Mathematisches Forschungsinstitut Oberwolfach

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Aperiodic Order

19.4. - 25.4.1997

Organized by Michael Baake (Tübingen), Ludwig Danzer (Dortmund) and Robert Moody (Edmonton) the workshop on Aperiodic Order at Oberwolfach was by all counts a great success. The warm open atmosphere of the conference and the extensive mutual interaction of the participants together with the evident emerging cohesion and depth of the subject left us all with a keen sense of excitement and optimism.

What makes the subject of aperiodic order so appealing is the extraordinary range of mathematical ideas that converge so beautifully around it and the genuine promise of mathematics to shed light on the significant problems of understanding the physical properties of real quasicrystalline materials.

The schedule was arranged to leave enough space for intensive discussion in various small groups besides the two plenary "Problem and Discussion"-sessions. Among the key ideas discussed and lucidly exposited in talks at the workshop we might mention:

- the gradual classification of the hierarchy of discrete geometrical structures that lead from strict periodic to aperiodic order
- the role of autocorrelation measures and diffraction in relation to long-range order
- the important connection between the theory of ergodic dynamical systems and diffraction, in particular for systems with pure point spectra
- the deep way in which C*-algebras enter into the band structure in the electronic theory of quasicrystals
- the considerable and various relationships of number theory to the subject, i.e., through algebraic number fields, zeta functions, or maximal orders in quaternionic algebras
- the first good intimations of an irrational wavelet theory
- the first rigorous results on stochastic model sets and random tilings and their connection to statistical mechanics and potential theory.

Part of the plan of the workshop has been to invite several experts in fields we knew ought to be part of the subject but who thought little about it. Without exception these people were entranced by the diversity and wealth of new ideas involved. There were numerous enthusiastic comments, even from seasoned "Oberwolfachers".

Part of the success of the workshop has to be attributed to the wonderful way in which the Institute is operated. The organizers are very grateful for having had the opportunity to hold this workshop under ideal conditions.

Author of the report: Gerrit van Ophuysen

VORTRAGSAUSZÜGE

(in chronologischer Reihenfolge)

On model sets

Robert V. Moody

The mathematics of aperiodic order concerns itself with extended geometric structures that are essentially discrete and exhibit long-range order, usually evidenced by discrete-like diffraction patterns. Of particular interest is the possibility of such structures to possess non-crystallographic symmetries.

One way of constructing point-sets with such properties is the cut and project formalism. In the setting originally devised by Y. Meyer (for entirely different reasons!) this looks like this:

$$\begin{array}{ccccc} \mathbb{R}^m & \xleftarrow{\pi} & \mathbb{R}^m \times H & \xrightarrow{\pi_{\text{int}}} & H \\ & & \cup & & \\ & \xleftarrow{1-1} & L & \xrightarrow{\text{dense image}} & \end{array}$$

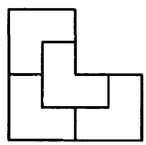
where $\mathbb{R}^m$ is the space of interest (physical space), H is a locally compact Abelian group, $L \subset \mathbb{R}^m \times H$ is a lattice, and the projection maps π and π_{int}, when restricted to L, are respectively $1-1$ and have dense image in H. Given a subset W of H which satisfies $\overline{W}$ is compact, $\text{int}(W) \neq \emptyset$, we construct the *model set*

$$\Lambda := \{\pi(x) \in \mathbb{R}^m \mid x \subset L,\ \pi_{\text{int}}(x) \in W\}.$$

Λ is a Delone set, is (generically) repetitive, and displays strong diffractive properties (see M. Schlottmann's contribution). We give two examples.

(1) Based on the maximal order of the quaternion algebra $\mathbb{H}(\mathbb{Q}(\sqrt{5}))$ we construct the Elser-Sloane model sets. These exhibit the symmetry of the non-crystallographic Coxeter group H_4 and the lattice $L \simeq E_8$, $m = 4$, $H \simeq \mathbb{R}^4$.

(2) Starting with the well known chair tiling we determine a model set interpretation of it where $m = 2$ and $H = (\hat{\mathbb{Z}}_2)^2 \sim$ the plane over the 2-adic integers.



Model sets and dynamical systems

Martin Schlottmann

Model sets can be generated by a cut and project scheme for arbitrary locally compact Abelian groups. There is a criterion formulated exclusively in terms of the "physical" space for a point set to be a regular model set. This criterion is highly non-local, i.e., cannot be checked on the set of finite patches.

Using the so-called torus parametrization, the dynamical system induced by a model set can be shown to be metrically equivalent to the action of a dense subgroup on a compact Abelian group. The pure point character and, by Dworkin's argument, the pure point diffraction is thereby established.

