Many Variables*

*Trevor D. Wooley, University of Michigan*

W : provide estimates for exponential sums over binary forms of strength close to that attainable by the classical version of Weyl's Inequality and Hua's Lemma in the diagonal situation. Our main results are as follows.

THEOREM 1. *Let $\Phi(x,y) \in \mathbb{Z}[x,y]$ be a non-degenerate binary form of degree $d \geq 3$, and let*

$$F(\alpha;P,Q) = \sum_{0\leq x\leq P}\sum_{0\leq y\leq Q} e(\alpha\Phi(x,y)).$$

*Suppose that $P \asymp Q$ are large. Let $\alpha \in \mathbb{R}$, and suppose that there exist $r \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(r,q)=1$ and $|\alpha - r/q| \leq 1/q^2$. Then*

$$F(\alpha;P,Q) \ll P^{2+\varepsilon}\left(q^{-1}+P^{-1}+qP^{-d}\right)^{2^{2-d}}.$$

THEOREM 2. *Let $\Phi(x,y)$ and $F(\alpha;P,Q)$ be defined as in the statement of Theorem 1. When $d=3$ or $4$, or when $d\geq 5$ and $j=1$ or $2$, one has*

$$\int_0^1 |F(\alpha;P,Q)|^{2^{j-1}}\,d\alpha \ll P^{2^j-j+\varepsilon}.$$

*When $d\geq 5$ and $3\leq j\leq d-2$ one has*

$$\int_0^1 |F(\alpha;P,Q)|^{2^{j-1}}\,d\alpha \ll P^{2^j-j+\frac{1}{2}+\varepsilon}.$$

*When $d\geq 5$ one has also*

$$\int_0^1 |F(\alpha;P,Q)|^{\frac{5}{16}2^{d-1}}\,d\alpha \ll P^{\frac{5}{8}2^d-d+1+\varepsilon},$$

*and*

$$\int_0^1 |F(\alpha;P,Q)|^{\frac{9}{16}2^{d-1}}\,d\alpha \ll P^{\frac{9}{8}2^d-d+\varepsilon}.$$

There are applications to the solubility of equations of the type

$$\Phi_1(x_1,y_1)+\ldots+\Phi_s(x_s,y_s)=0.$$

For example with each $\Phi_i$ a binary form of degree $d$ having integral coefficients, one may establish an asymptotic formula for the number of integral solutions within a box of size B large.

## *On an Extremal Problem Related to Gaussian Sums*

*András Biro, Math. Inst. of the Hungarian Academy of Sciences, Budapest*

We prove partial results concerning a modified version of a problem of Harvey Cohn on the "characterization of characters" (see Problem 39 of the book of Hugh L. Montgomery: Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis). We consider the problem only for the prime field. We show that there are only finitely many solutions in the complex case (for a fixed prime $p$), and solve the problem completely in the mod $p$ case.

## *Local Solubility in the Waring-Siegel*

*Morley Davidson, Kent State University*

Recent progress on the analytic side of the Hardy-Littlewood-Siegel circle method for number fields, as applied to the generalized Waring problem, has justified a re-examination of the algebraic side, dealing with local solubility. It was proved by C. P. Ramanujam that, for exponent $k$ in the Waring problem for a number field $K$, using at least $8k^5$ summands guarantees local solubility (hence convergence of the 'singular series' to a positive number). We are able to improve this to $k^3 \log k$ for almost all $k$ with only two distint prime divisors, and to $k^4 \log k$ for almost all squarefree $k$, by using results of R.-M. Stemmler on the density of primes of the form $(p^r - 1)/((p^d - 1)$ with $p$ prime. We conjecture that there is a constant $c$ independent of $k$ and $K$ such that $ck$ summands suffice. (Currently it is known that $4nk$ variables are sufficient, due indepedently to Stemmler and O. Körner.)

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# *PROBLEMS POSED*

## *Oberwolfach, 10 March 1998*

*1. (Jörg Brüdern)* In $\mathbb{F}_p^2$ we pick vectors $\binom{a_i}{b_i}$, $0 \le i \le p$, such that no three are on a line, which is to say that

$$\det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} \not\equiv 0 \,(\bmod\, p).$$

Is it true that there exist numbers $\varepsilon_i = 0$ or $1$, not all $0$, so that

$$\sum_{i=0}^{p} \varepsilon_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;(\bmod\, p)\ ?$$

If this is true, is there a generalization to dimension 3 and higher?

