MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 19/1998

# Fundamental Groups in Geometry

3.05.-09.05.1998

Die Tagung fand unter der Leitung von Herrn F. Bogomolov (Courant), Herrn S. Kollar (Utah), Frau M. Teicher (Bar Ilan) und Herrn M. Zaidenberg (Grenoble) statt. Die Teilnehmer kamen aus Deutschland, den USA, Russland, Frankreich, Israel und anderen Ländern. Sie vertraten einen breiten Themenkreis aus dem Gebiet der Fundamentalgruppen in der Geometrie, und es wurde vor allem jungen Mathematikern die Gelegenheit geboten, ihre Forschungsergebnisse einem interessierten Fachpublikum vorzustellen. Ein nicht allzu dicht gedrängtes Programm und die angenehme Atmosphäre des Instituts begünstigten den informellen Ideenaustausch und die angeregte Unterhaltung unter den Teilnehmern.



Gefördert durch
DFG Deutsche Forschungsgemeinschaft

# FUNDAMENTAL GROUPS AND GALOIS THEORY

*Shreeram S. Abhyankar* [1]

During my Ph.D. work, my guru Zariski advised me to use Chevalley's local rings to algebracize Jung's surface desingularization of 1908 for carrying it over from the complex domain to the case of positive characteristic. In my 1955 American Journal paper, I concluded that this cannot be done because in that case the algebraic local fundamental group above a normal crossing of the branch locus need not even be solvable. In my 1957 American Journal paper, by taking a section of the unsolvable surface covering, I was led to a conjecture about the structure of the algebraic fundamental group of an affine curve. After some initial work by myself, Nori and Serre, this conjecture was settled affirmatively by Raynaud and Harbater in their 1994 papers in volumes 116 and 117 of Inventiones Mathematicae. A chatty discussion of the curve case, including references, can be found in my 1992 paper on "Galois theory on the line in nonzero characteristic" in volume 27 of the AMS Bulletin, and also in my 1996 paper on "Factorizations over finite fields" in Number 233 of the LMS Lecture Note Series. In my 1997 paper on the "Local fundamental groups of algebraic varieties" in volume 125 of AMS Proceedings, this led me to explicitize the conjectures about higher dimensional algebraic fundamental groups which were implicit in my American Journal papers of 1955 and 1959-60. Leaving aside the global conjectures implicit in the 1959-60 papers, here I shall comment on the local conjecture implicit in the 1955 paper.

So let $N^d_{k,t}$ represent a neighborhood of a simple point on a $d$- dimensional algebraic variety, over an algebraically closed field $k$ of characteristic $p$, from which we have deleted a divisor having a $t$-fold normal crossing at the simple point. Also let $\pi^L_A(N^d_{k,t})$ be the corresponding algebraic local fundamental group, by which we mean the set of all Galois groups of finite unramified local Galois coverings of $N^d_{k,t}$. Finally let $P_t(p)$ be the set of all finite groups $G$ such that $G/p(G)$ is an abelian group generated by $t$ generators; here $p(G)$ denotes the subgroup of $G$ generated by all of its $p$-Sylow subgroups; in case of $p=0$ we take $p(G)=1$. Now we may state:

**Local Conjecture.** *For $d \geq 2$ and $t \geq 1$ we have $\pi^L_A(N^d_{k,t}) = P_t(p)$.*

Algebraically speaking, let $R$ be the formal power series ring $k[[X_1,\dots,X_d]]$, let $I$ be the quotient field $k((X_1,\dots,X_d))$ of $R$, let $\widehat{\Omega}$ be an algebraic closure of $I$, and let $\Omega$ be the set of all $J \in \widehat{\Omega}$ such that $X_1R,\dots,X_tR$ are the only height-one primes in $R$ which are possibly ramified in $J$. We may now identify $\pi^L_A(N^d_{k,t})$ with the set of all Galois groups $\mathrm{Gal}(J,I)$ with $J$ varying in $\Omega$.

In the 1955 paper I proved the inclusion $\pi^L_A(N^d_{k,t}) \subset N^d_{k,t}$ and by examples showed that, assuming $p$ to be nonzero, $\pi^L_A(N^d_{k,t})$ contains unsolvable groups. By refining these examples, in the 1997 paper I showed that $\pi^L_A(N^d_{k,t})$ contains $\mathrm{GL}(m,q)$ for every integer $m>1$ and every power $q>1$ of $p$.

[1] E-mail: ram@cs.purdue.edu

In a recent discussion, David Harbater has raised the question whether every member of $\pi_A^L(N_{k,t}^d)$ actually belongs to $P_t'(p)$, where $P_t'(p)$ is the set of all $G$ in $P_t(p)$ for which $p(G)$ has an abelian supplement in $G$, i.e., an abelian subgroup of $G$ which together with $p(G)$ generates $G$. To examine Harbater's question, I asked Gernot Stroth to make me some examples of groups in $P_t(p)$ which are not in $P_t'(p)$. Here are some of the beautiful examples produced by Stroth for $t = 3$, for which I have been scanning (so far unsuccessfully) the existence or nonexistence of suitable local coverings.

The first set of Stroth groups $G$ are for $p = 3$, and they are $G = E * \mathrm{GL}(2,3)$, where $*$ denotes central product, and where $E$ is either the dihedral group $D_8$ of order 8 or the quaternion group $Q_8$ of order 8. Moreover, $\mathrm{GL}(2,3)$ can be replaced by its flat version $\mathrm{GL}^\flat(2,3)$ by which we mean the other group which, like $\mathrm{GL}(2,3)$, is a nonsplit central $Z_2$ extension of $\mathrm{PGL}(2,3)$. Similarly, for any prime power $q \equiv 3(4)$ of any odd prime $p$, we get four Stroth groups by replacing $\mathrm{GL}(2,3)$ by the unique group $H$ (or its "flat version" $H^\flat$) such that $\mathrm{SL}(2,q) < H < \mathrm{GL}(2,q)$ with $[H : \mathrm{SL}(2,q)] = 2$. Turning to $p = 2$, we get Stroth groups $G = F * \mathrm{GL}(3,4)$ where $F$ is an extra-special group of order 27, i.e., $F$ is a nonsplit central $Z_3$ extension of $Z_3^2$ with $Z(F) = Z_3$; note that there are two versions of $F$, depending on whether it has only elements of order 3 (quaternion type) or also elements of order 9 (dihedral type); again, instead of $\mathrm{GL}(3,4)$ we can take its flat version $\mathrm{GL}^\flat(3,4)$.

## SOLVABLE KÄHLER GROUPS

*Donu Arapura*

Two related theorems, due to Madhav Nori and the author, were discussed. A weak form of the first theorem states the following: Let $\pi$ be the fundamental group of normal complex algebraic variety and $n$ a positive integer. Let $\Gamma = \pi/D^n\pi$ where $D^n\pi$ is the $n$th derived subgroup. Then if $\Gamma$ is solvable and admits an faithful representation into some $Gl_N(Q)$, it is virtually nilpotent. For the second theorem $\pi$ is the fundamental group of a Zariski open subset of a compact Kähler manifold and $\Gamma = \pi/D^n\pi$. If $\Gamma$ is polycyclic then it is virtually nilpotent. As a corollary, it follows that a polycyclic Kähler group must be virtually nilpotent. The proof of the first theorem is arithmetic, while that of the second is Hodge theoretic.

## ABELIAN CONNECTEDNESS OF COMPACT KÄHLER MANIFOLDS

*Frederic Campana*

Let $\mathcal{G}$ be the class of virtually abelian finitely generated groups (i.e. finitely generated with a subgroup of finite index which is abelian). Let $X$ be a compact Kähler manifold. Say that $X$ is *$\mathcal{G}$-connected* if any generic two points $x, y$ of $X$ are contained in a compact complex analytic subset $Z$ of $X$ such that for any irreducible component $Z_i$ of $Z$, one has: $\pi_1(\hat{Z}_i)_X = \mathrm{Im}(\pi_1(\hat{Z}_i) \to \pi_1(X))$ is in $\mathcal{G}$ (with $\hat{Z}_i$ the normalization of $Z_i$).

**Theorem.** *If $X$ (compact Kähler) is $\mathcal{G}$-connected, then $\pi_1(X) \in \mathcal{G}$.*

This result fails severely in the non-Kähler case and holds for other classes $\mathcal{G}$ (of finite, virtually nilpotent, virtually polycyclic groups). It was conjectured by Oguiso and Zaidenberg.

## FUNDAMENTAL GROUPS OF DISCRIMINANT COMPLEMENTS

*Jim Carlson*

What follows is a description of joint work with Daniel Allcock and Domingo Toledo.

The family of all hypersurfaces $\{X_t\}$ in complex projective space of degree $d$ and dimension $n$ is parameterized by a projective space $\mathbb{P}^N$. Let $\Delta$ be the *discriminant locus* of this family: the set $\{t \in \mathbb{P}^N \mid X_t \text{ is singular}\}$. Let $\Phi = \pi_1(\mathbb{P}^N - \Delta)$ be the fundamental group of the space of smooth hypersurfaces. Since the natural monodromy representation $\rho : \Phi \longrightarrow \mathrm{Aut}(H^n(X_o))$ is nontrivial in almost all cases, say, $d \geq 3$, we know that $\Phi$ is nontrivial. In fact, a result of Beauville (1986) shows that for $d \geq 3$, $(d, n) \neq (3, 0), (3, 2)$, the monodromy group $\Gamma = \rho(\Phi)$ is a lattice in the group $G = \mathrm{Aut}(H^n(X_o))$, which is noncompact. (A lattice is such that $G/\Gamma$ has finite volume). Now consider the kernel of the monodromy representation, which we denote by $K$. Dolgachev and Libgober (1981) showed that for $d = 3$, $n = 1$, the case of cubic curves, that $K$ is a finite group. In *alg-geom/9708002* we prove the following result:

**Theorem.** ( — , Toledo ) *For $d \geq 3$, $n \neq 0, 1$, the kernel of the monodromy representation is large.*

By definition a *large* group is one that admits a homomorphism to a noncompact almost simple Lie group $H$ with Zariski-dense image. Such groups are infinite and, by a theorem of Tits, contain a free subgroup of rank 2. The idea of the proof is to construct, for each $X_t$, a cyclic cover $Y_t$ of projective space branched along $X_t$. The family of $Y_t$'s has its own monodromy representation $\rho'$, and we show that under the stated hypotheses, $\rho'(K)$ is Zariski-dense. In the case of cubic surfaces, $d = 3$ and $n = 2$, we can show more: that $K$ is not finitely generated. This result follows from the main result of *alg-geom/970916*, which has appeared in C. R. Acad. Sci. Paris **326**:

**Theorem.** (Allcock, —, Toledo) *The moduli space of marked cubic surfaces is biholomorphic to $(B^4 - H)/\Gamma$, where $B^4$ is the unit ball in complex four-space, $\Gamma$ is a group of complex reflections acting on it, and $H$ is the collection of reflection hyperplanes for $\Gamma$.*

A cubic surface is marked by a choice of a system of six skew lines.

Using the above theorem one shows that the kernel $K$ is equivalent modulo finite groups to $\pi_1(B^4 - H)$: the groups are related by maps with finite kernel and cokernel. Such groups have the same rational homology. Since $H$ is an infinite collection of hyperplanes, the result follows. The identification of the moduli space with a Zariski-open subset of the ball quotient is given by the period map for triple covers of $\mathbb{P}^3$ branched along a cubic surface.

Note that the "hyperbolic hyperplane arrangement" $H$ contains points where four hyperplanes cross at right angles. From this one sees that $K$ contains free abelian subgroups of rank four. Consequently $K$, unlike the Torelli group for Riemann surfaces of genus two, studied by G. Mess, is not a free group.

## COHOMOLOGY OF FIBER-TYPE ARRANGEMENTS

*Daniel C. Cohen*[2] [3]

An arrangement of complex hyperplanes $\mathcal{A}$ is a finite collection of codimension one affine subspaces of Euclidean space $\mathbb{C}^\ell$. The cohomology of the complement, $M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$, with coefficients in a local system $\mathcal{L}$ arises in a number of contexts—generalized hypergeometric functions, Knizhnik-Zamolodchikov equations, representations of braid groups, etc.—and has been the subject of considerable recent interest (see, for instance, [1, 9, 5], and see Orlik and Terao [7] as a general reference for arrangements). For complex local systems satisfying certain genericity conditions, work of Esnault, Schechtman and Viehweg [6] shows that the cohomology of the complement is isomorphic to that of the Orlik-Solomon algebra (defined below), viewed as a complex with appropriate differential. It is then natural to ask: What is the relation between the two cohomology theories should these genericity conditions fail? I will give an answer to this question for the class of fiber-type arrangements (and complex local systems of rank one).

Write $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear polynomials $f_j$ with $H_j = \ker f_j$. The Orlik-Solomon algebra, $A = \bigoplus_{q=0}^{\ell} A^q$, is the graded $\mathbb{C}$-algebra generated by the differential forms $a_j = d\log(f_j)$. It is well-known that the algebra $A$ is isomorphic to the cohomology algebra of the complement $M$ of $\mathcal{A}$, and that $A$ is determined by combinatorial data, see [2, 7].