Perfect versus entropic order

Michael Baake

Quasicrystals are "real world" examples of structures with long-range orientational order. They show up by diffraction images containing sharp Bragg peaks (point measures) and usually display non-crystallographic symmetries. Various mathematical idealizations have been discussed in the literature. On the one hand, perfect quasicrystals have been described as model sets. Their structure is rather well understood and they show a pure point diffraction spectrum. On the other hand, random tilings are often considered to be a more realistic class of models. Their investigation is only in its infancy, and their diffraction properties are largely unknown. As an intermediate class, one can consider stochastically occupied model sets. With some standard tools from stochastics and ergodic theory, it is possible to formulate and prove a suitable generalization of lattice gases and determine their diffraction pattern. Finally, generalizing the usual symmetry concept to the action of semi-groups of self-similarities, one can connect a scaling symmetry on average with the existence of a unique invariant measure, determined as the unique solution of an integral equation (joint work with R. V. Moody). This shows in particular that there are many natural structures between lattices and random point sets worth further exploration. So, the conclusion is: we don't know yet what "order" is, neither physically nor mathematically.

Shelling icosahedral quasicrystals

Alfred Weiss

There are three non-equivalent $\mathbb{Z}[\tau]$-submodules of $\mathbb{R}^3$ which are stable under the Weyl group of a root system of type H_3. The usual shelling of a root lattice of type D_6 induces a shelling on these and then on the quasicrystal with spherical window of radius τ. In terms of the icosian ring $\mathbb{I}$ these shells can be decomposed into basic chunks

$$\left\{\tfrac{1}{2}x :\ x \in \mathbb{I},\ x^2 = -a,\ x \equiv \varepsilon \bmod 2\mathbb{I}\right\}$$

with $a \in \mathbb{Z}[\tau]$ totally positive and $\varepsilon = 0, 1, \tau, \tau^{-1}$. These are analyzed via optimal embeddings of the $\mathbb{Z}[\tau]$-orders of $\mathbb{Q}(\tau)(\sqrt{-a})$ which contain $\frac{\varepsilon+\sqrt{-a}}{2}$ and turn out to depend on the ideal class group of this field.

Dynamics and spectrum for tilings and Delone sets

Boris Solomyak

Delone sets, that is, sets which are uniformly discrete and relatively dense in $\mathbb{R}^d$, are used to model atomic structures. The dynamical systems approach starts with a Delone set Λ, such that $\Lambda - \Lambda$ is discrete. Then one can consider the orbit closure under the translation action to obtain a space Ξ_Λ. This space is compact in the natural topology. The group $\mathbb{R}^d$ acts by translations. Various geometric notions, such as repetitivity, have interpretations in dynamical terms. We give a description of eigenvalues with continuous eigenfunctions. Then we proceed to measure-preserving dynamical systems. Under the assumption of uniform patch frequencies, the invariant measure is unique and the methods of ergodic theory may be applied. The main emphasis in this talk is on the question when the resulting system has pure point spectrum. A sufficient condition is given in terms of "almost periodicity" of the original Delone set Λ. We then turn to the tiling setting which has much in common with the one based on Delone sets. If the tiling has the property of self-similarity, a number of strong conclusions can be made. In particular, a plane tiling dynamical system has non-trivial point spectrum if and only if the expansion is a complex Pisot number. We present a concrete combinatorial algorithm, the "overlap algorithm", which allows one to distinguish between pure point and mixed spectrum. Several examples are considered, among them the "chair", 3D chair, and the domino.

Long-range boundary effect in the lattice dimer model

Richard Kenyon

Joint work with H. Cohn, J. Propp. We consider random domino tilings of a large polyomino P. The boundary conditions on P have a long-range influence on local properties of the tiling, such as densities of local configurations. The influence of the boundary is "computed" via the *average height function* which can be thought of as a measure of local entropy. We show that the average height function satisfies a 2^{nd} order PDE, arising from entropy maximization subject to fixed boundary values. This permits us to compute the asymptotic number of tilings of large regions.

Local rules for quasiperiodic tilings

Franz Gähler

For many quasiperiodic tilings there exist conditions on the local tile clusters, which ensure that the tiling is perfectly ordered and quasiperiodic. After a short introduction to the relevant concepts (local isomorphism, local equivalence, quasiperiodicity), a fairly general theory of such local rules is sketched. This theory, the foundations of which have been developed by A. Katz and L. Levitov, gives an impression of the kind of conditions that are necessary and sufficient for the existence of such local rules.

Self-affine digit tiles and their boundaries

Andrew Vince

A self affine digit tile T is the attractor of an iterated function system $\{f_1, f_2, \ldots, f_N\}$ in the special case that the affine maps

$$f_i(x) = A^{-1}(x + d_i) \ : \ \mathbb{R}^d \to \mathbb{R}^d$$

satisfy

1. A is an expansive integer matrix;
2. $D = \{d_1, d_2, \ldots, d_N\}$ is a digit set, i.e. a set of residues for $\frac{Z^d}{AZ^d}$.