*2. (Imre Ruzsa)* Let $a_1, a_2, \dots$ be real numbers with $0 \le a_i \le 1$ for all $i$. We consider the sums $a_i + a_j$ for $1 \le i \le j \le n$, and ask how well-spaced these sums can be. Let $\delta(n)$ be the minimum dista. ce between any two of these $n(n+1)/2$ numbers. We know that $\delta(n) \le 3/n^2$. Is it true that $\liminf_{n\to\infty} n^2\delta(n) = 0$? It is known that the $a_i$ can be chosen so that $\delta(n) \gg 1/(n\log n)^2$.

*3. (Imre Ruzsa)* Let $\mathfrak{A}$ be a set of positive integers, and let $r(n)$ denote the number of ways of writing $n = a + b^2$ with $a \in \mathfrak{A}$. Can the set $\mathfrak{A}$ be chosen so that $\sum_{n \le N} |r(n) - 1| = o(N)$?

*4. (Imre Ruzsa)* Geometric problem. It is well-known that there is no finite set on the plane (not all points in a line) with the property that every line connecting two of the points passes throught a third. There are finite sets that have the following weaker property. If we connect two points, either this line passes through a third point, or there is a parallel line that passes through at least three of our points.
I have two examples. One has 7 points: the vertices of a triangle, the midpoints of the sides and the barycenter. The other has eleven: an affine regular pentagon, the crossing poins of the diagonals, and the center. Are there any further such configurations?

*5 (Jerzy Kaczorowski)* Let

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

be the factor in the functional equation for a function $F$ in the Selberg Class. We call $d_F = 2\sum_{j=1}^{r} \lambda_j$ the *degree* of $F$. The *Degree Conjecture* asserts that $d_F$ is a positive integer for all $F$ in the Selberg Class. We now formulate three conjectures that are equivalent if the Degree Conjecture is true.

*Conj. 1*: For every $F$ in the Selberg class, the numbers $\lambda_i$ are all rational.

*Conj. 2*: Call $\lambda_i$ and $\lambda_j$ *equivalent* if $\lambda_i/\lambda_j \in \mathbb{Q}$. For a given $F$ in the Selberg Class, let $h_F$ be the number of equivalence classes among the $\lambda_i$. We conjecture that $h_F = 1$.

*Conj. 3*: Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ and $p(\underline{\lambda}) \in \mathbb{C}$ an invariant of the functional equation. Then there is a function $f : \mathbb{R} \to \mathbb{C}$ such that $p(\underline{\lambda}) = f(d_F)$.

Given a functional equation

$$\Phi(s) = \omega\, \Phi(1-s) \qquad \text{with} \quad |\omega| = 1,$$

we can define the associated number

$$\omega^* = \omega\, e^{-i\pi(\eta+1)/2} \left(\frac{q}{(2\pi)^d}\right)^{\frac{i\theta}{d}} \prod_{j=1}^{r} \lambda_j^{-2i\,\mathrm{Im}\,\mu_j},$$

where

$$q = (2\pi)^d Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \qquad \eta + i\theta = \xi := 2\sum_{j=1}^{r} (\mu_j - \tfrac{1}{2}).$$

We conjecture that $\omega^*$ is an algebraic number.

David Farmer proposes the problem of showing that if $F$ is in the Selberg Class then $F(1+it) \neq 0$.

5. ***(Alberto Perelli)*** Suppose that $F$ is in the Selberg Class, and that $F$ is entire. Put $F_\theta(s) = F(s + i\theta)$. Show that if $F$ is primitive then $F_\theta$ is primitive for all $\theta$. (This would follow from the Selberg Orthonormality Conjecture.)

We know that members of the Selberg Class have unique factorization into primitive members of the class. Show that if $F$ and $G$ are members of the Selberg Class with $(F, G) = 1$, then there is a complex number $\varrho$ such that $m_F(\varrho) \neq m_G(\varrho)$. Here $m_F(\varrho)$ denote the multiplicity of vanishing of $F$ at $\varrho$.