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ be a "weight" vector. Associated to $\boldsymbol{\lambda}$, we have a representation $\rho : \pi_1(M) \to \mathbb{C}^*$ given by $\rho(g_j) = \exp(-2\pi i \lambda_j)$ for any meridian $g_j$ about the hyperplane $H_j$ of $\mathcal{A}$, an associated local system of coefficients $\mathcal{L}$ on $M$, and a distinguished element $\omega = \sum \lambda_j a_j$ in $A^1$. Right-multiplication by $\omega$ defines a map $\mu(\boldsymbol{\lambda}) : A^q \to A^{q+1}$. Clearly, $\mu \circ \mu = 0$, so $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is a complex. As noted above, if $\boldsymbol{\lambda}$ is sufficiently generic, the cohomology $H^*(M;\mathcal{L})$ of $M$ with coefficients in $\mathcal{L}$ is isomorphic to $H^*(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$, see [6], [8]. For arbitrary $\boldsymbol{\lambda}$, it is known that

$$\operatorname{rank} H^q(A,\mu) \le \operatorname{rank} H^q(M,\mathcal{L}) \le \operatorname{rank} H^q(M,\mathbb{C}).$$

The first of these inequalities was communicated to me by S. Yuzvinsky. The second may be obtained using stratified Morse theory [3, 4], and resolves a question raised by Aomoto and Kita [1].

For arrangements of fiber-type, more can be said. An arrangement $\mathcal{A}$ in $\mathbb{C}^{\ell+1}$ is *linearly fibered* if there is a choice of coordinates $(\mathbf{x}, z) = (x_1, \dots, x_\ell, z)$ so that the restriction, $p$, of the projection $\mathbb{C}^{\ell+1} \to \mathbb{C}^\ell$, $(\mathbf{x}, z) \mapsto \mathbf{x}$, to the complement $M(\mathcal{A})$ is a fiber bundle projection, with base $p(M(\mathcal{A})) = M(\mathcal{B})$, the complement of an arrangement $\mathcal{B}$ in $\mathbb{C}^\ell$, and fiber the complement of finitely many points in $\mathbb{C}$. An arrangement is *fiber-type* if sits atop a tower of linearly fibered arrangements. By a classical result of Fadell and Neuwirth, the

[2]Partially supported by grant LEQSF(1996-99)-RD-A-04 from the Louisiana Board of Regents.
[3]E-mail: cohen@math.lsu.edu; URL: http://math.lsu.edu/~cohen

braid arrangement $\mathcal{A}_\ell$, with complement the configuration space of $\ell$ ordered points in $\mathbb{C}$, is the prototypical example of a fiber-type arrangement.

The fundamental group, $G \cong \mathbb{F}_{d_\ell} \rtimes \cdots \rtimes \mathbb{F}_{d_1}$, of the complement of a fiber-type arrangement $\mathcal{A}$ admits the structure of an iterated semidirect product of finitely generated free groups, and the complement $M(\mathcal{A})$ is a $K(G,1)$-space. Given such a group, A. Suciu and I construct a finite free resolution $(C_\bullet(G), \Delta_\bullet)$ of the integers over the group ring $\mathbb{Z}G$ in [5]. This resolution may be used to compute the (co)homology of $M$ with coefficients in an arbitrary local system.

Let $\mathbf{t} = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ be a point in the complex torus. Associated to $\mathbf{t}$, we have a representation $\rho : G \to \mathbb{C}^*$ given by $\rho(g_j) = t_j$ (as above, $g_j$ is a meridian about $H_j \in \mathcal{A}$), which endows $\mathbb{C}$ with the structure of a $G$-module, and induces a local system $\mathcal{L}$ on $M$. For any $\mathbf{t}$, the homology of $M$ with coefficients in $\mathcal{L}$ is naturally isomorphic to the homology of the chain complex $C_\bullet(G) \otimes_{\mathbb{Z}G} \mathbb{C}$.

Denote the terms and boundary maps of this chain complex by $(\mathbf{C}_\bullet, \partial_\bullet(\mathbf{t}))$. The terms $\mathbf{C}_q = C_q(G) \otimes_{\mathbb{Z}G} \mathbb{C}$ are finite dimensional complex vector spaces. The boundary maps $\partial_q(\mathbf{t})$ may be viewed as "evaluations" of those of the resolution $C_\bullet(G)$ in the following way. If the matrix of $\Delta_q$ is $r \times s$, then $\partial_q(\mathbf{t}) : (\mathbb{C}^*)^n \to \mathrm{Mat}_{r\times s}(\mathbb{C})$ is the (smooth) map which takes a point $\mathbf{t}$ and yields the evaluation $\tilde{\rho}(\Delta_q)$, where $\tilde{\rho}$ denotes the extension of the representation $\rho$ to (matrices with entries in) the group ring $\mathbb{Z}G$. View the differentials of the Orlik-Solomon algebra complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ as maps $\mu^q(\boldsymbol{\lambda}) : \mathbb{C}^n \to \mathrm{Mat}_{r\times s}(\mathbb{C})$. The complexes $\mathbf{C}_\bullet$ and $A^\bullet$ are related by the following result.

**Theorem.** *Let $\mathcal{A}$ be a fiber-type arrangement, and for each $q$, let $\delta_q(\boldsymbol{\lambda})$ denote the derivative of the map $\partial_q(\mathbf{t})$ at the point $(1, \ldots, 1) \in (\mathbb{C}^*)^n$. Then the system of vector spaces and linear maps $(\mathbf{C}_\bullet, \delta_q(\boldsymbol{\lambda}))$ is a chain complex, which is dual to the Orlik-Solomon algebra complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ of $\mathcal{A}$. In other words, for each $q$, $A^q \cong \mathbf{C}_q$, and the map $\mu^q(\boldsymbol{\lambda}) : A_q \to A_{q+1}$ is the transpose of the map $\delta_{q+1}(\boldsymbol{\lambda}) : \mathbf{C}_{q+1} \to \mathbf{C}_q$, $\mu^q(\boldsymbol{\lambda}) = [\delta_{q+1}(\boldsymbol{\lambda})]^\top$.*

## References


[1] K. Aomoto, M. Kita, *Hypergeometric Functions* (in Japanese), Springer-Verlag, 1994.

[2] E. Brieskorn, *Sur les groupes de tresses*, In: Séminaire Bourbaki, 1971/72, Lect. Notes in Math., vol. 317, Springer-Verlag, 1973.

[3] D. Cohen, *Cohomology and intersection cohomology of complex hyperplane arrangements*, Advances in Math. **97** (1993), 231–266.

[4] ———, *Morse inequalities for arrangements*, Advances in Math. **134** (1998), 43–45.

[5] D. Cohen, A. Suciu, *Homology of iterated semidirect products of free groups*, J. Pure Appl. Algebra **126** (1998), 87–120.

[6] H. Esnault, V. Schechtman, V. Viehweg, *Cohomology of local systems on the complement of hyperplanes*, Invent. Math. **109** (1992), 557-561. Erratum, ibid. **112** (1993), 447.

[7] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Grundlehren Math. Wiss., vol. 300, Springer-Verlag, 1992.

[8] V. Schechtman, H. Terao, A. Varchenko, *Cohomology of local systems and the Kac-Kazhdan condition for singular vectors*, J. Pure Appl. Algebra **100** (1995), 93–102.

[9] V. Schechtman, A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991), 139–194.

# VON NEUMANN INVARIANTS FOR COHERENT ANALYTIC SHEAVES

*Philippe Eyssidieux*

In order to study the group of $L_2$ holomorphic sections of the pull-back to the universal covering space of an holomorphic vector bundle on a compact complex manifold, it is useful to have a cohomological formalism, generalizing Atiyah's $L_2$ index theorem [1]. The goal of this talk is to outline the construction of such a formalism.

To each coherent analytic sheaf $\mathcal{F}$ on a $n$-dimensional complex space $X^{(n)}$ and each Galois infinite unramified covering $\pi : \tilde{X} \to X$, whose Galois group is denoted by $\Gamma$, $L_2$ cohomology groups, denoted by $H_2^q(\tilde{X}, \mathcal{F})$ are attached, such that:

1. The $H_2^q(\tilde{X}, \mathcal{F})$ underly a cohomological functor on the abelian category of coherent analytic sheaves on $X$.
2. If $\mathcal{F}$ is locally free, $H_2^0(\tilde{X}, \mathcal{F})$ is the group of $L_2$ holomorphic sections of the pull-back to $\tilde{X}$ of the holomorphic vector bundle underlying $\mathcal{F}$.
3. $H_2^q(\tilde{X}, \mathcal{F})$ belongs to a category of $\Gamma$-modules on which a dimension function $\dim_\Gamma$ with real values is defined.
4. Atiyah's $L_2$ index theorem holds:

$$\sum_{q=0}^{n}(-1)^q \dim_\Gamma H_2^q(\tilde{X}, \mathcal{F}) = \sum_{q=0}^{n}(-1)^q \dim H_2^q(X, \mathcal{F})$$

The $L_2$-cohomology on the Galois covering $\tilde{X} \to X$ of a coherent analytic sheaf $\mathcal{F}$ on $X$ is the ordinary cohomology of a sheaf on $X$ obtained by an adequate completion of the tensor product of $\mathcal{F}$ by the locally constant sheaf on $X$ associated to the left regular representation of the discrete group $\mathrm{Gal}(\tilde{X}/X)$ in the space of $L_2$ functions on $\mathrm{Gal}(\tilde{X}/X)$.

$H_2^q(\tilde{X}, \mathcal{F})$ actually belongs to a very nice abelian category of $\Gamma$-modules, introduced by M.S. Farber and W. Lück [2] to give a new interpretation of Novikov-Shubin invariants. We sketch a proof of this fact in the smooth projective space. This enables us to construct, in addition to Von Neumann dimension, other invariants measuring the non-Hausdorffness of this topological vector space.

## References


[1] M. Atiyah, *Elliptic operators, discrete groups and Von Neumann algebras*, Soc. Math. Fr. Asterisque **32-33**, 1976, 43-72.

[2] M.S. Farber, *Homological algebra of Novikov-Shubin invariants and Morse inequalities*, GAFA, Vol. **6**, $n^o$ **4**, 1996.

# BOUNDARY MANIFOLDS OF ALGEBRAIC PLANE CURVES AND LINE ARRANGEMENTS

*Eriko Hironaka*

Let $\mathcal{C} \subset \mathbb{P}^2$ be an algebraic plane curve in the complex projective plane. The goal of my research is to study the fundamental group and homotopy type of the complement $E_{\mathcal{C}} = \mathbb{P}^2 \setminus \mathcal{C}$ and compare it to combinatorial and geometric information about $\mathcal{C}$ and its singularities.

The combinatorial information of $\mathcal{C}$ is captured in the decorated incidence graph of $\mathcal{C}$. The *incidence graph* is the bipartite graph $\Gamma_{\mathcal{C}}$ with *curve-vertices* $v_C$, one corresponding to each irreducible curve $C \subset \mathcal{C}$, and *point-vertices* $v_p$, one corresponding to each singular point $p \in \text{Sing}(\mathcal{C})$. A point-vertex $v_p$ and a curve-vertex $v_C$ are attached by edges $e_b$, one for each branch $b$ of $C$ at $p$. The *decorated incidence graph* $(\Gamma_{\mathcal{C}}, \mathcal{T})$ is the incidence graph $\Gamma_{\mathcal{C}}$ together with local information attached to the vertices and edges: if $C$ is an irreducible component of $\mathcal{C}$, then $\mathcal{T}(v_C)$ is the genus of $C$; if $p$ is a singularity on $\mathcal{C}$, then $\mathcal{T}(v_p)$ is the link type of the algebraic link associated to $(\mathcal{C}, p)$; and if $b$ is a branch of $C$ at $p$, then $\mathcal{T}(e_b)$ is a connected component of the link $\mathcal{T}(v_p)$.

Two algebraic plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are *combinatorially equivalent* if there is an isomorphism of graphs

$$\phi : \Gamma_{\mathcal{C}_1} \to \Gamma_{\mathcal{C}_2}$$

such that $\mathcal{T}(v) = \mathcal{T}(\phi(v))$ and $\mathcal{T}(e) = \mathcal{T}(\phi(e))$, for each vertex $v$ and edge $e$. The curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are *topologically equivalent* if there is a homotopy equivalence of pairs

$$(\mathbb{P}^2, \mathcal{C}_1) \simeq (\mathbb{P}^2, \mathcal{C}_2).$$

A general problem is to determine when a given decorated incidence graph can be realized, whether it can be realized in two or more topologically distinct ways (giving what is known as a *Zariski pair*), and what properties the realization spaces have.

As a tool for approaching this problem, we study the boundary 3-manifold $M_{\mathcal{C}}$ associated to an algebraic plane curve. The *boundary 3-manifold* of $\mathcal{C}$ is the boundary of a regular neighborhood of $\mathcal{C}$ in $\mathbb{P}^2$. It is not hard to show that the fundamental group and homotopy type of $M_{\mathcal{C}}$ are determined by $(\Gamma_{\mathcal{C}}, \mathcal{T})$. Furthermore, $M_{\mathcal{C}}$ and its fundamental group can be realized as a graph manifold and a graph of groups over $\Gamma_{\mathcal{C}}$ in the sense of Waldhausen and Serre.

Our goal is to describe the homotopy type of the complement $E_{\mathcal{C}} = \mathbb{P}^2 \setminus \mathcal{C}$ explicitly in terms of $M_{\mathcal{C}}$ and extra data. This extra data is necessary because of the existence of Zariski pairs, shown by Zariski and later by several authors. The simplest example is the case when $\mathcal{C} = \mathcal{L}$ is a finite union of planar lines. The combinatorial type of $\mathcal{L}$ is then determined by $\Gamma_{\mathcal{L}}$, since the only singularities of $\mathcal{L}$ are the points of multiple intersection, and if $d$ lines of $\mathcal{L}$ come together at the point $p$, then $\mathcal{T}(v_p)$ is the positively oriented $d$-component Hopf link.