Given such a tile T, there always exists a tiling of $\mathbb{R}^d$ by translates of T. Sometimes this tiling $\mathcal{T}$ satisfies a global self-replicating property: the image of each tile in $\mathcal{T}$ under A is, in turn, tiled by elements of $\mathcal{T}$. We give several conditions, some algorithmic, equivalent to this self-replicating property.

If one (hence all) of these conditions hold, then there is a simple formula for the Hausdorff dimension of the boundary of the tile T in the case that A is a similarity:

$$\dim_H(\partial T) = \frac{\log \lambda}{\log c}$$

where c is the expansion ratio of A and λ is the largest eigenvalue of a certain easily computable matrix.

Also in the self-replicating case, there is a method, based on Dekking's recurrent sets, for constructing the boundary of a self-affine digit tile.

Average unit cell in diffraction analysis

Janusz Wolny

Average unit cells for the Fibonacci chain, modulated structures, Thue-Morse sequence and hexagonal layers have been constructed in physical space. The positions of atoms have been replaced by a statistical probability of atomic displacements from the nearest point of the reference lattice. Analytical expressions for diffraction peak intensities have been derived.

It has been shown that:

- the diffraction pattern can be calculated as a Fourier transform of the probability distribution of distances from the reference lattice positions
- the diffraction pattern consists of periodic series of peaks which are described by an envelope function
- the reference lattice approach can be successfully used instead of higher-dimensional analysis.

Mathematical quasicrystals

Jeffrey Lagarias

This talk describes idealized mathematical structures representing atomic positions for solid state structures such as quasicrystals. A Delone set or (r, R)-set is a set in $\mathbb{R}^n$ with a finite packing radius r with equal spheres and a finite covering radius R with equal spheres. A Delone set X is of finite type if $X - X$ is a discrete closed set, and is a Meyer set if $X - X$ is a Delone set; cut-and-project sets are special cases of Meyer sets. Delone sets of finite type have an additive address map $\phi : X \to \mathbb{Z}^s \subset \mathbb{R}^s$ with a projection $\pi : \mathbb{R}^s \to \mathbb{R}^n$ with $\pi \circ \phi = \mathrm{id}$. Properties of such sets can be described by properties of the address map: A Delone set is of finite type if $||\phi(x_1) - \phi(x_2)||_{\mathbb{R}^s} \leq C||x_1 - x_2||_{\mathbb{R}^n}$ for a constant C, and is a Meyer set if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^s$ with $||\phi(x) - L(x)|| \leq C$.

Properties of minimality of associated dynamical systems are studied. If a set is linearly repetitive, that is, minimal with all patches of size T represented in any patch of size cT, then it has limiting patch frequencies and a well-defined diffraction measure. Delone sets of finite type X have pure point autocorrelation measures $\gamma = \sum_{x \in X-X} c_x \delta_x$. If they have pure discrete diffraction spectrum $\hat{\gamma}$, the Fourier transform of γ, then one obtains a summation formula. Poisson's summation formula is a special case.

Constructing quasicrystals with given symmetry group

Peter Pleasants

In 1987 Fred Lunnon and myself showed that for every finite subgroup G of $\mathbb{O}(n)$ there is a G-invariant cut-and-project quasicrystal (model set) generated by an inflation. In my talk I described a more direct method of obtaining this result. The method relies on the fact that there is a representation of G over a real algebraic number field K. A G-invariant O-module can be found, where O is the ring of integers of K. The higher-dimensional lattice for the cut-and-project is then obtained as the image of this O-module in the direct product of the d different homomorphisms of K into $\mathbb{C}$, where d is the degree of K. The resulting quasicrystal has an inflation and the multiplier can be chosen to be any Pisot-Vijayaraghavan-number that takes the O-module into itself. When K is quadratic the inflation is local and can be used to generate the quasicrystal. In other cases it should be possible to construct a closely related quasicrystal whose inflation is local. It should also be possible, when G has no large crystallographic subgroup, to prove the existence of weak local rules, that force non-periodicity but not necessarily determine the local isomorphism class. The natural approach to both these questions seem to be "Ammann hyperplanes" or some similar analog of Ammann lines, but the details are not yet worked out. Another construction method has recently been found by N. Cotfas and J.-P. Verger-Gaugry.