6. ***(Yoichi Motohashi)*** Find a direct proof (without using Kloosterman sums) for the spectral decomposition of

$$\int_{-\infty}^{+\infty} \left|\zeta(\tfrac{1}{2} + it)\right|^4 g(t)\, dt$$

with suitable weights $g$.

7. ***(Aleksandar Ivić)*** Let $\varrho$ be a simple zero of $\zeta(s)$. Bound $|\zeta'(\varrho)|$ from below, in terms of $|\varrho|$.

8. ***(Aleksandar Ivić)*** (due to Kuropa, 1971) If $p > 2$ then

$$0! + 1! + \cdots + (p-1)! \not\equiv 0 \,(\mathrm{mod}\, p)\ ?$$

True for $p < 8 \cdot 10^6$.

9. ***(Antal Balog)*** Let $c_n$ be real or complex numbers such that $c_n \mu(n) = 0$ for all $n$. How small can

$$\sup_{\alpha \in [0,1]} \left| \sum_{n \le x} e(n\alpha) - \sum_{n \le x} c_n e(n\alpha) \right|$$

be? It is known that there exist $c_n$ so that the above is $\ll x^{3/4} \log^2 x$, and that the above is $\gg x^{2/3}$ for any choice of the $c_n$.

*10. (Trevor Wooley)* Let $\mathbb{Q}^{\text{rad}}$ be the maximal radical extension of $\mathbb{Q}$. Thus, if $\alpha \in \mathbb{Q}^{\text{rad}}$ then $\alpha^{1/n} \in \mathbb{Q}^{\text{rad}}$ for all positive integers $n$. Let $d$ be a given positive integer. How large must $s$ be, in order that for any homogeneous $F \in \mathbb{Q}^{\text{rad}}[x_1, \dots, x_s]$ of degree $d$, there is a $\underline{y} \in (\mathbb{Q}^{\text{rad}})^s \setminus \{\underline{0}\}$ such that $F(\underline{y}) = 0$? For $d = 1, 2, 3, 4$, $s = 2$ is enough. For $d \geq 5$ one needs at least $s \geq d+1$. It is also known that $s = 2^{2^{d-2}} + 1$ is enough.

Also, in this connection, find an absolutely irreducible polynomial $P$ in three variables with coefficients in $\mathbb{Q}^{\text{rad}}$ such that $P$ has no non-trivial zero in $(\mathbb{Q}^{\text{rad}})^3$.

*11. (Trevor Wooley)* (due to Novák) Prove that there is a $\delta > 0$ such that the number of solutions of the equation

$$\frac{x^k - y^k}{u^k - v^k} = \frac{p^k}{q^k}$$

in variables $x, y, u, v, p, q$ satisfying $1 \leq x, y, u, v \leq X$, $(x, y) = (u, v) = (p, q) = 1$, $|p/q| \neq 1$ is $\ll X^{2-\delta}$.

This would have the following application: If $k$ is odd then the number of lattice points $(u, v)$ such that $|u|^k + |v|^k \leq T^{k/2}$ is $cT - bT^{1/2 - 1/k} + \Omega_+(T^{1/4}(\log\log)^{1/4})$.

*12. (Gérald Tenenbaum)* Is it true that the number of perfect powers between $x$ and $x + y$ is $\ll \sqrt{y}$ uniformly in $x$? Even stronger, is it true that the number of square-full integers between $x$ and $x + y$ is $\ll \sqrt{y}$ uniformly in $x$? The estimate $\ll \sqrt{y} + \log x$ is trivial.

*13. (Dieter Wolke)* Let $C$ be a sufficiently large constant. An odd integer $N$ is called *rich* if for every prime $p \in (2, N - C)$ the number $N - p$ can be written as a sum of two primes. Do there exist infinitely many rich integers? If so, give a lower bound for their frequency.