Descriptions of the homotopy type of the complement of a *real line arrangement* $\mathcal{L}$ (where $\mathcal{L}$ is defined by real equations) by Libgober, Orlik, Salvetti and Falk imply the following result.

**Theorem 1.** *The homotopy type of $E_{\mathcal{L}}$ is determined by $\Gamma_{\mathcal{L}}$ and an ordering of the edges emanating from each vertex.*

Work of Rybnikov gives an example of combinatorially equivalent complex line arrangements whose complements have non-isomorphic fundamental group. However, to the author's knowledge, there are still no known examples of two combinatorially equivalent real line arrangements whose complements have different homotopy type.

In a recent preprint, we give a new proof of Theorem 1. We exhibit the boundary manifold $M_{\mathcal{L}}$ as a graph manifold over the incidence graph $\Gamma_{\mathcal{L}}$, then explicitly find a "lifting"

$$f : \Gamma_{\mathcal{L}} \to M_{\mathcal{L}},$$

taking each vertex in $\Gamma_{\mathcal{L}}$ to the corresponding vertex manifold in $M_{\mathcal{L}}$, such that $E_{\mathcal{L}}$ is homotopy equivalent to the space

$$M_{\mathcal{L}}/f(\Gamma_{\mathcal{L}}),$$

given by collapsing the image of $f(\Gamma_{\mathcal{L}})$ to a point. Here, for $e$ an edge in $\Gamma_{\mathcal{L}}$, $f(e)$ need not be homotopic to a path in the edge manifold of $M_{\mathcal{L}}$ corresponding to $e$. We give the homotopy class of the map $f$ explicitly in terms of the incidence graph $\Gamma_{\mathcal{L}}$ endowed with orderings on the edges emanating from each vertex of $\Gamma_{\mathcal{L}}$. These orderings depend on the slopes of the lines and on the image of the points of intersection under a 1-dimensional linear projection.

## HARMONIC MAPS AND REPRESENTATIONS OF FUNDAMENTAL GROUPS

*Jürgen Jost*
(Joint work with Kang Zuo)

Given a representation $\rho$ of the fundamental group $\pi_1(X)$ of a compact Kähler manifold in a linear algebraic group $G$ defined over $\mathbb{C}$ or a p-adic field, one constructs a $\rho$–equivariant harmonic map

$$u : \tilde{X} \to G/K \quad \text{or} \quad \Delta(G),$$

where in the archimedean case, $G/K$ is the symmetric space associated to $G$, and in the nonarchimedean one, $\Delta(G)$ is a Euclidean Bruhat-Tits building on which $G$ operates isometrically. The strategy then is to exploit properties of $u$ in order to draw conclusions about $\rho$. $u$ is pluriharmonic by work of Siu and Sampson and defines a holomorphic foliation by work of Jost-Yau.

In the archimedean case, one pulls back convex functions form $G/K$ to produce plurisubharmonic functions on $\tilde{X}$, and in the nonarchimedean as well as in the Euclidean case where $u$ is the Albanese map, one pulls back holomorphic 1-forms.

If the representation is generically large, one produces in this manner a semi-Kähler form on $\tilde{X}$ that is positive definite on a Zariski open set. A vanishing theorem for $L^2$–cohomology is used to show that in that situation, $H^0_{L^2}(\tilde{X}, \Omega^i) = 0$ for $0 \le i < \dim_{\mathbb{C}} X$.

Combined with Atiyah's $L^2-$index theorem, this implies Kollár's conjecture that $\chi(K_X) \geq 0$ assuming that $\pi_1(X)$ admits a generically large representation as described above.

## REPRESENTATIONS OF BRAID GROUPS AND MAPPING CLASS GROUPS APPEARING IN CONFORMAL FIELD THEORY

*Toshitake Kohno*

Geometrically, conformal field theory is formulated as a vector bundle over the moduli space of Riemann surfaces equipped with a projectively flat connection. Let $\Sigma$ be a compact Riemann surface of genus $g$ with marked points $p_1, \cdots, p_n$. We fix a positive integer $k$, affine Lie algebra $\hat{\mathfrak{g}}$, integrable highest weight modules of level $k$ $\mathcal{H}_{\lambda_j}$, $1 \leq j \leq n$, and local coordinates at $p_1, \cdots, p_n$. We denote by $\mathcal{M}_{\mathbf{p}}$ the space of meromorphic functions on $\Sigma$ with poles at $p_1, \cdots, p_n$, which acts diagonally on $\otimes_{j=1}^n \mathcal{H}_{\lambda_j}$ by Laurent expansion at $p_1, \cdots, p_n$. The space of conformal blocks is by definition

$$\mathcal{H}_\lambda(\mathbf{p}) = \mathrm{Hom}_{\mathfrak{g}(\mathbf{p})}\left(\otimes_{j=1}^n \mathcal{H}_{\lambda_j}, \mathbb{C}\right),$$

which forms a vector bundle over the moduli space of Riemann surface of genus $g$ with $n$ marked points. A projectively flat connection is defined by means of the action of the Virasoro algebra.

In the case $g = 0$, we have an embedding

$$i^* : \mathcal{H}_\lambda(\mathbf{p}) \to \mathrm{Hom}_{\mathfrak{g}}\left(\otimes_{j=1}^n V_{\lambda_j}, \mathbb{C}\right),$$

and for a section $\Psi$ of the above vector bundle over the configuration space of $n$ points, we can show that

$$\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)}\Psi$$

is again a section. Here $L_{-1}^{(j)}$ denotes the action of the Virasoro operator $L_{-1}$ on the $j$-th component. This construction gives us an explicit form of the connection, which is called the Knizhnik-Zamolodchikov connection.

As the monodromy of Knizhnik-Zamolodchikov connection, we get an irreducible unitary representation of the braid group. In the case of higher genus, we obtain a projective representation of the mapping class group. This representation was used by the author to give a Heegaard splitting formula of the Witten invariant of 3-manifolds. In the case $\mathfrak{g} = sl_2(\mathbb{C})$, we can apply a method of abelianization of the conformal field theory using Prym variety due to T. Yoshida to show that the monodromy group is finite.

## ON A CHISINI CONJECTURE

*Vik.S. Kulikov*

Let $B \subset \mathbb{P}^2$ be an irreducible curve over $\mathbb{C}$ with $c$ ordinary cusps and $n$ nodes, as the only singularities. Put $\deg B = 2d$, and let $g$ be the genus of its desingularization. The curve $B$ is *the discriminant curve of a generic morphism* $f : S \to \mathbb{P}^2$, $\deg f \geq 3$, if:

$(i)$ $S$ is a non-singular irreducible projective surface;
$(ii)$ $f$ is unramified over $\mathbb{P}^2 \setminus B$;
$(iii)$ $f^*(B) = 2R + C$, where $R$ is irreducible and non-singular, and $C$ is reduced;
$(iv)$ $f_{|R} : R \to B$ is the normalization of $B$.

Chisini's Conjecture asserts that for the discriminant curve $B$ of a morphism $f$ of $\deg f \geq 5$ this generic morphism is unique up to equivalence.

We prove that if $B$ is the discriminant curve of a morphism $f$ of

$$\deg f > \frac{4(3d + g - 1)}{(2(3d + g - 1) - c)},$$

then, for $B$, the generic morphism $f$ is unique, i.e., Chisini's Conjecture holds for $B$.

This inequality holds for almost all generic morphisms. More precisely, let $S$ be a projective non-singular surface and $L$ an ample divisor on $S$, $f : S \to \mathbb{P}^2$ a generic morphism given by a three-dimensional subsystem $\{E\} \subset |mL|$, $m \in \mathbb{Q}$, and $B$ its discriminant curve. Then there exists a constant $m_0$ (depending on $S$ and $L$) such that, for $B$, the generic morphism $f$ is unique if $m \geq m_0$.

As a consequence we prove that if the canonical bundle $K_S$ of $S$ is ample, $f$ is a generic morphism such that $f^{-1}(\mathbb{P}^1) \equiv mK_S$, $m \in \mathbb{N}$, then, for $B$, the generic morphism $f$ is unique.

## HOLOMORPHIC FUNCTIONS OF SLOW GROWTH ON COVERING SPACES OF PROJECTIVE MANIFOLDS

*Finnur Lárusson* [4]

Let $Y \to M$ be an infinite covering space of a projective manifold $M \subset \mathbb{P}^N$ of dimension $n \geq 2$. Let $C$ be the intersection with $M$ of at most $n-1$ generic hypersurfaces of degree $d$ in $\mathbb{P}^N$. The preimage $X$ of $C$ in $Y$ is a connected submanifold. Let $\phi : Y \to [0, \infty)$ be the smoothed distance from a fixed point in $Y$ in a metric pulled up from $M$. Let $\mathcal{O}_\phi(X)$ be the Hilbert space of holomorphic functions $f$ on $X$ such that $f^2 e^{-\phi}$ is integrable on $X$, and define $\mathcal{O}_\phi(Y)$ similarly. We get a continuous linear restriction map $\rho : \mathcal{O}_\phi(Y) \to \mathcal{O}_\phi(X)$.

**Theorem.** *$\rho$ is an isomorphism for $d$ large enough.*

As an application, we obtain new examples of Riemann surfaces and domains of holomorphy in $\mathbb{C}^n$ with corona.

[4] E-mail: larusson@uwo.ca

In some sense, the Theorem reduces the problem of constructing holomorphic functions on $Y$ to the 1-dimensional case. There, we have the following result.

**Theorem.** *Let $X$ be a Galois covering space of a compact Riemann surface with a non-elementary hyperbolic covering group. Then either*

1. *every positive harmonic function on $X$ is the real part of a holomorphic function, or*
2. *if $u \geq 0$ is the real part of an $H^1$ function on $X$, then the boundary decay of $u$ at a zero on the Martin boundary of $X$ is no faster than its radial decay.*

*In case (1), $X$ is $H^p$-convex for each $p < \infty$.*

The condition in (2) is a geometric obstruction to a harmonic $h^p$ function being the real part of a holomorphic function, which arises simply and naturally in higher-dimensional settings. Here, however, it is the *only* obstruction.

How severe is the hyperbolicity restriction? Olshanskii has proved that almost every finitely presented group is hyperbolic. Is the same true for fundamental groups of projective manifolds?

Papers are available on the Web at

**http://www.math.uwo.ca/~larusson.**

## COHOMOLOGY OF LOCAL SYSTEMS ON THE COMPLEMENTS TO PLANE CURVES AND POSITION OF SINGULARITIES

*Anatoly Libgober*

The purpose of this talk is to discuss several invariants of fundamental groups of the complements of plane algebraic curves and describe them in terms of the local type and *position* of singularities in $\mathbb{P}^2$. Let $C \subset \mathbb{C}^2$ be a curve with $r$ components. Characteristic variety $V_i(C)$ can be defined as a subvariety of the torus of local systems on the complement to $C$ (i.e. $\mathrm{Hom}(\pi_1(\mathbb{C}^2 - C), \mathbb{C}^*)$) consisting of local systems $\mathcal{L}$ such that $\dim H^1(\mathbb{C}^2 - C, \mathcal{L}) \geq i$. Characteristic varieties are invariants of $\pi_1(\mathbb{C}^2 - C)$ and can be calculated via Fox calculus. We show that the local type of singularities of $C$ and geometry of the subset of $\mathbb{P}^2$ consisting of singularities of $C$ determine the characteristic varieties in an explicit way. More precisely the local type of singularities of $C$ defines a natural partition of the unit cube $U \subset \mathbb{R}^r$ into a union of polytopes (polytopes of quasiadjunction) so that each face $\Delta$ of such polytope defines in a natural way the ideal sheaf $\mathcal{I}_\Delta \subset \mathcal{O}_{\mathbb{P}^2}$ such that the support of $\mathcal{O}_{\mathbb{P}^2}/\mathcal{I}_\Delta$ is the singular locus of $C$. Face $\Delta$ called $i$-contributing if:

1. it belongs to a hyperplane $d_1x_1 + \cdots + d_rx_r = \ell(\Delta)$ for some integer $\ell(\Delta)$ depending on the face $\Delta$ (here $d_1, \ldots, d_r$ are the degrees of components of $C$);
2. $\dim H^1\big(\mathbb{P}^2, \mathcal{I}_\Delta\big(\sum_{i=1}^{r} d_i - 3 - \ell(\Delta)\big)\big) = i$.

**Theorem.** *Let $\mathcal{U} \to (\mathbb{C}^*)^r$ be the exponential map*

$$(s_1, \ldots, s_r) \to (\exp 2\pi s_1, \ldots, \exp 2\pi s_i).$$

*Then the closure of the image of each i-contributing face of quasiadjunction is a component of characteristic variety $V_i(C)$. Moreover, all components of characteristic variety $V_i(C)$ can be obtained in this way.*

**Example.** Let $C = C_1 \cup C_2$ be a union of two irreducible nonsingular curves such that the only singularities of $C$ are ordinary tacnodes. Partition into polytopes of quasiadjunction contains two polytopes:

$$\Delta_1 = \{(x,y) | 0 \le x, 0 \le y, x+y < \frac{1}{2}\}$$

$$\Delta_2 = \{(x,y) | x+y \ge \frac{1}{2}, x \le 1, y \le 1\}$$

where the face of quasiadjunction is given by $x+y = \frac{1}{2}$. It is contributing if and only if for some $k$ one has $d_1 = 2k$, $d_2 = 2k$ and $\dim H^1\big(\mathcal{I}(d-3-k)\big) \neq 0$ where $\mathcal{I}$ is the ideal of the union of singular points of $C_1 \cup C_2$.