Remarks on the theory of quasicrystals

Peter Kramer

Mathematical concepts for aperiodic orientational long-range order are applied to the physics of icosahedral quasicrystals. For the root lattice $D_6 \subset \mathbb{R}^6$, the icosahedral group $A(5)$ determines an invariant decomposition $\mathbb{R}^6 = \mathbb{R}^3_{||} \oplus \mathbb{R}^3_{\perp}$. Decompose the lattice points as $\tilde{x} = x_{||} + x_{\perp} \subset D_6$ and fix in $\mathbb{R}^3_{\perp}$ the compact window W as the projection $V_{\perp}$ to $\mathbb{R}^3_{\perp}$ of the Voronoi domain V of D_6. Then $V_{\perp}$ becomes Kepler's triacontahedron, and the set $\{x_{||} \mid \tilde{x} = x_{||} + x_{\perp}, x_{\perp} \in V_{\perp}\}$ determines a tiling $\mathcal{T}^{*(D_6)}$ (Kramer et al. 1992) by six tetrahedral tiles and with global icosahedral orientational symmetry. To describe atomic models for the physics, we choose module positions from three classes of holes in D_6 and their projections and fix them on the tiles in $\mathbb{R}^3_{||}$. The window W for the bare tiling is now replaced by a composite window in $\mathbb{R}_{\perp}$ for each class of holes. For the icosahedral quasicrystals i-AlFeCu and i-AlMnPd we construct the composite windows. Local atomic neighbourhoods and their frequencies are obtained by geometric analysis. They provide the basis for quantum computations of local observables like Mößbauer data or the local density of electron states. Perfect quasicrystals of type i-AlPdRe (1994) show zero conductivity at $T \to 0$. In the absence of periodicity and hence of electronic band structure, the computation of electronic structure is a great challenge for the quantum theory of quasicrystals. Approximate computations illustrate the problems. Tunnel microscopy was used in 1994 to scan the surface of i-AlPdMn on the atomic level. It yields a terrace structure in Fibonacci spacing perpendicular to 5-fold icosahedral axes. The composite tiling model yields a detailed and quantitative explanation of this terrace structure, based on Kepler's decomposition of triacontahedron into decagonal prism and two caps. Future efforts in the mathematical physics of quasicrystals are needed in order to pass from the description of the geometric structure to the quantum theory and thermodynamics of these systems.

Wulff shape for quasiperiodic arrangements

Károly Böröczky

The well-known Wulff-shape construction for crystals is generalized for the classical quasi-periodic point sets. This way the "quasi-crystal" growth with icosahedral symmetry can be modeled, leading e.g. to the regular dodecahedron, which actually appears as the preferred shape of certain quasi-crystals.

Wavelets for planar quasiperiodic tilings

Jean-Pierre Gazeau

Wavelet analysis is challenging Fourier analysis because of its better suitability to self-similar structures. The wavelet approach is based on affine actions in $\mathbb{R}^d$, and discrete wavelet families are usually obtained from a probe function through dyadic translations/contractions combined with lattice translations. We shall present a construction of discrete "τ-adic" wavelets, $\tau = \frac{1+\sqrt{5}}{2}$, which instead makes use of dilations/contractions by powers of τ combined with quasilattice translations. Ten-fold rotations are also included in these affine actions when dealing with the planar case. The method is illustrated by considering the elementary example of the so-called τ-Haar family, which is the counterpart of the well-known dyadic Haar basis of $L^2(\mathbb{R})$ or $L^2(\mathbb{C})$.

Such wavelet basis with irrational scaling are thought to play an important role in quasicrystalline studies (image processing for diffraction patterns or scanning microscopy, spectral problems).

Deformation of model sets and icosahedral quasicrystals

Michel Duneau

Icosahedral quasicrystals are currently described by model sets as introduced by Y. Meyer. However, such models can be generalized by small deformations which preserve the Delone property, the quasiperiodicity of the autocorrelation function and the icosahedral symmetry. These deformations are analyzed by means of the theory of univariant and covariant polynomials. They are generated by finitely many fundamental modes and it is shown that some of these deformations could hardly be detected by the analysis of experimental diffraction data.

Group theory and the dynamics of trace maps and cat maps

John A.G. Roberts

In this talk, we review recent work that utilizes the group structure of the integer matrix groups $\mathrm{Sl}(2,\mathbb{Z})$, $\mathrm{Gl}(2,\mathbb{Z})$ and their projective counterparts to help prove results concerning the dynamics of related groups of automorphisms.

In particular, we consider *cat maps*, which are hyperbolic toral automorphisms induced by an element of $\mathrm{Gl}(2,\mathbb{Z})$ acting mod 1. Also we consider the set $\mathcal{A} = \{A \in \mathbb{C}[x,y,z]^3 : I(A(x,y,z)) = I(x,y,z)\}$ where $I(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1$. The set of such 3D polynomial mappings is a group (Peyriere, Wen; Wen 1992) isomorphic to a semi-direct product of the Klein-4 group and $\mathrm{PGl}(2,\mathbb{Z})$. The mappings have integer coefficients and so can be considered on $\mathbb{R}^3$ or $\mathbb{C}^3$. We call the subgroup $G \equiv \mathrm{PGl}(2,\mathbb{Z})$ of such polynomial maps *trace maps* because they arise when we calculate the trace of a word in two $\mathrm{Sl}(2,\mathbb{Z})$ matrices A and B, where the word is $\sigma^n(A)$ and σ is an invertible two-letter substitution rule. Various problems like the 1D tight-binding model with quasiperiodic potential can be studied utilizing the trace map and its dynamics.