*14. (Daniel Goldston)*
Let

$$\lambda_Q(n) = \sum_{q \leq Q} \frac{\mu^2(q)}{\varphi(q)} \sum_{\substack{d|q \\ d|n}} \mu(d)\, d = \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} c_q(n),$$

and set

$$\Lambda_Q(n) = \sum_{\substack{d|n \\ d \leq Q}} \mu(d) \log(Q/d).$$

We believe that $\lambda_Q(n)$ and $\Lambda_Q(n)$ are close for most $n$. Graham (JNT 10, 1978) proved that

$$\sum_{n \leq x} \Lambda_Q(n)^2 = x \log Q + O(x)$$

for $1 \leq Q \leq x$. Prove the same for $\lambda_Q(n)$.

*15. (Yoichi Motohashi)* In the notation above, can one show that

$$\sum_{n \leq x} \Lambda_Q(n)^{2k} \ll x(\log Q)^{2k-1}$$

when $k$ is a fixed integer $> 1$?

*16. (Yoichi Motohashi)* The Brun–Titchmarsh inequality asserts that if $(q, \ell) = 1$ then

$$\pi(x; q, \ell) \leq (2 + o(1)) \frac{x}{\varphi(q) \log(x/q)} .$$

In addition, it is known that if $q \leq x^{1/3}$ then the $\log(x/q)$ in the denominator can be replaced by $\log(x/q^{3/16})$. We have two problems:
(a) Derive this improvement with the restriction $q \leq x^{1/3}$ relaxed, to allow larger values of $q$. (b) Replace $q^{3/16}$ by something smaller, even if only for a more restricted range, say $q < x^{1/100}$.

*17. (Eduard Wirsing)* Among the abstracts of the conference of Nov. 9–15, 1972 one finds the following entry:

> *It is easy to see that the set $\mathbb{P}$ of all primes cannot be represented in the form $\mathbb{P} = \mathcal{A} + \mathcal{B}$ with $\#\mathcal{A}, \#\mathcal{B} \geq 2$. Similarly $\mathbb{P} \setminus \{2\} \neq \mathcal{A} + \mathcal{B}$.*
>
> *The "Inverse Goldbach Problem" consists in showing that even*
>
> $$\mathbb{P} \cap [n, \infty) = (\mathcal{A} + \mathcal{B}) \cap [n, \infty), \quad \#\mathcal{A} \geq 2, \quad \#\mathcal{B} \geq 2$$
>
> *with any $n \in \mathbb{N}$ is impossible.*

At that occasion I proved

THEOREM. *Let $N$ be a natural number and sets $\mathcal{A}, \mathcal{B} \subset [0, N]$ such that $\mathcal{A} + \mathcal{B} \subset \mathbb{P}$. Then $\#\mathcal{A} \cdot \#\mathcal{B} \ll N$.*

The proof is a simple application of the Davenport–Halberstam inequality.

The Inverse Goldbach Problem would obviously be settled if one could prove $\#\mathcal{A} \cdot \#\mathcal{B} = o(\frac{N}{\log N})$ instead, provided that $\#\mathcal{A} \geq 2$, $\#\mathcal{B} \geq 2$.

*18. (Jürgen Sander)* A result of Erdős and Selfridge from 1975 shows that

$$y^k = (x+1)(x+2) \cdots (x+m). \tag{1}$$

has no integer solutions $x, y \neq 0$ for $k \geq 2$ and $m \geq 2$. From the viewpoint of algebraic geometry, equation (1) represents a plane curve for fixed $k$ and $m$, which is an elliptic curve for $k = 2$ and $m = 3$. Therefore, it is natural to ask for rational solutions. For $k > 1$, $m > 1$ and $k + m > 6$, we know from Faltings' proof of Mordell's conjecture that equation (1) has at most finitely many rational solutions. We have proved that for $k \geq 2$ and $2 \leq m \leq 4$, rational points $x$, $y \neq 0$ on the superelliptic curve (1) exist only for $k = m = 2$. They are given by

$$x = \frac{2c_1^2 - c_2^2}{c_2^2 - c_1^2}, \qquad y = \frac{c_1 c_2}{c_2^2 - c_1^2}$$

with coprime integers $c_1 \neq \pm c_2$. We conjecture that for other $k \geq 2$ and $m \geq 2$ no rational points $x$ and $y \neq 0$ on (1) exist.

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