These results generalize previous description of the Alexander polynomials of plane curves in terms of ideals of quasiadjunction [Proc. Symp. of Pure Math., 1983, vol. 40, A. Libgober, Alexander invariants of plane algebraic curves].

## LIOUVILLE TYPE PROPERTIES AND AUTOMORPHISM GROUPS

*Vladimir Lin*

The classical Liouville theorem says that $\mathbb{C}^n$ carries no nonconstant bounded holomorphic functions. More generally, any holomorphic function of polynomial growth on $\mathbb{C}^n$ is a polynomial. The same is true for harmonic functions on $\mathbb{R}^n$. By *Liouville type properties* of complex (or Riemannian) manifolds I mean the properties of similar nature. I wish to explain certain relationship between Liouville type properties of a complex (or Riemannian) manifold $X$ and the "reachness" of the action of its automorphism group $\operatorname{Aut} X$.

The following simple fact illustrates the idea: if the natural action of $\operatorname{Aut} X$ in $X$ is 2-transitive then $X$ is *Liouville* (that is, it carries no nonconstant bounded holomorphic functions). The conclusion holds true under the following weaker condition: *the diagonal* $\operatorname{Aut} X$*-action* $\Delta$ *in* $X \times X$,

$$\Delta\colon\ \operatorname{Aut} X \ni g \mapsto [X \times X \ni (x,y) \mapsto (gx, gy) \in X \times X]\,,$$

*admits a dense orbit* $\Gamma$. Indeed, if $f\colon\ X \to \mathbb{C}$ is holomorphic and bounded, then the function $F(x,y) \overset{\text{def}}{=} \sup_{g\in \operatorname{Aut} X} |f(gx) - f(gy)|$ is bounded, continuous, and plurisubharmonic on $X \times X$; it is $\Delta$-invariant and hence constant on $\Gamma$. Density of $\Gamma$ and the evident relation $F(x,x) \equiv 0$ imply $F = 0$ and, thereby, $f = \text{const}$.

Consider a subgroup $G \subseteq \operatorname{Aut} X$ (a typical case when $X$ comes together with certain automorphism group $G \subseteq \operatorname{Aut} X$ is the case of a Galois covering $X \to Y$). The natural

$G$-action in $X$ is *cocompact* if $G(K) = X$ for some compact $K \subseteq X$ (this is weaker than transitivity condition $G$(a single point set) $= X$).

**Theorem 1.** *A complex space (or Riemannian manifold) is Liouville whenever it admits a cocompact action of a virtually nilpotent group.*

For Riemannian manifolds this theorem is due to Y. Guivarch (1981) and (in more general form) to T. Lyons & D. Sullivan (1983). Holomorphic version appeared in my paper (1986).

**Example.** *a)* $X = \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, $G = \{z \mapsto az+b \mid a > 0,\, b \in \mathbb{R}\} = \mathrm{Aff}_+(\mathbb{R})$ (a two–step–solvable group). $G$ is transitive in $\mathbb{H}$; thus, $\mathbb{H}$ carries no nonconstant $G$-invariant functions. However, $\mathbb{H}$ is not Liouville. Actually, $\mathbb{H}$ is even *Carathéodory hyperbolic*, that is, bounded holomorphic functions separate points of $\mathbb{H}$.

*b)* Let $A \in \mathrm{SL}(3; \mathbb{Z})$ be a matrix with one real eigenvalue $\alpha > 1$ and two nonreal eigenvalues $\beta, \overline{\beta}$. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be a real and a complex eigenvectors of $A$ corresponding to $\alpha$ and $\beta$ respectively. Take $X = \mathbb{H} \times \mathbb{C}$ and consider the subgroup $G \subset \mathrm{Aut}\, X$ generated by the following four automorphisms $g_j$: $g_0(z,w) = (\alpha z, \beta w)$, $g_j(z,w) = (z + a_j, w + b_j)$, $1 \leq j \leq 3$, $(z,w) \in X = \mathbb{H} \times \mathbb{C}$.

$G$ is a two–step–solvable polycyclic group, and $X \to X/G$ is a polycyclic Galois $G$-covering over a smooth compact complex surface $\mathcal{I} \overset{\mathrm{def}}{=} X/G$, which is one of the *Inoue surfaces.*

These examples show that for groups $G$ "bigger" than virtually nilpotent Theorem 1 falls (even in the simplest case of Galois $G$-coverings over compact manifolds). However, certain weaker Liouville type properties are held for relatively big groups even under weaker transitivity type conditions.

Let $X$ be a complex space (or Riemannian manifold), $\mathcal{BO}(X)$ be the space of all bounded holomorphic (respectively harmonic) functions on $X$, and $\mathcal{P}(X)$ be the convex cone of all bounded continuous plurisubharmonic (respectively subharmonic) functions on $X$. $G$ acts in $\mathcal{BO}(X)$ and $\mathcal{P}(X)$ $(G \ni g \mapsto [f \mapsto f^g];\ f^g(x) = f(gx))$. An element $g \in G$ is a *$f$-period* if $f^g = f$. $G$-action in $X$ is *$\mathcal{P}(X)$-ergodic* if $\mathcal{P}(X)$ does not contain nonconstant $G$-invariant functions. (By the maximum principle, a cocompact action is $\mathcal{P}(X)$-ergodic.)

**Theorem 2** (V. L., 1986). *Let $G$ be amenable and $G$-action in $X$ be $\mathcal{P}(X)$-ergodic. Let $s \in G$, $f \in \mathcal{BO}(\mathcal{X})$, and $[s,h] = shs^{-1}h^{-1}$ be a $f$-period for each $h \in H$ in some finite index subgroup $H \subseteq G$, Then $s$ is a $f$-period. In particular, each $s \in G$ whose conjugacy class $s^G = \{gsg^{-1} \mid g \in G\}$ is finite acts trivially in $\mathcal{BO}(X)$, that is, $f^s = f$ for all $f \in \mathcal{BO}(X)$.*

**Corollary.** *Let an amenable group $G$ with nontrivial center act on a Carathéodory hyperbolic complex space $X$. Then the quotient space $X/G$ (if it exists in complex category) cannot be a Zariski open subset of a compact complex space. In particular, if such $G$ acts in a bounded domain $U \subset \mathbb{C}^n$, the quotient $U/G$ cannot be a quasiprojective variety.*

The proof of Theorem 2 involves some special $G$-invariant probability measure $\mu$ (related to the element $s$ and the function $f \in \mathcal{BO}(X)$ under consideration) on the Stone-Čech compactification $\beta G$ of $G$, and then the corresponding space $L_2(\beta G, \mu)$.

G. Margulis (1986) noted that all statements of Theorem 2 remain true for *arbitrary*, not necessary amenable, group $G$ whenever its action in $X$ is cocompact; his argument is based on a special form of the Harnack inequality related to the $G$-action.

Some parts of Theorem 2 were recently generalized as follows:

**Theorem 3** (V. L. & M. Zaidenberg, 1998). *Suppose that either $G$ is amenable and its action in $X$ is $\mathcal{P}(X)$-ergodic or the $G$-action in $X$ is cocompact. Then any FC-hypercentral element $s \in G$ acts trivially in $\mathcal{BO}(X)$.*

**Theorem 4** (V. L. & M. Zaidenberg, 1998). *Let $X \to Y$ be a Galois $G$-covering over a compact Riemannian or Kähler manifold $Y$. If $G$ is an extension of a FC-hypernilpotent group by a Varopoulos group, then $X$ is Liouville.*

On the other hand, the following result (strengthening a theorem of T. Lyons & D. Sullivan) is valid:

**Theorem 5** (V. L. & M. Zaidenberg, 1998). *Every compact Riemann surface $Y$ of genus $g \geq 2$ admits a Carathéodory hyperbolic metabelian covering $X \to Y$.*

## References


[1] Y. Guivarch, *Mouvement brownien sur les revêtements d'une variété compacte*, C. R. Acad. Sci. Paris, Sér. I **292** (1981), 851–853.

[2] Masahisa Inou, *On surfaces of class* $\mathrm{VII}_0$, Invent. Math. **24** (1974), 269–310.

[3] V. Ya. Lin, *Liouville coverings of complex spaces and amenable groups*, Math. USSR Sbornik **60:1** (1988), 197-216.

[4] V. Ya. Lin and M. Zaidenberg, *Liouville and Carathéodory coverings in Riemannian and complex geometry*, Voronezh Winter Mathematical School (P. Kuchment & V. Lin eds.), Amer. Math. Soc. Transl. (2) Vol. **184** (1998), 111–130.

[5] R. Lyons and D. Sullivan, *Function theory, random paths and covering spaces*, J. Different. Geom. **19** (1984), 299–323.


# DEGENERATING FAMILIES OF BRANCHED COVERINGS OF THE COMPLEX PROJECTIVE LINES AND BALLS


*Makoto Namba* [5]


**1** It is known that a finite branched covering of a given complex manifold $M$ is determined uniquely (up to isomorphisms) by its branch locus and (permutation) monodromy representation. However it is a difficult problem to determine the covering from them concretely (algebraically, analytically), even if $M = \mathbb{P}^1$ the complex projective line. We introduce two kinds of pictures, a Klein picture and a Riemann picture, each of which determines the covering topologically for the cases $M = \mathbb{P}^1$ and $M = \Delta(0, a) = \{z \in \mathbb{C} \mid |z| < a\}$, a ball.

Let $X \longrightarrow \mathbb{P}^1$ be a branched covering of $\mathbb{P}^1$ of degree $d$, where $X$ is a compact Riemann surface. We denote by $B_f = \{q_1, \ldots, q_n\}$ and $\Phi_f$ the branch locus and the monodromy


[5] E-mail: namba@math.wani.osaka-u.ac.jp

representation of $f$ respectively. $\Phi_f$ is determined uniquely up to its representation class $[\Phi_f]$.

Let $\Gamma$ be a simple oriented loop in $\mathbb{P}^1$ passing through the points $q_1, \ldots, q_n$ in this order surrounding a domain $\Omega$ clockwisely. We regard $\Omega$ and $\mathbb{P}^1 - \Omega$ as a continent and an ocean respectively. We pull $\Gamma$, $\Omega$ and $\mathbb{P}^1 - \Omega$ back over $f$ and get a checked pattern consisting of $d$-continents and $d$-oceans, which we call a Klein picture of $f$. The Klein picture determines $f$ topologically.

On the other hand, take a reference point $q_0$ in $\Omega$. We take disjoint paths connecting $q_0$ and $q_j$ $(j = 1, \ldots, n)$. Let $T_0$ be the graph consisting of the points $q_0, q_1, \ldots, q_n$ and these paths. We pull $T_0$ back over $f$ and get a graph $T$ on $X$. $T$ gives a cellular decomposition of $X$, which we call a Riemann picture of $f$. The Riemann picture also determines $f$ topologically. The Klein picture and the Riemann picture for a finite branched covering of a ball are defined in a similar way.

**2** A finite branched covering $f : X \longrightarrow \Delta(0, a) \times \mathbb{P}^1$ is called *a degenerating family of finite branched coverings of* $\mathbb{P}^1$ if (1) every fiber $t \times \mathbb{P}^1$ is not contained in $B_f$ and (2) every fiber $t \times \mathbb{P}^1, (t \neq 0)$ meets transversally with $B_f$ at $n$ ($n$ : *fixed*) points $\{q_1, \ldots, q_n\}$. Put $X_t = f^{-1}(t \times \mathbb{P}^1)$ and $f_t = f : X_t \longrightarrow t \times \mathbb{P}^1$. $f$ is then identified with the family $\{f_t\}$. Assume for simplicity, $a > 1$ and $(\Delta(0, a) \times \infty) \cap B_f = \emptyset$. Let $\delta = \{t = e^{is} \mid 0 \leq s \leq 2\pi\}$ be the unit circle. $\delta$ induces a braid $\theta(\delta)$ on $\{q_1, \ldots, q_n\}$ which is called *the braid monodromy*. By the theorem of Zariski- van Kampen, the equality $\Phi_g \theta(\delta) = \Phi_g$, where $g = f_1$. We show that $f = \{f_t\}$ is topologically determined by the pair $([\Phi_g], \theta(\delta))$, while the central fiber $f_0$ is determined topologically only by $[\Phi_g]$. We can observe the degeneration through the Klein picture.

We can also define a degenerating family of finite branched coverings of balls and get a similar theory to the case of that of $\mathbb{P}^1$.

**3** Every complex 2-dimensional normal singularity $(X, x)$ can be regarded as a degenerating family of branched coverings of balls. We can compute the local fundamental group $\pi_1(X - x, p_0)$ using the Zariski-van Kampen theorem and the Reidemeister-Schreier method. The Riemann picture is very useful to carry out the computation correctly.

This method works for the computation of the fundamental group of every 3-dimensional oriented compact manifold $Y$. In fact, By the theorem of Hilden-Montesinous, there is a covering $h : Y \longrightarrow S^3$ of degree 3 of the 3-sphere $S^3$ branching at a knot $B_h$ whose monodromy $\Phi_h(\gamma_j)$, ($\gamma_j$: generators of the fundamental group $\pi_1(S^3 - B_h)$), consist of only transpositions.