We use the free product structure of $\mathrm{PSl}(2,\mathbb{Z})$ and the amalgamated free product structure of $\mathrm{PGl}(2,\mathbb{Z})$ to give

1. necessary and sufficient conditions for escaping orbits under related iteration of $g \in G$, g hyperbolic, which generalize results obtained previously for e.g. the Fibonacci trace map

2. the structure of the reversing symmetry group $\mathcal{R}(M)$ of a cat map M, which consists of symmetries S which commute with M and (possibly) reversing symmetries G which conjugate M into M^{-1}. We give examples where M has no reversing symmetry (joint work with M. Baake and S. Wilson).

Finally, we consider the existence for cat maps of generalizations of symmetries and time-reversal symmetries (called k-(reversing) symmetries). We show how this problem is equivalent to the way the cat map induces a partition into periodic orbits of each rational sublattice of the torus. We give a number-theoretic solution of this problem.

Inflation centers in the n-dimensional cut and project quasicrystals

Jiri Patera

Cut and project quasicrystals with $\sqrt{5}$ irrationality and convex acceptance windows in any dimension were discussed.

Inflation centers inside and outside of the quasicrystal point set were described for all dimensions. Minimal distances in 1D quasicrystals were given as functions of the length of the acceptance window. Expressions for $\Omega_{\min}$ and $\Omega_{\max}$, the minimal and maximal acceptance window respectively, which reproduce a given 1D quasicrystal fragment, were given.

Gap labeling theorem for quasicrystals

Jean Bellissard

Electronic motion in a quasicrystal is described accurately through a Schrödinger operator, the potential of which is modulated according to the position of atoms, located at the vertices of a quasiperiodic set of points in $\mathbb{R}^d$ ($d = 1, 2, 3$ in practice). The absence of periodicity forbids the use of Bloch's theorem to analyze the spectrum of such an operator.

We have developed, since the early eighties, a formalism based upon non commutative geometry, in which the Brillouin zone admits the structure of a non commutative manifold.

Most of the models built by physicists have in common an intricate band spectrum, sometimes nowhere dense with zero Lebesgue measure. They may have infinitely many gaps in a finite interval of the energy spectrum.

To label these gaps physicists use the so called integrated density of states (IDS), namely the number of eigenstates of energy less than a given energy E, per unit volume. The Shubin formula relates this IDS to the integral over the (non commutative) Brillouin zone $\mathbb{B}$ of the eigenprojection of the Schrödinger operator corresponding to the spectral part contained in $(-\infty, E]$. The IDS is constant on gaps, and corresponds to the integral over $\mathbb{B}$ of a projection in the C^*-algebra of continuous function on $\mathbb{B}$. These numbers are computed through computing the Grothendieck K_0-group of $\mathbb{B}$.

The calculations have been performed for quasicrystal in dimension $d = 1, 2$. The result is the set

$$\left\{ \int d\mathbb{P}(\omega) f(\omega); f \in C(\Omega, \mathbb{Z}) \right\}.$$

where Ω is the "acceptance zone" or "window" defining the quasicrystal and enclosed with a topology such that $\Omega \cup T_a\Omega$ is closed and open for any translation a in the virtual higher dimensional lattice $\tilde{L}$ defining the point set of the quasicrystal (ideal location of atoms). $\mathbb{P}$ is the unique $\tilde{L}$ invariant measure on Ω.

For $d = 1$, a QC built from $\mathbb{Z}^2 \subset \mathbb{R}^2$ projected on a line of slope $\beta = \frac{\alpha}{1-\alpha}$ ($\alpha \in [0, 1] \setminus \mathbb{Q}$), this set of numbers is $\mathbb{Z} + \alpha\mathbb{Z}$. For $d = 2$, in the case of the octagonal Ammann-Beenker tiling, it gives

$$\left\{ \frac{m + n\sqrt{2}}{8} ; m, n \in \mathbb{Z}, m + n \text{ even} \right\}.$$

Level-spacing distributions of planar quasiperiodic tight-binding models

Uwe Grimm

Tight-binding models yield a simplified description of the motion of non-interacting electrons in a quasiperiodic background. Given a quasiperiodic tiling, electrons can move from one vertex to another along the edges of the tiling, resulting in a Hamiltonian matrix that is essentially the adjacency or incidence matrix of the quasiperiodic graph.