We may regard $S^3$ as the boundary of $\Delta(0, 1) \times \Delta(0, b)$ in $\mathbb{C}^2$ and the knot $B_h$ as a braid in $\delta \times \Delta(0, b)$. Let $B$ be the cone connecting the origin of $\mathbb{C}^2$ and every point of $B_h$. We construct a topological covering $f : X \longrightarrow \Delta(0, a) \times \Delta(0, b)$ of degree 3 branching at $B$ which is an extension of $h$ ($X$ is in fact a cone of $Y$). Then $f$ is a topological degenerating family of branched coverings of balls and

$$\pi_1(Y, p_0) \simeq \pi_1(X - x, p_0).$$

Thus the computation of $\pi_1(Y, p_0)$ can be done by our method.

To construct 3-dimensional oriented compact manifolds is reduced to find pairs $([\Phi], \sigma)$ such that $\Phi\sigma = \Phi$, where $\Phi$ is a transitive representation of the free group $< \gamma_1, \ldots, \gamma_n >$ to the 3rd symmetric group such that every $\Phi(\gamma_j)$ is a transposition and $\sigma$ is a braid of $n$-strings.

We prove that there are 3 canonical forms for $[\Phi]$. The braids $\sigma$ with $\Phi\sigma = \Phi$ form a subgroup $K_\Phi$ of the n-th braid group $B_n$ of finite index. It is important to analyze the subgroup $K_\Phi$ for the canonical $\Phi$.

## WEAK LEFSCHETZ THEOREMS

*Terrence Napier and Mohan Ramachandran*

Joint work with Mohan Ramachandran on an approach to Nori's weak Lefschetz theorem is described. The approach, which involves the $\bar{\partial}$-method, avoids moving arguments and gives much stronger results. In particular, it is proved that if $X$ and $Y$ are connected smooth projective varieties of positive dimension and $f : Y \to X$ is a holomorphic immersion with ample normal bundle, then the image of $\pi_1(Y)$ in $\pi_1(X)$ is of finite index. This result is obtained as a consequence of the following direct generalization of Nori's theorem:

**Theorem.** *Suppose $\Phi : U \to X$ is a holomorphic map from a connected complex manifold $U$ into a connected smooth projective variety $X$ of dimension at least 2 which is a submersion at some point. Let $Y \subset U$ be a connected compact analytic subspace such that $\dim H^0(\widehat{U}, \widehat{\mathcal{L}}) < \infty$ for every locally free analytic sheaf $\mathcal{L}$ on $U$. Then, for every Zariski open subset $Z$ of $X$, the image $G$ of $\pi_1(\Phi^{-1}(Z))$ in $\pi_1(Z)$ is of finite index.*

The idea of the proof is to form a covering space $\widetilde{Z} \to Z$ with fundamental group equal to $G$ and then to construct $L^2$ holomorphic sections of a suitable line bundle which separate the sheets of the covering. This construction is a standard application of the $L^2$ $\bar{\partial}$-method (Andreotti-Vesentini, Hörmander, Skoda, Demailly). Pulling these sections back to $\Phi^{-1}(Z)$ by a lifting of $\Phi$, the finite dimensionality of the space of holomorphic sections on the formal completion gives a bound on the dimension of the space of sections on $\widetilde{Z}$ and hence a bound on the degree of the covering space (i.e. on the index of $G$).

## INVARIANTS AT INFINITY OF POLYNOMIAL MAPS AND SUPERABUNDANCE

*András Némethi*

Let $f = f_d + f_{d-1} + \cdots : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial map, where $f_i$ is homogeneous of degree $i$. We will assume that $X^\infty = \{f_d = 0\} \subset \mathbb{P}^n$ has only isolated singularities with Milnor number $\{\mu_i\}_{i=1}^k$ and local monodromy operators $\{T_i\}_{i=1}^k$.

**Theorem.** *If $X^\infty \cap \{f_{d-1} = 0\} = \emptyset$, then $f$ is quasi-tame (in particular "good at infinity"). All its topological (and discrete Hodge theoretical) invariants at infinity depend only on the hypersurface $X^\infty \subset \mathbb{P}^n$. E.g. the generic fiber has the homotopy type of $\vee_\mu S^n$, where*

$\mu = (d-1)^{n+1} - \sum_i \mu_i$, *the characteristic polynomial of the monodromy at infinity* $T_f^\infty$ *is completely determined by the local monodromies* $\{T_i\}_i$.

The interesting fact is that the Jordan block structure of the monodromy at infinity $T_f^\infty$ is not local. In order to describe it, we use the following notations: If $V$ is a vector space and $T : V \to V$ is an endomorphism, then $(V_\lambda, T_\lambda)$ denotes the generalized $\lambda$-eigenspace with the restriction of $T$ on it; $\#_l T_\lambda$ denotes the number of $l \times l$-Jordan blocks of $T_\lambda$, and $\#T_\lambda := \sum_l \#_l T_\lambda$. We define the (equivariant) superabundance of $Sing(X^\infty)$ as follows: $\beta_0 := \dim P^n(X^\infty)$ and $\beta_s := \dim(P^{n+1}(X_n), \text{Galois action})_{\exp(2\pi i s/d)}$ for $1 \leq s \leq d-1$, where $X_d \to \mathbb{P}^n$ is the $d$-cyclic covering of $\mathbb{P}^n$ branched along $X^\infty$, and $P^*$ denotes the primitive cohomology. Moreover, we consider the local invariants $\chi_s$ as well: $\chi_0 = -\sum_i \mu_i + (-1)^{n+1} + [(-1)^n + (d-1)^{n+1}]/d$, and $\chi_s = \chi_0 + (-1)^n$ for $1 \leq s \leq d-1$.

**Theorem.** *I. If* $\alpha = e^{2\pi i s/d}$, $s = 0, \ldots, d-1$, *then:*

$\#_1(T_f^\infty)_\alpha = \chi_s + 2\beta_s - \Sigma_{i=1}^k \#(T_i)_\alpha$.

$\#_2(T_f^\infty)_\alpha = -\beta_s + \Sigma_{i=1}^k \#_1(T_i)_\alpha$.

$\#_{l+1}(T_f^\infty)_\alpha = \Sigma_{i=1}^k \#_l(T_i)_\alpha$ *for* $l \geq 2$.

*II. If* $\alpha^d \neq 1$, *then* $\#_l(T_f^\infty)_\alpha = \Sigma_{i=1}^k \#_l(T_i)_{\alpha^{1-d}}$ *for all* $l \geq 1$.

In particular, $T_f^\infty$ depends on the position of the singular points of $X^\infty$.

Similarly, the limit mixed Hodge structure at infinity associated with $f$ can be computed from the Hodge data of $X^\infty$, $X_d$ and the local (hypersurface) singularities of $X^\infty$.

The talk is based on the joint work with Ricardo García López. Some of the results are the global version of the results proved by E. Artal Bartolo, I. Luengo and A. Melle-Hernández in the local situation.

## GEOMETRY OF CUSPIDAL SEXTICS AND THEIR DUAL CURVES

*Mutsuo Oka* [6]

Let $C$ be a given irreducible plane curve of degree $n$ defined by $f(x,y) = 0$ where $f(x,y)$ is an irreducible polynomial. $C$ is called a *torus curve of type* $(p,q)$ if $p, q | n$ and $f(x,y)$ is written as $f(x,y) = f_{n/p}(x,y)^q + f_{n/q}(x,y)^p$ for some polynomials $f_{n/p}, f_{n/q}$ of degree $n/p$ and $n/q$ respectively. This terminology is due to Kulikov [K2]. Torus curves have been studied by many authors ([Z], [O1],[K2], [D],[T]).

In the process of studying Zariski pairs in the moduli of plane curves of degree 6 with 3 cusps of type $y^4 - x^3 = 0$, we have observed that *there exist exactly two irreducible components* $\mathcal{N}_{3,1}$ *and* $\mathcal{N}_{3,2}$ *which corresponds to torus curves and non-torus curves respectively. Their dual curves are sextics with six cusps and three nodes.* Starting from this observation, we study the moduli space of sextic with 6 cusps and 3 nodes which we denote by $\mathcal{M}$ and the moduli of their dual curves. It turns out that $\mathcal{M}$ has a beautiful symmetry. The "regular part" (=Plücker curves) of $\mathcal{M}$ is stable by the dual curve operation. On the other hand, the moduli of 3 (3,4)-cuspidal sextics $\mathcal{N}_3$ is on the "boundary" of $\mathcal{M}$ in a nice

[6]E-mail: oka@math.metro-u.ac.jp

way. By the dual operation, this moduli is isomorphic to a "singular" stratum $\mathcal{M}_3$ of $\mathcal{M}$, which consists of 6 cuspidal 3 nodal sextics with 3 flexes of order 2. The moduli space $\mathcal{M}$ is a disjoint union of torus curves and non-torus curves. For a curve $C \in \mathcal{M}$, the generic Alexander polynomial $\Delta_C(t)$ of $\mathbb{P}^2 - C$ is determined by the type $C$. If $C$ is a torus curve, $\Delta_C(t) = t^2 - t + 1$ and $\pi_1(\mathbb{P}^2 - C) = \mathbf{Z}_2 * \mathbf{Z}_3$, while for non-torus curve $C$, $\Delta_C(t) = 1$ and $\pi_1(\mathbb{P}^2 - C) = \mathbf{Z}_6$. Moreover we show that $C^*$ *is a torus curve if and only if* $C$ *is a torus curve.*

In this talk, we study dual curves and their singularities. We show a lemma which gives explicitly the defining polynomials of the dual curves and then we give a duality theorem which describes the dual singularity in terms of the original singularity. Then we study the moduli space $\mathcal{M}$ and other moduli spaces which appear on the canonical stratification of the "closure" $\widehat{\mathcal{M}}$ of $\mathcal{M}$. Main Theorem describes the stratification structure and the topological properties on $\widehat{\mathcal{M}}$. Then we compute the Alexander polynomial, using the method of Esnault and Artal ([E],[A1]). After that, we compute the moduli space of sextics with 3 (3,4) cusps. We can compute the fundamental groups of the complements of 3 (3,4) cuspidal sextics of torus type and non-torus type. Finally we give a new Zariski triple of plane curves of degree 12 with 12 (3,4) cusps, as an application of Main Theorem.

## References


[A1] E. Artal, *Sur les couples des Zariski*, J. Algebraic Geometry, vol 3 (1994), 223-247.

[A2] E. Artal and J. Carmona, *Zariski pairs, fundamental groups and Alexander polynomials*, preprint, 1995

[B-K] E. Brieskorn and H. Knörrer, *Ebene Algebraische Kurven*, Birkhäuser (1981), Basel-Boston - Stuttgart.

[E] H. Esnault, *Fibre de Milnor d'un cône sur une courbe algébrique plane*, Invent. Math. 68 (1982), 477-496.

[F] R.H. Crowell and R.H. Fox, *Introduction to Knot Theory*, Ginn and Co. (1963).

[D] A. Degtyarev, *Alexander polynomial of a curve of degree six*, J. Knot Theory and its Ramification, Vol. 3, No. 4 (1994), 439-454.

[D-L] I. Dolgachev and A. Libgober, *On the fundamental group of the complement to a discriminant variety*, in: Algebraic Geometry, Lecture Note 862 (1980), 1-25, Springer, Berlin Heidelberg New York.

[G-H] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, 1978 A Wiley-Interscience Publication, New York-Chichester-Brisbane-Toronto.

[K1] Vik. S. Kulikov, *The Alexander polynomials of algebraic curves in* $\mathbb{C}^2$, Algebraic geometry and its applications, Vieweg, Braunschweig, 1994, 105-111

[K2] Vik. S. Kulikov, *On plane algebraic curves of positive Albanese dimension*, preprint.

[Le] D.T. Lê, *Calcul du nombre de cycles évanouissants d'une hypersurface complexe*, Ann. Inst. Fourier, vol. 23,4 (1973), 261-270.

[L-O] D.T. Lê and M. Oka, *On the Resolution Complexity of Plane Curves*, Kodai J. Math. Vol. 18 (1995), 1-36.

[Li1] A. Libgober, *Alexander invariants of plane algebraic curves*, Proceeding of Symposia in Pure Math., Vol. 40, (1983), 135-143.

[Li2] A. Libgober, *Alexander polynomial of plane algebraic curves and cyclic multiple planes*, Duke Math. J., vol. 49, no. 4 (1982), 833-851.

[O1] M. Oka, *Some plane curves whose complements have non-abelian fundamental groups*, Math. Ann., vol 218 (1975), 55-65.
[O2] M. Oka, *On the fundamental group of the complement of certain plane curves*, J. Math. Soc. Japan, vol 30 (1978), 579-597.
[O3] M. Oka, *Symmetric plane curves with nodes and cusps*, J. Math. Soc. Japan, vol 44, no. 3 (1992), 375-414.
[O4] M. Oka, *Two transforms of plane curves and their fundamental groups*, J. Math. Sci. Univ. Tokyo, vol 3 (1996), 399-443.
[O5] M. Oka, *Flex curves and their applications*, preprint (1997).
[O6] M. Oka, *Non-degenerate complete intersection singularity*, Hermann, Paris, 1997.
[N] M. Namba, *Geometry of projective algebraic curves*, Decker, New York, 1984
[R] R. Randell, *Milnor fibers and Alexander polynomials of plane curves*, Proceeding of Symposia in Pure Math., Vol. 40, part 2 (1983), 415-419.
[S] I. Shimada, *A note on Zariski pairs*, Compositio Math., no.104 (1996), 125-133.
[T] H. Tokunaga, *(2,3) torus decompositions of plane sextics and their applications*, preprint (1997).
[W1] R. Walker, *Algebraic curves*, Dover Publ. Inc., New York (1949).
[W2] C.T.C. Wall, *Duality of singular plane curves*, J. London Math. Soc. (2) 50 (1994), 265-275.
[Z] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. vol 51 (1929), 305-328.