In this joint work with J.X. Zhong, R.A. Römer, and M. Schreiber, we investigate the statistical properties of the eigenvalue spectrum of this Hamiltonian for the octagonal Ammann-Beenker tiling by numerical methods. Approximating the infinite tiling by different sequences of finite patches, and by taking into account their symmetries, we find that the underlying universal level-spacing distribution of eigenvalues agrees with that of the Gaussian orthogonal random matrix ensemble. Contradictory results in the literature we attribute to an "almost symmetry" of the finite patches that had been considered, which is not present in the generic case. The agreement between our numerical data and the random matrix result is astonishingly good, one can even see that the numerical data fit the exact random matrix distribution better than the well-known Wigner surmise.

Generalized Meyer sets with toric internal spaces

Jean-Louis Verger-Gaugry

We show that one-dimensional aperiodic sets of points having the Delaunay (=Delone) property can be associated with Meyer sets, for which

1. the internal space is toric, $\frac{\mathbb{R}}{\lambda\mathbb{Z}}$, with a selection rule based on a congruence mode with respect to the frequencies λ producing punctuated windows,
2. a scaling exponent in $[0,1]$ can be uniquely defined for each element of the window, related to the scaling properties of the intensity function and the point density measure on canonical one-dimensional sublattices of period λ, where a scaling exponent of 1 corresponds to Bragg-peaks,
3. the projection mappings are adapted to global average lattice and are not orthogonal.

The Thue-Morse quasicrystal, arising from the Thue-Morse automaton, is studied modulo $\lambda\mathbb{Z}$ to show that it produces punctuated windows in the one-dimensional torus, for values of λ selected by congruence. The construction of Y. Meyer allows arbitrary locally compact Abelian groups as internal spaces but it seems to be the first time that a toric component is explicitly used, linked to the search of scaling exponents and spectral analysis.

Crystallographic clusters

Nikolai Dolbilin

In the talk a group-free approach to crystallographic point sets and tilings is discussed. The *local theorem* (obtained over 20 years ago) and the recently proved *global criterion* describe certain well-defined conditions providing Delone sets with crystallographic symmetry.

Nevertheless, these two theorems relate to an ideal infinite crystal which never exists in nature. A goal of the talk is to discuss several versions of the so-called *extension theorem*. The point version of this theorem describes necessary and sufficient conditions for a non-numerous point set (a cluster) to admit an extension to an infinite ideal crystal.

The polyhedral version of the extension theorem presents a criterion of a polyhedron to tile space in an isohedral way. The last version generalizes well-known statements on polyhedra that isohedrally tile space (e.g., fundamental domains for Coxeter groups, a theorem of Venkov, Alexandrov and McMullen on convex polyhedra designed to tile space by translations).

A species of planar triangular tilings with inflation-factor $\sqrt{-\tau}$

Ludwig Danzer & Gerrit van Ophuysen

Consider $A :=$ , $X :=$ , $\mathfrak{F} := \{A, X\}$ and the inflation rule given by $\mathrm{infl}(A) := X$, $\mathrm{infl}(X) := X\dot{\cup}A =$ . Interpreting $\mathbb{E}^2$ as $\mathbb{C}$ the inflation-factor η becomes $i\sqrt{\tau}$, which is a complex PV-number. The species $\mathcal{S}(\mathfrak{F}, \mathrm{infl})$ of all global $\mathfrak{F}$-tilings created by infl has a unique deflation ("infl^{-1}") and hence is aperiodic. The set of all vertices can be shown to be a "model set" (Robert Moody), so the Fourier-transformation of the autocorrelation function is "pure point" with the Bragg-peaks located on the $\mathbb{Z}$-module $\frac{3-\tau}{5}\left\langle \binom{\sqrt{\tau}}{0}, \binom{\sqrt{\tau}^{-1}}{0}, \binom{0}{1}, \binom{0}{\tau} \right\rangle_{\mathbb{Z}}$.

With $\mathfrak{F}_c := \{A, B, C, X, Y, Z\}$, where B and C are congruent to A, while Y and Z are congruent to X, but all differently coloured, and

$$\begin{aligned} \mathrm{infl}_c(A) &:= X & \qquad \mathrm{infl}_c(X) &:= Y\dot{\cup}A \\ \mathrm{infl}_c(B) &:= Y & \qquad \mathrm{infl}_c(Y) &:= X\dot{\cup}C \\ \mathrm{infl}_c(C) &:= Z & \qquad \mathrm{infl}_c(Z) &:= Z\dot{\cup}B \end{aligned}$$

we receive the species $\mathcal{S}(\mathfrak{F}_c, \mathrm{infl}_c)$ of coloured tilings. In contrast to $\mathcal{S}(\mathfrak{F}, \mathrm{infl})$, the coloured species can also be defined by a perfect local matching rule. In fact the 42 coloured vertex-stars may serve as an atlas.

Problems

1. A *polyomino* is a topological disc, which can be tiled by congruent squares.

Question 1 Does there exist an aperiodic polyomino?

Question 2 Given a polyomino P, does there exist an algorithm to determine whether or not there is a P-tiling?

Theorem (Keating, A.V.) Given a polyomino P, there is a polynomial time algorithm to determine whether there exists an isohedral P-tiling.

Corollar Given a polyomino P there exists an algorithm to determine whether there is a P-tiling by translation.

Question 3 Given a polyomino P, does there exist an algorithm to determine whether there is a periodic P-tiling?

Andrew Vince

2. Solution to a related problem of A. Vince:
Let P be a polytope that tiles $\mathbb{R}^3$ by translation.

Question: Does there exist a lattice tiling of $\mathbb{R}^3$ with P?

Known:

(1) If P is convex then there always exits a lattice tiling (P. McMullen).

(2) If P is star-shaped the answer is *no*: take as tile the union of one center cube, six congruent cubes each placed face to face next to the centered one and place on the opposite face of each such cube a squared pyramid of hight half the edge length of the cubes. This star-shaped polytope tiles whole space by translation but cannot be arranged to a lattice tiling.

Egon Schulte

3. Is the presented triangular tiling in $\mathbb{E}^2$ with inflation factor $\sqrt{-\tau}$ in the same MLD-class as the Ammann-chair-tiling with the same inflation factor (c.f. Grünbaum and Shepard: Tilings and Patterns)?

Ludwig Danzer

Preliminary results by Reinhard Lück and myself indicate that this is not the case and that there is no local derivability in either direction.

Gerrit van Ophuysen

4. Is there an example for a tiling in $\mathbb{E}^2$ which has perfect local matching rules but there is no tiling with an inflation in its MLD-class?

Ludwig Danzer

5. **Question 1:** Does there exist a self-similar tiling of the plane with two tile types up to translation, whose expansion factor $\lambda \in \mathbb{C}$ is of algebraic degree 4? If λ is real, can λ have degree 2?

By self-similar we mean,

a) upon expansion $z \mapsto \lambda z$ each tile maps over existing tiles,
b) two tiles of the same type subdivide in the same way,
c) the tiling is repetitive.

One is free to replace the exact self-similarity (condition (1)) with the property that the inflated tiling is MLD to the original tiling.
Thurston's theorem on self-similar tilings implies that when λ is not real, λ is of algebraic degree $1, 2$, or 4; and when λ is real, it must be of degree 1 or 2.
If there is only one tile type, it is known that λ must be of degree 1 if real and of degree 1 or 2 if not real, and each such λ occurs. Furthermore there exist SSTs with two tile types for such λ. But these are all MLD with SSTs having only one tile type.
So we could rephrase our question as follows:

Question 1': Is there a SST with two tile types which is not MLD with a SST having only one tile type?

Richard Kenyon

6. A region R is a compact set in $\mathbb{R}^n$ which is
(1) the closure of its interior and
(2) the boundary ∂R of it is of Lebesgue measure 0. Suppose R tiles $\mathbb{R}^n$ by translations, i.e.

$$\mathbb{R}^n = \bigcup_{t \in \mathcal{F}} (R + t)$$

and

$$\mu((R+t) \cap (R+t')) = 0 \qquad \forall t, t' \in \mathcal{F} \quad t \neq t'.$$

a) Does R always have a fully periodic tiling? (One can take $\mathcal{F} = \Lambda + \{\text{finite set}\}$.)
b) Same question with R being a finite union of lattice cubes.
c) Same question only allowing tilings $\mathcal{F} \subseteq \mathbb{Z}^n$.

Jeffrey Lagarias

7. There are (e.g. in $\mathbb{E}^2$) protosets and substitution rules which cannot be described by similarities and lack a uniform inflation factor. Even in the hyperbolic plane there are tilings which appear to be "hierarchical". This leads to the following

Problem: Give a useful and precise definition of what should be called a "hierarchical tiling" in d dimensions, at least for $d = 2$.

You may employ the idea of substitution, but neither translations nor similarities. The definition shall be applicable as well to $\mathbb{E}^d$ as to $\mathbb{H}^d$.

Ludwig Danzer

8. Problem A:
Two positive self-adjoint operators A and B on a Hilbert space are called Λ-unitarily equivalent for Λ a positive $n \times n$-matrix, if $\Lambda \otimes A$ is unitarily equivalent to $\Lambda \otimes B$. Classify self-adjoint operators up to Λ-unitary equivalence.
Remarks:

a) In finite dimensions Λ-unitary equivalence is the same as unitary equivalence.
b) The simplest example for a non-trivial pair of Λ-equivalent operators are on $\ell^2(\mathbb{Z})$ the multiplication operators M_f and M_g with $f: \mathbb{Z} \ni n \mapsto \lambda^{2n}$, $g: \mathbb{Z} \ni n \mapsto \lambda^{2n+1}$, $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda > 0$.