## HYPERGEOMETRIC INTEGRALS AND HYPERPLANE ARRANGEMENTS

*Peter Orlik*

This talk was an exposition of the Aomoto-Gelfand theory of multivariable hypergeometric integrals with emphasis on the role played by the theory of hyperplane arrangements. Let $\mathbb{C}^\ell$ be affine space and $\mathcal{A}$ an arrangement of affine hyperplanes. Let $N = \cup_{H\in\mathcal{A}} H$ be its divisor and $M = \mathbb{C}^\ell - N$ its complement. Let $\alpha_H$ be a linear polynomial with kernel $H$ and let $\lambda_H \in \mathbb{C}$. Define a rank one local system $\mathcal{L}$ on $M$ to have monodromy $\exp(-2\pi i\lambda_H)$ around $H$. Let $\mathcal{L}^\vee$ be its dual local system. The hypergeometric pairing is

$$H_p(M, \mathcal{L}^\vee) \times H^p(M, \mathcal{L}) \longrightarrow \mathbb{C}$$

In order to interpret the result of this pairing as an integral, $M$ is given a smooth, locally finite triangulation and a holomorphic de Rham theorem is proved

$$H^p(M, \mathcal{L}) \simeq H^p(\Gamma(M, \Omega^\cdot), \nabla).$$

Here $\Omega^\cdot$ is the holomorphic de Rham complex of $M$ and

$$\nabla = d + \omega_\lambda = d + \sum_{H\in\mathcal{A}} \lambda_H \frac{d\alpha_H}{\alpha_H}.$$

If $\mathcal{L}$ is sufficiently generic, then Esnault-Schechtman-Viehweg proved

$$H^p(M, \mathcal{L}) \simeq H^p(B^\cdot, \omega_\lambda)$$

where $B^\cdot$ is the $\mathbb{C}$-algebra generated by the logarithmic forms $\{d\alpha_H/\alpha_H \mid H \in \mathcal{A}\}$ and

$$H^p(M, \mathcal{L}) = 0 \text{ for } p \neq \ell, \quad \dim H^\ell(M, \mathcal{L}) = |e(M)|$$

where $e(M)$ is the Euler characteristic of the complement.

Arrangement theory provides a combinatorial calculation of $e(M)$ in terms of the characteristic polynomial. The **nbc** set of $\mathcal{A}$ is a basis for the algebra $B^{\cdot}$ and a subset called $\beta$**nbc** was used by Falk-Terao to construct a basis for $H^{\ell}(M, \mathcal{L})$

## FUNDAMENTAL GROUPS OF COMPACT KÄHLER MANIFOLDS WITH NUMERAICALLY EFFECTIVE RICCI CLASS

*Mihai Paun*

In the paper "Compact Kähler manifolds with numerically effective Ricci class", the authors (J.- P. Demailly, T. Peternell and M. Schneider) raised the following two conjectures:

**Conjecture 1.** *Let $(X, \omega)$ be a compact Kähler manifold with numerically effective Ricci class. Then $\pi_1(X)$ is almost-nilpotent.*

**Conjecture 2.** *Let $(X, \omega)$ be a compact Kähler manifold with numerically effective Ricci class. Then the Albanese morphism of $X$ is surjective.*

In geometrical terms, the hypothesis on the Ricci class translates as follows: there exists a sequence of Kähler metrics $(\omega_k)_k$ on $X$ such that $\omega_k \in \{\omega\}$ and $\mathrm{Ricci}_{\omega_k} \geq -1/k\omega_k$ (here we denote by $\{\omega\}$ the cohomology class of the metric $\omega$).

Now in the setting of complete Riemannian manifolds with Ricci curvature bounded from below, the following (deep) result was recently proved by Cheeger–Colding:

**Theorem (Cheeger–Colding).** *There exists a positive number $\delta_m$ such that for each complete manifold $(M, g)$ with $\dim M = m$ and $\mathrm{Ricci}_g \geq -g$, the image*

$$\mathrm{Im}\left(\pi_1(B_p(\delta_m)) \mapsto \pi_1(X)\right)$$

*is a almost-nilpotent group.*

By combining this result with the techniques developed by Demailly et al. in the paper mentioned above, we prove:

**Theorem 1.** *Conjecture 1 is true.*

Actually, the essential observation is that one can bound the norm of the generators of $\pi_1$ uniformly with respect to all the Kähler metrics sitting in a compact set of the Kähler cone.

As for the second problem, it has been recently settled by Qi Zhang for $X$ a projective manifold. In the Kähler case we have obtained the next partial result:

**Theorem 2.** *Let $(X, \omega)$ be a compact Kähler manifold endowed with a sequence $(\omega_k)_k$ of Kähler metrics with the following properties:*

1. $\omega_k \in \{\omega\}$
2. $\mathrm{Ricci}_{\omega_k} \geq -1/k\omega_k$.

*Then:*

i) $b_1(X) \leq 2n = \dim_{\mathbb{R}}(X)$.

ii) *Under the additional hypothesis* $\operatorname{diam}(X, \omega_k)/\sqrt{k} \mapsto 0$ *as* $k$ *goes to infinity, the Albanese map of* $X$ *is surjective.*

The proof of the first point of this result uses some ideas of Gromov. As for the proof of *ii)*, it rests on a "Toponogov $L^2$" theorem of Cheeger-Colding and on some *ad-hoc* Kählerian arguments. The (undesired) hypothesis concerning the growth of the sequence of diameters is needed to keep some global estimates of the functions of type $x \mapsto |\beta|^2_{k,x}$ (where $\beta$ is a holomorphic 1–form) uniform with respect to $k$.

**Remark.** As for the moment we do not know any "honest" example for which the sequence of diameters goes to infinity, we suspect that for a manifold with nef Ricci class, the additional hypothesis in *ii)* is always satisfied for an "optimal" sequence of metrics with the properties 1 and 2 of theorem 2. This is actually the case if the anticanonical class of $X$ is numerically effective and contains a closed positive current with small enough Lelong numbers.

## COMPLETE UNFOLDINGS OF CLOSED POSITIVE BRAIDS AND OREVKOV'S GENERALIZATION OF THE ZARISKI CONJECTURE

*Lee Rudolph*

In [1], Orevkov proved a generalization of the so-called Zariski Conjecture (Theorem of Fulton–Deligne), of which a slightly special case is the following.

**Theorem.** *If* $V \subset \mathbb{C}^2 \subset \mathbb{CP}^2$ *is an affine complex plane curve such that the link-at-infinity* $L_\infty(V) \subset S^3_{1/\epsilon}$ *is a closed positive braid, and if the only singularities of* $V$ *in* $\mathbb{C}^2$ *are nodes, then the fundamental group* $\pi_1(\mathbb{C}^2 \setminus V)$ *is abelian.*

The original Zariski Conjecture is the much more special case that $L_\infty(V)$ is a link of $\deg V$ components of a positive Hopf fibration $S^3_{1/\epsilon} \to \mathbb{P}^1 = S^2$; the general case of Orevkov's theorem allows $L_\infty(V)$ to be a split sum of several closed positive braids, in which case the conclusion has to modified a bit.

While Orevkov's ingenious proof uses the structure of $L_\infty(V)$ as a closed positive braid (and of $V$ as a sort of "quasipositive nodal braided surface", cf. [3]) repeatedly and profoundly, it does not (at least not explicitly: nor implicitly as far as I can tell) use what is, topologically, perhaps the most salient fact about closed positive braids, namely, that they are fibered links. In my May 1998 talk at Oberwolfach, I proposed a generalization of the quoted Theorem to all closed positive braids (not just those which appear as links-at-infinity). In the light of subsequent conversations with Orevkov, Teicher and other participants at that meeting, I am now emboldened to a further generalization.

**Conjecture.** *If* $p : (D^4, D^4 \setminus \operatorname{Int}(N(K), K)) \to (D^2, S^1, 0)$ *is an unfolding of a fibered link* $L \subset S^3 = \partial D^4$*, such that:*

1. $V = p^{-1}(0)$ *is immersed with no singularities but nodes, and*
2. *each critical point* x *of* $p$ *with* $p(\mathrm{x}) \neq 0$ *is a positive quadratic singularity,*

*then $\pi_1(D^4 \setminus V)$ is abelian.*

Here "unfolding" is to be understood in the fibered-knot-theoretical sense introduced in [2].

The proposed method of proof, in the case that $L$ is a closed positive braid, or more generally a quasipositive Hopf-plumbed fibered link, is to extend the methods of [4] and [5] to analyze Hopf-plumbed fiber surfaces from the point of view of unfoldings (which are, *a priori*, more general than the Murasugi sums discussed there). As to the general case, no proof method is proposed at this time; it just seems like a good guess!

## REFERENCES


[1] S. Yu. Orevkov, *The fundamental group of the complement of a plane algebraic curve.*, Mat. Sb., Nov. Ser. **137(179)** (1988), no. 2(10), 260–270; English transl. in Math. USSR, Sb. **65** (1990), no. 1, 267–277.

[2] W. D. Neumann and Lee Rudolph, *Unfoldings in knot theory*, Math. Annalen **278** (1987), 409–439; corrigendum, ibid. **282** (1988), 349–351

[3] Lee Rudolph, *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helvetici **58** (1983), 1–37.

[4] ———, *Quasipositive plumbing* (*Constructions of quasipositive knots and links*, V), Proc. A.M.S. **126** (1998), 257-267.

[5] ———, *Hopf plumbing, baskets, arborescent surfaces, and $\mathcal{T}$-homogeneous braids*, in preparation (1998).


# NORMAL CROSSING SYSTEMS OF PLANE ONE-PLACE CURVES

*P. Russell*

In the study of simple connectedness of certain affine 3-folds, the following question arose: If $G, H$ are "embedded lines" in $\mathbb{C}^2$ meeting normally in $r \geq 1$ points, is $\pi_1(\mathbb{C}^2 - G \cup H)$ abelian? The answer is yes, in essence since a line $G$ is a *curve with one place at infinity*, i.e. the closure $\overline{G}$ in $\mathbb{P}^2$ has a unique unibranch point at infinity. Let $G$ be such a curve. For $c \in \mathbb{C}$, put $G_c = g^{-1}(c)$, where $G = g^{-1}(0)$. Let $L_\infty = \mathbb{P}^2 - \mathbb{C}^2$.

Put

$$MNC(G) = \text{exceptional locus } + L'_\infty$$

in the minimal normal crossing ($MNC$) resolution of $\overline{G} + L_\infty$ at infinity. Here, "$\,'\,$" denotes proper transform.

**Fundamental theorem** on one-place curves (Abhyankar-Moh):

*(1)* $\forall c \in \mathbb{C}$, $MNC(G) = MNC(G_c)$;

*(2)* $G'^2 > 0$ ($G' =$ *proper transform of* $\overline{G}$ *after* $MNC$*-resolution).*

**Corollary.** *If $G$ is smooth, then $G$ is a general fibre of $g : \mathbb{C}^2 \to \mathbb{C}$.*

Let $G, H$ be one-place curves. We **define**: $H < G \Leftrightarrow$ the proper transform of $\overline{H}$ has NC with $MNC(G)$.

**Lemma.** $H \cap G = \emptyset \Leftrightarrow G = g^{-1}(0),\ H = h^{-1}(0)$ *with* $h - g = c \in \mathbb{C}^*$.

**Proposition.** *Suppose $H \cap G \neq \emptyset$ and $H < G$. Then $G'^2 > 0$ and $H'^2 \geq 0$, where $G', H'$ are the proper transforms of $\overline{G}, \overline{H}$ in the MNC-resolution at infinity of $\overline{G} + \overline{H} + L_\infty$.*

By a result of Nori, one has immediately

**Theorem 1.** *Let $G_1, \cdots, G_k$ be smooth one-place curves in $\mathbb{C}^2$ such that $G_1 + \cdots + G_k$ is a NC-divisor and $G_i \cap G_j \neq \emptyset \ \forall i, j$. Then $\pi_1(\mathbb{C}^2 - G_1 \cup \cdots \cup G_k)$ is abelian, i.e. $\mathbb{Z}^k$.*

More generally:

**Theorem 2.** *Let $G_{11}, \cdots, G_{1s_1}, \cdots, G_{k1}, \cdots, G_{ks_k}$ be smooth one-place curve such that $\Sigma G_{ij}$ is a NC-divisor and $G_{ij} \cap G_{i'j'} = \emptyset \Leftrightarrow i = i'$. Then*

$$\pi_1(\mathbb{C}^2 - \cup G_{ij}) = F_{s_1} \times \cdots \times F_{s_k}.$$

*($F_s$ = free group on $s$ generators.)*

It is an interesting question to what extent these results can be extended to nodal one-place curves. There may be a chance with theorem 1, or the "commuting" part of theorem 2. This is true, for instance (again by a theorem of Nori), for **generic rational curves** with one place at infinity (they are nodal of finite distance). However,

**Example.** *Let $G_0 = g^{-1}(0)$ be a generic rational one-place curve with $t$ nodes. Then $g : \mathbb{C}^2 \to \mathbb{C}$ has $t$ additional singular fibres $G_1, \cdots, G_t$, each with one node, and $U = \mathbb{C}^2 - \cup G_i \to \mathbb{C}^{(1+t)*}$ is a fibration. So*

$$1 \to F_{2t} \to \pi_1(U) \to F_{t+1} \to 1$$

*is exact and $\pi_1(U)$ is not generated by one vanishing loop each for each $G_i$. In case $t = 1$, $U = \mathbb{C}^2$-(two nodal cubics). The examples show that a conjecture of Orevkov on the generation of the fundamental group of the complement of one-place curves needs some modification.*

## ZARISKI HYPERPLANE SECTION THEOREM AND CHOW FORMS

*Ichiro Shimada* [7]

We extend Zariski's hyperplane section theorem to relative cases and Grassmannians.

Let $V$ be a complex vector space of dimension $m$, and let $U := \mathrm{Grass}(r, V)$ be the Grassmannian variety of all $r$-dimensional linear subspaces of $V$, where $1 \leq r \leq m - 2$. Let the group $G := \mathrm{GL}(V)$ acts on $U$ from left in the natural way. Suppose that we are given a morphism $f : X \to U$ from a nonsingular connected quasi-projective variety $X$, and a non-zero reduced effective divisor $D$ of $U$. For $\gamma \in G$, let ${}^\gamma f : X \to U$ be the composite of $f$ with the action $\gamma : U \to U$ of $\gamma$ on $U$, and let ${}^\gamma F : {}^\gamma f^{-1}(U \setminus D) \to (U \setminus D) \times X$ denote the morphism given by $x \mapsto ({}^\gamma f(x), x)$. We put $\mathbb{P}_*(V) := \mathrm{Grass}(1, V)$, and consider $U$ as the variety of all $(r-1)$-dimensional projective linear subspaces of $\mathbb{P}_*(V)$. For a point $p \in U$, let $\Pi(p) \subset \mathbb{P}_*(V)$ denote the projective linear subspace corresponding to $p$. Let $S \subset \mathbb{P}_*(V)$ be a reduced irreducible closed subvariety. For a point $s \in S$, the Zariski tangent space

[7] E-mail: shimada@math.sci.hokudai.ac.jp

$T_sS$ to $S$ at $s$ is regarded as a projective linear subspace of $\mathbb{P}_*(V)$. For a non-negative integer $k$, we define the *generalized Chow form* $C[S,k] \subset U$ to be the Zariski closure of the locus

$$\left\{ p \in U \; ; \; \begin{array}{l} \text{there exists a nonsingular point } s \in S \\ \text{such that } s \in \Pi(p) \text{ and } \dim(T_sS \cap \Pi(p)) \geq k. \end{array} \right\}$$

When $\dim S = m - r - 1$ and $k = 0$, the variety $C[S,0]$ is the classical Chow form of $S$. When $\dim S = m - r - 1 + k$, the variety $C[S,k]$ is called *a tangential Chow form of* $S$, which was introduced and studied by Green and Morrison ([3]).

Let $U^*$ be the Grassmannian variety of all $(m-r)$-dimensional linear subspaces of $V^* := \mathrm{Hom}(V, \mathbb{C})$. We have a natural isomorphism

$$\delta \; : \; U^* \xrightarrow{\sim} U.$$

For a reduced irreducible closed subvariety $S^*$ of $\mathbb{P}^*(V) := \mathbb{P}_*(V^*)$, we have its generalized Chow forms $C[S^*,k]^* \subset U^*$.

We consider the following conditions on $f : X \to U$ and $D \subset U$. Let $\mathrm{Sing}\, D$ denote the singular locus of $D$.

(AI) There is an $r$-dimensional projective linear subspace $M$ of $\mathbb{P}_*(V)$ such that $\Pi(f(x))$ is contained in $M$ for all $x \in X$.

(AII) There is an $(r-2)$-dimensional projective linear subspace $N$ of $\mathbb{P}_*(V)$ such that $\Pi(f(x))$ contains $N$ for all $x \in X$.

(B) There exists an irreducible component $D_i$ of $D$ and a reduced irreducible closed subvariety $S \subset \mathbb{P}_*(V)$ such that $D_i$ coincides with the tangential Chow form $C[S,k]$, where $k = \dim S - (m - r - 1)$.

(CI) There exists an irreducible component $(\mathrm{Sing}\, D)_j$ of $\mathrm{Sing}\, D$ with codimension 2 in $U$ and a reduced irreducible closed subvariety $S \subset \mathbb{P}_*(V)$ with $\dim S = m - r - 2$ such that $(\mathrm{Sing}\, D)_j = C[S,0]$.

(CII) There exists an irreducible component $(\mathrm{Sing}\, D)_j$ of $\mathrm{Sing}\, D$ with codimension 2 in $U$ and a reduced irreducible closed subvariety $S^* \subset \mathbb{P}^*(V)$ with $\dim S^* = r - 2$ such that $(\mathrm{Sing}\, D)_j = \delta(C[S^*,0]^*)$.

Let $T$ be an oriented connected topological manifold, and let $\alpha$ be an element of $H^2(T;\mathbb{Z})$. Then there is a topological line bundle $L \to T$, unique up to isomorphisms, such that $c_1(L) = \alpha$. Let $L^\times \subset L$ be the complement to the zero section of $L$. We have the homotopy exact sequence

$$\longrightarrow \pi_2(T) \xrightarrow{\partial_L} \pi_1(\mathbb{C}^\times) \longrightarrow \pi_1(L^\times) \longrightarrow \pi_1(T) \longrightarrow 1 \quad .$$

such that the image of $\pi_1(\mathbb{C}^\times) \to \pi_1(L^\times)$ is contained in the center. Thus we obtain a central extension of $\pi_1(T)$ by a cyclic group $\mathrm{Coker}\, \partial_L$, which we shall call *the central extension associated with* $\alpha \in H^2(T;\mathbb{Z})$. It is easy to write down the cohomology class of $H^2(\pi_1(T); \mathrm{Coker}\, \partial_L)$ corresponding to $\pi_1(L^\times)$ in terms of $\alpha$ (see [1]).

Let $c \in H^2(U;\mathbb{Z})$ be the first Chern class of the positive generator of Pic$(U)$. Let ext $\in H^2((U \setminus D) \times X;\mathbb{Z})$ be the cohomology class

$$-(\mathrm{incl} \circ \mathrm{pr}_1)^* c + (f \circ \mathrm{pr}_2)^* c,$$

where $\mathrm{pr}_1$ and $\mathrm{pr}_2$ are projections from $(U \setminus D) \times X$ to $U \setminus D$ and $X$, respectively, and incl is the inclusion of $U \setminus D$ into $U$.

Our main theorem, which contains the classical theorem of Zariski ([4], [6]) as a special case when $r = 1$ and $f : X \to U = \mathbb{P}^{m-1}$ is a linear embedding of a projective plane, is as follows.

**Main Theorem.** *Suppose that* $\dim f(X) \geq 2$. *Let* $\gamma$ *be a general element of* $G$. *Then either one of the following holds;*

- *the homomorphism*

$$^{\gamma}F_* \ : \ \pi_1(^{\gamma}f^{-1}(U \setminus D)) \longrightarrow \pi_1(U \setminus D) \times \pi_1(X)$$

*gives* $\pi_1(^{\gamma}f^{-1}(U \setminus D))$ *a structure of the central extension of* $\pi_1(U \setminus D) \times \pi_1(X)$ *by the cokernel of* $f_* : \pi_2(X) \to \pi_2(U)$, *and this extension is associated with the cohomology class* ext, *or*

- $r \geq 2$ *and the condition*

(1) ( (AI) and ( (B) or (CI) )) or ( (AII) and ( (B) or (CII) ))

*is satisfied.*

There are examples which shows that the conditions that the condition (1) should not be satisfied for the desired isomorphisms between fundamental groups to be valid.

This type of theorem has been proved by Goresky-MacPherson's stratified Morse theory ([2]) for various other situations. Our method is completely different and based on the monodromy argument of Zariski-van Kampen type. The central idea is the following observation. Suppose that we are given a family of algebraic varieties over an affine space $\mathcal{A}^N$ such that, outside a Zariski closed subset $Z \subset \mathcal{A}^N$ of codimension $\geq 2$, we have a local section. Under certain mild conditions, the triviality of the local monodromies on the fundamental group of a general fiber implies that the fundamental group of the general fiber is isomorphic to the fundamental group of the total space. We apply this observation to the affine space End$(V)$.

## References


[1] F. Catanese and F. Tovena, *Vector bundles, linear systems and extensions of* $\pi_1$, Lecture Notes in Math. **1507**, Springer-Verlag, Berlin, 1992, pp. 51 – 70.

[2] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Springer-Verlag, Berlin, 1980.

[3] M. L. Green and I. Morrison, *The equations defining Chow varieties*, Duke Math. J. **53** (1986), 733 – 747.

[4] H. A. Hamm and D. T. Lê, *Un théorème de Zariski du type de Lefschetz*, Ann. Sci. École Norm. Sup. **6** (1973), 317 – 355.

[5] I. Shimada, *Zariski hyperplane section theorem and Chow forms*, preprint.

[6] O. Zariski, *A theorem on the Poincaré group of an algebraic hypersurface*, Ann. Math. **38** (1937), 131 – 141.

## ON THE DIFFERENCE BETWEEN THE FUNDAMENTAL GROUPS OF REAL AND COMPLEX ARRANGEMENTS

*Alexander I. Suciu*

The $k^{\text{th}}$ Fitting ideal of the Alexander invariant of an arrangement $\mathcal{A}$ of $n$ complex hyperplanes defines a *characteristic* subvariety, $V_k(\mathcal{A})$, of the complex algebraic torus $(\mathbb{C}^*)^n$. The characteristic varieties of an arrangement provide rather subtle and effectively computable homotopy-type invariants of its complement. In joint work with Daniel Cohen, we show that the tangent cone at the identity of $V_k(\mathcal{A})$ coincides with $\mathcal{R}^1_k(\mathcal{A})$, one of the cohomology support loci of the Orlik-Solomon algebra. Using work of Arapura and Libgober, we conclude that all positive-dimensional components of $V_k(\mathcal{A})$ are combinatorially determined, and that $\mathcal{R}^1_k(\mathcal{A})$ is the union of a subspace arrangement in $\mathbb{C}^n$, thereby resolving a conjecture of Falk.

If $\mathcal{A}$ is a real 2-arrangement (in the sense of Goresky and McPherson), the characteristic varieties are no longer subtori through the origin. The nature of these varieties vividly illustrates the difference between real and complex arrangements. In joint work with Daniel Matei, we study the homotopy types of complements of arrangements of $n$ transverse planes in $\mathbb{R}^4$, obtaining a complete classification for $n \leq 6$, and lower bounds for the number of homotopy types in general. Furthermore, we show that the homotopy type of the complement of a 2-arrangement in $\mathbb{R}^4$ is not determined by its cohomology ring, thereby answering a question of Ziegler.

## AN OPEN QUESTION IN FUNDAMENTAL GROUPS OF COMPLEMENTS OF BRANCH CURVES

*Mina Teicher*

We want to use fundamental groups of complements of branch curves to distinguish among surfaces lying in different connected components of moduli spaces.

This topic started with Zariski who proved in the 30's that for a cubic surface in $\mathbb{CP}^3$, $\overline{G} \simeq Z_2 \star Z_3$ (see [26]). In the late 70's Moishezon proved that if $X$ is a deg $n$ surface in $\mathbb{CP}^3$ then $G \simeq B_n$, $\overline{G} \simeq B_n/\text{Center}$ (see [5]). In fact, Moishezon's result for $n = 3$ is the same as Zariski's result since $B_3/\text{Center} \simeq Z_2 \star Z_3$. The next example was $V_2$ (Veronese of order 2) (see [9]). In all the above examples we have $G \supset F_2$ where $F_2$ is a free noncommutative subgroup with 2 elements. We call a group $G$ "big" if $G \supset F_2$.

Since 1991 we have discovered the following new examples: $V_3$, the Veronese of order 3, generalized later to general $V_n$; $X_{ab}$, the embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ into $\mathbb{CP}^N$ w.r.t. a linear system $|a\ell_1 + b\ell_2|$; and $CI$, the complete intersection (see [13], [14], [15], [16], [17], [21]).

Unlike previous expectations, in all the new examples $G$ is not "big". Moreover, $G$ is "small", i.e., $G$ is "almost solvable", i.e., it contains a subgroup of finite index which is solvable. During our research we discovered a new quotient of the braid group (by a subgroup of the commutant), namely $\tilde{B}_n$ s.t. all new results give $G = \tilde{B}_n$-group and $\bar{G} = G/$central element ($\tilde{B}_n$*-group* is a group on which $\tilde{B}_n$ act) (See [21], [22]). For $CI$, $G$ is $\tilde{B}_n$ itself. So the old examples were exceptions ($V_2$ often turns out to be an exception) and fundamental groups of complements of branch curves are not "big". They are surprisingly "small". Moreover, in all the new examples $G, \tilde{G}$ are an extension of a solvable group by a symmetric one. In addition, in *all* our computations the decomposition series had quotients of the type $(\mathbb{Z}^t \oplus \mathbb{Z}_p)^q$. Thus we can attach to the embedding a discrete invariant consisting of $(\ldots t_i, p_i, q_i, \ldots)$ which will distinguish surfaces in different components! The globality of the new invariant depends on the answer to the following question:

**Question.** *For which families of simply connected algebraic surfaces of general type is the fundamental group of the complement of the branch curve of a generic projection to $\mathbb{CP}^2$ an extension of a solvable group by a symmetric group?*

## References


[1] Chen, Z., *On the geography of surfaces (simply connected surfaces with positive index)*, Math. Ann. **277**, 141-164 (1987).