After some reductions, Problem A essentially reduces to the following
Problem B:
Classify pairs of subsets $A, B \subseteq \mathbb{Z}^n$ such that

i) $A \cap B = \emptyset$
ii) $A \cup S_1(A) \cup \cdots \cup S_n(A) = B \cup S_1(B) \cup \cdots \cup S_n(B)$ where S_i on $\mathbb{Z}^n$ is the shift in the i-th coordinate

Remarks:

a) For $n = 1$ the only such pair is given by $A = 2\mathbb{Z}$, $B = 2\mathbb{Z} + 1$ (or conversely).
b) For $n \geq 2$ there are more possibilities which nevertheless show a high degree of regularity.

Burkhard Kümmerer

9. Consider edge-to-edge tilings of the plane by rhombi with internal angles of the form $2\pi k/n$, for some fixed even $n > 2$, and $0 < k < n-1$. The tilings are required to satisfy the alternation condition (AC): along any lane of tiles sharing edges in a fixed direction, congruent tiles in different orientations must alternate (see [1]). If n is not twice an odd prime, we assume in addition that all rhombi of this form are actually used. The vertex set $V(T)$ of such a tiling T generates a module M of finite rank r. Indexing V with respect to a fixed basis of M defines a map $F : V \to \mathbb{Z}^r$. From the work of Socolar [1] it follows that there exists a linear map $L : V \to \mathbb{Z}^r$ such that $F - L$ is uniformly bounded. In other words: V is a Meyer set (but not a model set, in general). If n is not divisible by 4, the same approximating map L can be used for all tilings satisfying the AC. Otherwise, maps from a one-parameter family are needed.

Conjecture 1: *If $n = 10$, and T satisfies the AC, then $V(T)$ is a model set. Every such tiling is a generalized Penrose tiling* [2].

Conjecture 2: *If $n = 12$, and $60°$-rhombi are welded together to regular hexagons, the vertex set of every hexagon-square-rhombus tiling satisfying the AC is a model set.*

Conjecture 3: *If $n = 12$, every rhombus tiling satisfying the AC has a vertex set which is a model set.*

Remarks: Even if the AC enforces tilings with a model set as vertex set, it admits tilings in uncountably many local isomorphism classes (unless $n = 4$ or 6). Matching rules based on the hierarchical structure of a tiling cannot achieve this. Le [3] claims to have a proof of a slightly weaker version of Conjecture 1, but a written proof does not seem to be available. A proof of Conjecture 2 should be doable with the methods of [4], where the analogous result for $n = 8$ is proved. Conjecture 3 combines the difficulties of both Conjectures 1 and 2. A proof of Conjecture 1 would imply that 3D icosahedral rhombohedron tilings satisfying an analogous AC are model sets.

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Franz Gähler

10. Let $A = \{a, b, c\}$ be an alphabet of 3 letters. Consider all words of length n that are repeat-free or *square-free* (i.e., no substring is a square), and define f_n to be the number of such words (length n). Define $s_n := \frac{1}{n}\log(f_n)$, the *entropy density*.
It is known that $s = \lim_{n\to\infty} s_n$ exists, and the best numerical estimate (to my knowledge) is $s \approx 0.263719(1)$, where (1) means the uncertainty in the last digit.

1. problem: Can s be determined exactly?

2. problem: Consider the generating function $F(x) = \sum_{n\geq 1} f_n x^n$. Then $s = \log(\frac{1}{\rho})$, $\rho =$ radius of convergence of $F(x)$ (around $x = 0$). Is $F(x)$ analytically continuable beyond its circle of convergence? (My guess: it is *not*.)

If we consider an alphabet in N letters ($N \geq 3$), the same type of question occurs. Furthermore,

$$\tilde{s}^{(N)} = \log \frac{(N-1) + \sqrt{(N-1)^2 - 4}}{2} = \operatorname{arcosh} \frac{N-1}{2} \qquad \circledast$$

seems to be a lower bound for the entropy density $s^{(N)}$. It is pretty bad for $N = 3$ ($\tilde{s} = 0$ while $s \approx 0.263\ldots$), but improves rapidly – being asymptotically exact for $N \to \infty$ (idea available, but no complete proof; see [1]).

3. problem: Prove & improve $\circledast$.

[1] M. Baake, V. Elser, U. Grimm; The Entropy of Square-Free Words, Mathl. Comput. Modelling **26** (1997) 13-26.

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