[2] Ciliberto, C., Lopez, A. and Miranda, R., *Projective degenerations of K3 surfaces, Gaussian maps and Fano threefolds*, Invent. Math. **114**, 641-667 (1993).

[3] Feustel / Holzapfel, *Symmetry points and Chern invariants of Picard-modular-surfaces*, Math. Nach. **111**, 7-40 (1983).

[4] Freitag, P., Robb, A. and Teicher, M., *The fundamental group of Galois covers of Hirzebruch surfaces*, preprint.

[5] Moishezon,, B., *Stable branch curves and braid monodromies*, Lect. Notes in Math. **862**, 107-192 (1981).

[6] Moishezon, B., Robb, A. and Teicher, M., *On Galois covers of Hirzebruch surfaces*, Math. Ann. **305**, 493-539 (1996).

[7] Moishezon, B. and Teicher, M., *Existence of simply connected algebraic surfaces of positive and zero indices*, Proceedings of the National Academy of Sciences, United States of America **83**, 6665-6666 (1986).

[8] Moishezon, B. and Teicher, M., *Simply connected algebraic surfaces of positive index*, Invent. Math. **89**, 601-643 (1987).

[9] Moishezon, B. and Teicher, M., *Galois coverings in the theory of algebraic surfaces*, Proc. of Symp. in Pure Math. **46**, 47-65 (1987).

[10] Moishezon, B. and Teicher, M., *Braid group technique in complex geometry, I*, Contemp. Math. **78**, 425-555 (1988).

[11] Moishezon, B. and Teicher, M., *Finite fundamental groups, free over $\mathbb{Z}/c\mathbb{Z}$, for Galois covers of* $\mathbb{CP}^2$, Math. Ann. **293**, 749-766 (1992).

[12] Moishezon, B. and Teicher, M., *Braid group technique in complex geometry, II, From arrangements of lines and conics to cuspidal curves*, Algebraic Geometry, Lecture Notes in Math. **1479**, (1990).

[13] Moishezon, B. and Teicher, M., *Braid group techniques in complex geometry, III, Projective degeneration of $V_3$*, Contemp. Math. **162**, 313-332 (1994).

[14] Moishezon, B. and Teicher, M., *Braid group techniques in complex geometry, IV, Braid monodromy of the branch curve $S_3$ of $V_3 \to \mathbb{CP}^2$ and application to $\pi_1 : (\mathbb{CP}^2 - S_3, *)$*, Contemp. Math. **162**, 338-358 (1994).

[15] Moishezon, B. and Teicher, M., *Braid group techniques in complex geometry, V, the fundamental group of a complement of a branch curve of a Veronese generic projection*, Communications in Analysis and Geometry, to appear.

[16] Moishezon, B. and Teicher, M., *Braid Groups, Singularities and Algebraic Surfaces*, Birkhauser, to appear.

[17] Moishezon, B. and Teicher, M., *Fundamental groups of complements of curves as solvable groups*, IMCP (AMS) **9**, 329-345 (1996).

[18] Robb, A. and Teicher, M., *Applications of braid group techniques to the decomposition of moduli spaces, new examples*, Topology and its Applications **78**, 143-151 (1997).

[19] Seiberg, N. and Witten, E., *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nuclear Physics B **426**, 19-52 (1994).

[20] Shustin, E., *Real plane algebraic curves with prescribed singularities*, Topology **32(4)**, 845-856 (1993).

[21] Teicher, M., *The fundamental groups of a $\mathbb{CP}^2$-complement of a branch curve as an extension of a solvable group by a symmetric group*, preprint.

[22] Teicher, M., *Braid groups, algebraic surfaces and complements of braid curves*, Proceedings of the Santa Cruz Summer Institute, 1995, Proceedings of Symposia in Pure Mathematics **62(1)**, 127-150 (1997).

[23] Teicher, M., *On the quotient of the braid group by commutators of transversal halftwists and its group actions*, Topology and its Applications, 153-186 (1997).

[24] Teicher, M., *Chern classes of fibered products of surfaces*, preprint.

[25] Witten, E., *Monopoles and four-manifolds*, Math. Research Letters **1**, 769-796 (1994).

[26] Zariski, O., *Algebraic Surfaces* (Ch. VIII), Second Edition, Springer (1971).


## A LOCAL LEFSCHETZ THEOREM


*Heinke Wagner*


For analytic subsets $X, Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ with codim $X = q$, $\dim Y = d$, a result of Faltings (1980) implies, that $H^i(Y, Y \cap X; \mathbb{C}) = 0$ for $i \le \frac{d+1}{q} - 2$, provided $Y - Y \cap X$ is nonsingular. This bound is sharp. The analogous result for homotopy groups was proved by M. Peternell (1983). We presented a proof of the local generalization of this homotopy result: Let $(X, 0), (Y, 0) \subseteq (\mathbb{C}^n, 0)$ be germs of analytic subsets, $S_\varepsilon$ a sphere of sufficiently small radius $\varepsilon > 0$ centered around 0 and consider $X_\varepsilon := X \cap S_\varepsilon$, $Y_\varepsilon := Y \cap S_\varepsilon$. If $Y_\varepsilon - Y_\varepsilon \cap X_\varepsilon$ is nonsingular, then

$$\pi_i(Y_\varepsilon, Y_\varepsilon \cap X_\varepsilon) = 0 \quad \text{for} \quad i \le \frac{\dim Y}{\text{codim } X} - 2.$$

# VANISHING THEOREMS FOR $L^2$-COHOMOLOGY AND SOME APPLICATIONS IN ALGEBRAIC GEOMETRY

*Kang Zuo*
(Joint work with Jürgen Jost)

Let $X$ denote a compact Kähler manifold. Suppose that the fundamental group of $X$ admits a big reductive linear representation. A representation is called *big* if its Shafarevich map defined by Campana and Kollár is birational (roughly speaking, if it does not factor through any holomorphic map on $X$ of positive dim of generic fibre).

Using pluriharmonic map on $X$ we construct a bounded singular Kähler form on the universal covering of $X$, which is d-exact of a 1-form of at most linear growth. Furthermore, we use this Kähler form and some important idea due to Gromov and show that all $L^2$-holomorphic forms on the universal covering except the top forms must vanish. Consequently we prove a conjecture of Kollár in the representation case, namely, if the fundamental group of $X$ is big then the holomorphic Euler characteristic of canonical line bundle of $X$ is non negative.

Edited by: D. Garber.

# Tagungsteilnehmer


Prof.Dr. Shreeram Abhyankar
Dept. of Mathematics
Purdue University

West Lafayette , IN 47907-1395
USA

Prof.Dr. Jaume Amoros
Departamento de Matematicas
ETSEIB - UPC
Diagonal 647

E-08028 Barcelona

Prof.Dr. Donu Arapura
Dept. of Mathematics
Purdue University

West Lafayette , IN 47907-1395
USA

Prof.Dr. Enrique Artal Bartolo
Dept. Matematicas
Universidad de Zaragoza
Facultad de Ciencias

E-50009 Zaragoza

Dr. Ingrid Bauer
Fachbereich Mathematik
Universität Göttingen
Bunsenstr. 3/5

37073 Göttingen

Prof.Dr. Stefan Alois Bauer
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131

33501 Bielefeld

Prof.Dr. Fedor A. Bogomolov
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street

New York , NY 10012-1110
USA

Prof.Dr. Paul Bressler
Department of Mathematics
Pennsylvania State University
218 McAllister Building

University Park , PA 16802
USA

Prof.Dr. Frederic Campana
Dept. de Mathematiques
Universite de Nancy I
B.P. 239

F-54506 Vandoeuvre-les-Nancy Cedex

Prof.Dr. Jim Carlson
Dept. of Mathematics
University of Utah

Salt Lake City , UT 84112
USA

Prof.Dr. Fabrizio Catanese
Mathematisches Institut
Universität Göttingen
Bunsenstr. 3-5

37073 Göttingen

Dr. Denis Cheniot
Centre de Mathematiques et
d'Informatique
Universite de Provence
39, Rue Joliot Curie

F-13453 Marseille Cedex 13

Prof.Dr. Daniel C. Cohen
Dept. of Mathematics
Louisiana State University

Baton Rouge , LA 70803-4918
USA

Prof.Dr. Olivier Debarre
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes

F-67084 Strasbourg Cedex

Dr. Gerd E. Dethloff
UFR Sciences et Techniques
Department de Mathematiques
Universite de Bretagne Occidentale
6, avenue Le Gorgeu, BP 452

F-29275 Brest Cedex

Prof.Dr. Igor Dolgachev
Department of Mathematics
The University of Michigan
3220 Angell Hall

Ann Arbor , MI 48109-1003
USA

Dr. Philippe Eyssidieux
Laboratoire Emile Picard
CNRS UMR 5580
UFR MIG Universite Paul Sabatier
118 route de Narbonne

F-31000 Toulouse Cedex 4

Prof.Dr. Hubert Flenner
Institut f. Mathematik
Ruhr-Universität Bochum
Gebäude NA

44780 Bochum

Prof.Dr. Eriko Hironaka
Department of Mathematics
The Florida State University

Tallahassee , FL 32306
USA

Prof.Dr.Dr.h.c. Alan T. Huckleberry
Institut f. Mathematik
Ruhr-Universität Bochum
Gebäude NA

44780 Bochum

Prof.Dr. Jürgen Jost
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26

04103 Leipzig

Prof.Dr. Shulim Kaliman
Dept. of Mathematics and Computer
Science
University of Miami
P.O. Box 248011

Coral Gables , FL 33124
USA

Prof.Dr. Jyoichi Kaneko
Graduate School of Mathematics
Kyushu University
6-10-1 Hakozaki

Fukuoka 812-8581
JAPAN

Prof.Dr. Ludmil Katzarkov
Dept. of Mathematics
University of California at Irvine

Irvine , CA 92697-3875
USA

Prof.Dr. Meeyoung Kim
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26

53225 Bonn

Prof.Dr. Toshitake Kohno
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku

Tokyo 153
JAPAN

Prof.Dr. Janos Kollar
Dept. of Mathematics
University of Utah

Salt Lake City , UT 84112
USA

Prof.Dr. Viktor S. Kulikov
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26

53225 Bonn

Prof.Dr. Herbert Kurke
Institut für Reine Mathematik
Humboldt-Universität Berlin

10099 Berlin

Prof.Dr. Finnur Larusson
Department of Mathematics
The University of Western Ontario

London , Ont. N6A 5B7
CANADA

Prof.Dr. Anatoly Libgober
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street

Chicago , IL 60607-7045
USA

Prof.Dr. Vladimir Lin
Department of Mathematics
Technion
Israel Institute of Technology

Haifa 32000
ISRAEL

Prof.Dr. Makoto Namba
Dept. of Mathematics
Osaka University
Toyonaka

Osaka 560
JAPAN

Prof.Dr. Terrence Napier
Dept. of Mathematics
Lehigh University
Christmas-Saucon Hall
14 E. Packer Avenue

Bethlehem , PA 18015-3174
USA

Prof.Dr. Andras Nemethi
Department of Mathematics
Ohio State University
231 West 18th Avenue

Columbus , OH 43210-1174
USA

Prof.Dr. Mutsuo Oka
Dept. of Mathematics
Faculty of Sciences
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji

Tokyo 192-03
JAPAN

Dr. Stepan Orevkov
System Studies Institute
Russian Academy of Sciences
Avtozavodskaya 23

Moscow
RUSSIA

Prof.Dr. Peter Orlik
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive

Madison WI, 53706-1388
USA

Prof.Dr. Mihai Paun
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74

F-38402 Saint-Martin-d'Heres Cedex

Prof.Dr. Lee Rudolph
P.O.Box 251

Adamsville , RI 02801
USA

Prof.Dr. Peter Russell
Dept. of Mathematics and Statistics
McGill University
805, Sherbrooke Street West

Montreal , P.Q. H3A 2K6
CANADA

Heinke Wagner
Institut f. Mathematik
Ruhr-Universität Bochum
Gebäude NA
Universitätsstr. 150

44801 Bochum

Prof.Dr. Ichiro Shimada
Dept. of Mathematics
Hokkaido University
Kita-ku

Sapporo 060
JAPAN

Prof.Dr. Bronislaw Wajnryb
Department of Mathematics
Technion
Israel Institute of Technology

Haifa 32000
ISRAEL

Prof.Dr. Alexander I. Suciu
Dept. of Mathematics
Northeastern University
567 Lake Hall

Boston , MA 02115
USA

Prof.Dr. Mikhail Zaidenberg
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74

F-38402 Saint-Martin-d'Heres Cedex

Prof.Dr. Mina Teicher
Dept. of Mathematics
Bar-Ilan University

52 900 Ramat-Gan
ISRAEL

Prof.Dr. Kang Zuo
Fachbereich Mathematik
Universität Kaiserslautern

67653 Kaiserslautern

Prof.Dr. Hiro-o Tokunaga
Dept. of Mathematics
Kochi University
Faculty of Sciences
2-5-1, Akebono

Kochi 780-8520
JAPAN

Gefördert durch
DFG Deutsche Forschungsgemeinschaft